



A note on Karlsson-Srivastava terminating series ${}_3F_2(4)$ and ${}_3F_2\left(\frac{1}{4}\right)$ with applications

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Abstract

The aim of this note is to derive two general double-series identities having the bounded sequences, by employing Karlsson-Srivastava terminating summation theorems ${}_3F_2(4)$, ${}_3F_2\left(\frac{1}{4}\right)$ along with series rearrangement technique. Further, two reduction formulas for Srivastava-Daoust double hypergeometric functions with arguments $(z, -4z)$ and $\left(z, \frac{z}{4}\right)$ are also obtained, in terms of single generalized hypergeometric functions with arguments $\pm(27)^{(D-E-1)}z^3$ respectively, associated with suitable convergence conditions.

Keywords: Srivastava-Daoust double hypergeometric function, terminating hypergeometric summation theorem, generalized hypergeometric function

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1. Introduction, definitions and preliminaries

In our investigations, we use the following standard notations:

$\mathbb{Z}_0^- := \{0, -1, -2, -3, \dots\}$ and the symbols \mathbb{C}, \mathbb{R} for the sets of complex numbers and real numbers respectively.

For the definitions of Pochhammer symbol and generalized hypergeometric series (or function) ${}_pF_q$, which is a natural generalization of the Gaussian hypergeometric series ${}_2F_1$, we refer the beautiful monographs (see, e.g., [1, 7, 10, 11, 16]).

In an earlier paper [16, p.199] Srivastava and Daoust defined a generalization of the Kampé de Fériet function [2,

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p. 150] by means of the double hypergeometric series (see also [12]-[15]):

$$F_{C: D: D'}^{A: B: B'} \left(\begin{matrix} [(a_A) : \vartheta, \varphi] : [(b_B) : \psi] ; [(b'_{B'}) : \psi'] ; \\ [(c_C) : \delta, \varepsilon] : [(d_D) : \eta] ; [(d'_{D'}) : \eta'] ; \end{matrix} \middle| x, y \right) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^A (a_j)_{m\vartheta_j+n\varphi_j} \prod_{j=1}^B (b_j)_{m\psi_j} \prod_{j=1}^{B'} (b'_j)_{n\psi'_j}}{\prod_{j=1}^C (c_j)_{m\delta_j+n\varepsilon_j} \prod_{j=1}^D (d_j)_{m\eta_j} \prod_{j=1}^{D'} (d'_j)_{n\eta'_j}} \frac{x^m y^n}{m! n!}, \quad (1.1)$$

where the coefficients

$$\left\{ \begin{matrix} \vartheta_1, \dots, \vartheta_A; \varphi_1, \dots, \varphi_A; \psi_1, \dots, \psi_B; \psi'_1, \dots, \psi'_{B'}; \delta_1, \dots, \delta_C; \\ \varepsilon_1, \dots, \varepsilon_C; \eta_1, \dots, \eta_D; \eta'_1, \dots, \eta'_{D'} \end{matrix} \right. \quad (1.2)$$

are real and positive.

$$\Delta_1 = 1 + \sum_{j=1}^C \delta_j + \sum_{j=1}^D \eta_j - \sum_{j=1}^A \vartheta_j - \sum_{j=1}^B \psi_j, \quad (1.3)$$

$$\Delta_2 = 1 + \sum_{j=1}^C \varepsilon_j + \sum_{j=1}^{D'} \eta'_j - \sum_{j=1}^A \varphi_j - \sum_{j=1}^{B'} \psi'_j. \quad (1.4)$$

Case I. The double power series in (1.1) converges for all complex values of x and y when $\Delta_1 > 0$ and $\Delta_2 > 0$.

Case II. The double power series in (1.1) is convergent for suitably constrained values of $|x|$ and $|y|$ when $\Delta_1 = 0$ and $\Delta_2 = 0$.

Case III. The double power series in (1.1) would diverge except when, trivially, $x = y = 0$ when $\Delta_1 < 0$ and $\Delta_2 < 0$.

The following results will be needed in our investigations.

