


About on distinguished chains

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Abstract

Let v be a valuation of a field K and $a = a_0, a_1, \dots, a_m$ be a complete distinguished chain for $a \in \bar{K} \setminus K$. Let v_a be the extension of the valuation v to the field $K(a)$. In this study some properties provided by distinguished pairs and lifting polynomials used to define a residual algebraic free extension of v to the rational function field $K(x)$ are obtained. The value group and residue field of the valuation v_a are written with the help of constants $\delta_K(a_i)$ and minimal polynomials of the elements in the chain. In addition to the relations between value groups and residue fields of the valuations v_{a_i, b_i} of the fields $K(a_i, b_i)$ are studied by considering two chains $a = a_0, a_1, \dots, a_m$ and $b = b_0, b_1, \dots, b_m$.

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1. Introduction



Along this study, v is a Henselian valuation of K , G_v is value group, k_v is residue field, O_v is valuation ring of v , $p_v : O_v \rightarrow k_v$ is the canonical homomorphism and \bar{v} is the fixed extension of v to \bar{K} which is the algebraic closure of K . For any λ in the valuation ring of v , $p_v(\lambda) = \lambda^*$ will indicate its v -residue. If λ isn't an element of O_v , for an element $b \in K$ such that $v(\lambda) = v(b)$, λ^* will indicate the v -residue of λ/b . For $\theta \in \bar{K} \setminus K$, let $M(\theta, K) = \{\bar{v}(\theta - \alpha) \mid \alpha \in \bar{K}, [K(\alpha) : K] < [K(\theta) : K]\}$ be the subset of $G_{\bar{v}}$. In 2002, K. Aghigh, S. K. Khanduja have proved that $M(\theta, K)$ has a maximum element for every $\theta \in \bar{K} \setminus K$ if and only if each simple extension of (K, v) is defectless. The supremum $\delta_K(\theta)$ of $M(\theta, K)$ is called the main invariant of θ . $\delta_K(\theta) = \{\bar{v}(\theta - \alpha) \mid \alpha \in \bar{K}, [K(\alpha) : K] < [K(\theta) : K]\}$ satisfies a fundamental principle. Also they studied these pairs (b, a) of elements of \bar{K} where a is an element of smallest degree over K satisfying $\bar{v}(b - a) = \delta_K(b)$. This kind pairs are known as distinguished pairs (d.p.). It means that, a pair (b, a) of elements of \bar{K} is called a d.p. if the followings are satisfied:

$$\deg b > \deg a, \quad \bar{v}(b - a) = \delta_K(b) \quad \text{and} \quad \bar{v}(b - c) < \bar{v}(b - a)$$

for $c \in \bar{K}$, $\deg c < \deg a$ (cf. [1]).

A chain $b = b_0, b_1, \dots, b_m$ of elements of \bar{K} is called as a complete distinguished chain (c.d.c.) for $b \in \bar{K} \setminus K$ if (b_i, b_{i+1}) is a d.p. for $0 \leq i \leq m - 1$ and $b_m \in K$.

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N. Popescu and A. Zaharescu presented the notion of distinguished pairs (d.p.) and distinguished chains (d.c.) at firstly (cf. [9]). In 1995 they studied the relations between roots of lifting polynomials and minimal pairs defining residual transcendental extensions to $K(x)$.

Let v be a valuation on K , and w be an extension of v to $K(x)$.

An element $(a, \delta) \in \bar{K} \times G_{\bar{v}}$ is known a minimal pair with respect to (K, v) if each $b \in \bar{K}$, $v(a - b) \geq \delta$ satisfies $[K(a) : K] \leq [K(b) : K]$. The valuation w is known a residual transcendental extension of v if k_w is a transcendental extension of k_v . Then w is defined by minimal pair $(a, \delta) \in \bar{K} \times G_{\bar{v}}$. Let $f(x) = Irr(a, K)$ be the minimal polynomial of a over K and $\gamma = \sum_{a'} \inf(\delta, \bar{v}(a - a'))$ where a' runs over all roots of $f(x)$.

If $F(x) \in K[x]$ and $F(x) = \sum_i F_i(x)f(x)^i$, $\deg F_i < \deg f$ is the f -expansion of $F(x)$ then the valuation $w = w_{(a,\delta)}$ on $K[x]$ is defined as $w_{(a,\delta)}(F(x)) = \inf_i \{ \bar{v}(F_