Summation theorems of Karlsson-Srivastava for Clausen’s series [6, Eq. (9), p. 86]:

$${}_3F_2 \left[\begin{matrix} -n, b - \frac{1}{2}, 1 - b - n; \\ 2b - 1, 2 - 2b - 2n; \end{matrix} \middle| 4 \right] = \begin{cases} 0 & ; n = 3r + 1 \text{ and } 3r + 2, \\ \frac{(3r)!(b-\frac{1}{2})_{2r}}{4^r r! (b)_r (b-\frac{1}{2})_{3r}} & ; n = 3r, \end{cases} \quad (1.5)$$

where $r = 0, 1, 2, 3, \dots$

By the reversal of order of the terms in finite summation of above terminating hypergeometric series ${}_3F_2(4)$ and replacing b by $b + \frac{1}{2}$, we get the following result of Karlsson [5, Entry (iv), p. 178]:

$${}_3F_2 \left[\begin{matrix} -n, 2b + n, 1 - 2b - n; \\ b + \frac{1}{2}, 1 - b - n; \end{matrix} \middle| \frac{1}{4} \right] = \begin{cases} 0 & ; n = 3m + 1 \text{ and } 3m + 2, \\ \frac{(-1)^m (3m)!}{4^m m! (b+\frac{1}{2})_m (b+2m)_m} & ; n = 3m, \end{cases} \quad (1.6)$$

where $m = 0, 1, 2, 3, \dots$

Cauchy’s double-series identity (see[10, p. 57, Lemma (11)]; see also [16, p. 100, Lemma (2)]):

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Theta(n, m) = \sum_{n=0}^{\infty} \sum_{m=0}^n \Theta(n - m, m). \quad (1.7)$$

Decomposition of unilateral series:

$$\sum_{n=0}^{\infty} \Phi(n) = \sum_{n=0}^{\infty} \Phi(3n) + \sum_{n=0}^{\infty} \Phi(3n + 1) + \sum_{n=0}^{\infty} \Phi(3n + 2), \quad (1.8)$$

provided that both sides of identities (1.7) and (1.8) are absolutely convergent.

Motivated by the work of Henrici’s triple product theorem [4], Bailey [3], Karlsson [5], Karlsson-Srivastava [6], Qureshi et al. [8] and [9], we derive two double-series identities involving a bounded sequence of arbitrary complex numbers in section 2. Further, in Section 3, we obtain two reduction formulas for Srivastava-Daoust double hypergeometric function in terms of generalized hypergeometric functions with arguments $\pm(27)^{(D-E-1)}z^3$ respectively, associated with suitable convergence conditions.

Remark 1.1. Any values of parameters and arguments in sections 2 and 3, leading to the results which do not make sense, are tacitly excluded.

2. Applications of Karlsson-Srivastava theorems (1.5) and (1.6)

In this section, we derive two double-series identities involving a bounded sequence:

Theorem 2.1. *Let us assume that $\{\Psi(\mu)\}_{\mu=1}^\infty$ be a bounded sequence of essentially arbitrary complex numbers or real numbers such that $\Psi(0) \neq 0$. Then, the following general double-series identity holds true:*

$$\sum_{n=0}^\infty \sum_{m=0}^\infty \left\{ \Psi(n+m) \frac{(2b-1)_{2n+m} (b)_{n+m} \left(b - \frac{1}{2}\right)_m (-4)^m z^{n+m}}{(2b-1)_{2n+2m} (b)_n (2b-1)_m n! m!} \right\} = \sum_{n=0}^\infty \left\{ \Psi(3n) \frac{\left(\frac{2b-1}{4}\right)_n \left(\frac{2b+1}{4}\right)_n z^{3n}}{(b)_n \left(\frac{2b-1}{6}\right)_n \left(\frac{2b+1}{6}\right)_n \left(\frac{2b+3}{6}\right)_n (27)^n n!} \right\}, \tag{2.1}$$

where $2b - 1 \in \mathbb{C} \setminus \mathbb{Z}_0^-$, then the remaining numerator and denominator parameters are also neither zero nor negative integers. Infinite series occurring on both sides of equation (2.1) are absolutely convergent.

Proof of the assertion (2.1). Let

$$\Lambda(z) := \sum_{n=0}^\infty \sum_{m=0}^\infty \left\{ \Psi(n+m) \frac{(2b-1)_{2n+m} (b)_{n+m} \left(b - \frac{1}{2}\right)_m (-4)^m z^{n+m}}{(2b-1)_{2n+2m} (b)_n (2b-1)_m n! m!} \right\}. \tag{2.2}$$

Replacing n by $(n - m)$ in (2.2) and using Cauchy’s double-series identity (1.7), we have

$$\Lambda(z) = \sum_{n=0}^\infty \sum_{m=0}^n \left\{ \Psi(n) \frac{(b)_n (2b-1)_{2n-2m} \left(b - \frac{1}{2}\right)_m (-4)^m z^n}{(2b-1)_{2n} (b)_{n-m} (2b-1)_m (n-m)! m!} \right\}. \tag{2.3}$$

Using the algebraic property of Pochhammer symbol in equation (2.3), we get

$$\begin{aligned} \Lambda(z) &= \sum_{n=0}^\infty \left\{ \Psi(n) \frac{z^n}{n!} \sum_{m=0}^n \frac{(-n)_m \left(b - \frac{1}{2}\right)_m (1-b-n)_m 4^m}{(2-2b-2n)_m (2b-1)_m m!} \right\} \\ &= \sum_{n=0}^\infty \left\{ \Psi(n) \frac{z^n}{n!} {}_3F_2 \left[\begin{matrix} -n, b - \frac{1}{2}, 1-b-n; \\ 2b-1, 2-2b-2n; \end{matrix} \quad 4 \right] \right\}. \end{aligned} \tag{2.4}$$

Now applying the decomposition identity (1.8) on the right-hand side of equation (2.4), we find

$$\begin{aligned} \Lambda(z) &= \sum_{n=0}^\infty \left\{ \Psi(3n) \frac{z^{3n}}{(3n)!} {}_3F_2 \left[\begin{matrix} -3n, b - \frac{1}{2}, 1-b-3n; \\ 2b-1, 2-2b-6n; \end{matrix} \quad 4 \right] \right\} \\ &+ \sum_{n=0}^\infty \left\{ \Psi(3n+1) \frac{z^{3n+1}}{(3n+1)!} {}_3F_2 \left[\begin{matrix} -(3n+1), b - \frac{1}{2}, -b-3n; \\ 2b-1, -2b-6n; \end{matrix} \quad 4 \right] \right\} \\ &+ \sum_{n=0}^\infty \left\{ \Psi(3n+2) \frac{z^{3n+2}}{(3n+2)!} {}_3F_2 \left[\begin{matrix} -(3n+2), b - \frac{1}{2}, -b-3n-1; \\ 2b-1, -2b-6n-2; \end{matrix} \quad 4 \right] \right\}. \end{aligned} \tag{2.5}$$

Finally, use of summation theorem (1.5) of Karlsson-Srivastava leads to

$$\Lambda(z) = \sum_{n=0}^{\infty} \left\{ \Psi(3n) \frac{\left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{2b-1}{4}\right)_n \left(\frac{2b+1}{4}\right)_n z^{3n}}{(1)_{3n} (b)_n \left(\frac{2b-1}{6}\right)_n \left(\frac{2b+1}{6}\right)_n \left(\frac{2b+3}{6}\right)_n} \right\}, \tag{2.6}$$

which, after simplification, yields the required result (2.1). □

Theorem 2.2. Let us assume that $\{\Psi(\mu)\}_{\mu=1}^{\infty}$ be a bounded sequence of essentially arbitrary complex numbers or real numbers such that $\Psi(0) \neq 0$. Then, the following general double-series identity holds true:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ \Psi(n+m) \frac{(2b)_{n+2m} (b)_n z^{n+m}}{(b)_{n+m} (2b)_n \left(b + \frac{1}{2}\right)_m (-4)^m n! m!} \right\} = \sum_{n=0}^{\infty} \left\{ \Psi(3n) \frac{\left(\frac{b}{2}\right)_n \left(\frac{b+1}{2}\right)_n (-z)^{3n}}{\left(\frac{b}{3}\right)_n \left(\frac{2b+1}{2}\right)_n \left(\frac{b+1}{3}\right)_n \left(\frac{b+2}{3}\right)_n (27)^n n!} \right\}, \tag{2.7}$$

where $2b \in \mathbb{C} \setminus \mathbb{Z}_0^-$, then the remaining numerator and denominator parameters are also neither zero nor negative integers. Infinite series occurring on both sides of equation (2.7) are absolutely convergent.

Proof of the assertion (2.7). The proof of (2.7) follows exactly the same procedure as in the assertion (2.1). Use of a summation theorem (1.6) is made here. □

3. Applications of Double-Series identities (2.1) and (2.7)

In this section, we establish two results for the reducibility of the Srivastava-Daoust double hypergeometric function defined in (1.1).

Theorem 3.1. The following result holds true:

$$F_{E+1:1:1}^{D+2:0:1} \left(\begin{matrix} [(d_D) : 1, 1], [2b - 1 : 2, 1], [b : 1, 1] : \text{---} ; \left[\frac{2b-1}{2} : 1\right] ; \\ [(e_E) : 1, 1], [2b - 1 : 2, 2] : [b : 1]; [2b - 1 : 1]; \end{matrix} \middle| z, -4z \right) = {}_{2+3D}F_{4+3E} \left[\begin{matrix} \Delta[3; (d_D)], \Delta\left(2; \frac{2b-1}{2}\right) ; \\ \Delta[3; (e_E)], \Delta\left(3; \frac{2b-1}{2}\right), b ; \end{matrix} \middle| (27)^{(D-E-1)} z^3 \right], \tag{3.1}$$

where $e_1, e_2, \dots, e_E, 2b - 1 \in \mathbb{C} \setminus \mathbb{Z}_0^-$, then the remaining numerator and denominator parameters are also neither zero nor negative integers.

- (i) When $D \leq E$, then it is always convergent for all finite values of z .
- (ii) When $D = E + 1$, then both sides of equation (3.1) will be convergent for suitable constrained values of $|z|$.

Theorem 3.2. The following result holds true:

$$F_{E+1:1:1}^{D+1:1:0} \left(\begin{matrix} [(d_D) : 1, 1], [2b : 1, 2] : [b : 1]; \text{---} ; \\ [(e_E) : 1, 1], [b : 1, 1] : [2b : 1]; \left[\frac{2b+1}{2} : 1\right]; \end{matrix} \middle| z, \frac{-z}{4} \right) = {}_{2+3D}F_{4+3E} \left[\begin{matrix} \Delta[3; (d_D)], \Delta(2; b) ; \\ \Delta[3; (e_E)], \Delta(3; b), \frac{2b+1}{2} ; \end{matrix} \middle| -(27)^{(D-E-1)} z^3 \right], \tag{3.2}$$

where $e_1, e_2, \dots, e_E, 2b \in \mathbb{C} \setminus \mathbb{Z}_0^-$, then the remaining numerator and denominator parameters are also neither zero nor negative integers.

- (i) When $D \leq E$, then it is always convergent for all finite values of z .

(ii) When $D = E + 1$, then both sides of equation (3.2) will be convergent for suitable constrained values of $|z|$.

Proof of the assertion (3.1). Put

$$\Psi(\mu) = \frac{(d_1)_\mu (d_2)_\mu \dots (d_D)_\mu}{(e_1)_\mu (e_2)_\mu \dots (e_E)_\mu} = \frac{\prod_{i=1}^D (d_i)_\mu}{\prod_{j=1}^E (e_j)_\mu}, \quad (\mu = 0, 1, 2, 3, \dots)$$

on both sides of general double-series identity (2.1) to obtain

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\prod_{i=1}^D (d_i)_{n+m} (b)_{n+m} (2b-1)_{2n+m} \left(b - \frac{1}{2}\right)_m (-4)^m z^{n+m}}{\prod_{j=1}^E (e_j)_{n+m} (2b-1)_{2n+2m} (b)_n (2b-1)_m n! m!} = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^D (d_i)_{3n} \left(\frac{2b-1}{4}\right)_n \left(\frac{2b+1}{4}\right)_n z^{3n}}{\prod_{j=1}^E (e_j)_{3n} (b)_n \left(\frac{2b+1}{6}\right)_n \left(\frac{2b-1}{6}\right)_n \left(\frac{2b+3}{6}\right)_n (27)^n n!}. \quad (3.3)$$

Applying the definition of double hypergeometric function (1.1) of Srivastava-Daoust in the left-hand side of (3.3), also recalling the well-known definition of generalized hypergeometric function together with some algebraic properties of Pochhammer symbols in the right-hand side of equation (3.3), we obtain the desired result (3.1). \square

The proof of assertion (3.2) follows exactly on the same procedure. Use of double-series identity (2.7) is made here.

4. Conclusion

To close the exposition in above article, we have obtained two general double-series identities by employing the Karlsson-Srivastava summation theorems ${}_3F_2(4)$ and ${}_3F_2\left(\frac{1}{4}\right)$. On the contrary, the reduced form of double hypergeometric functions of Srivastava-Daoust having arguments $(z, -4z)$ and $\left(z, \frac{-z}{4}\right)$ are also derived. On replacing Karlsson-Srivastava terminating summation theorems by any other terminating summation theorem, similar types of results can be obtained in an analogous manner. It is hoped that the results derived in this paper will find useful application.

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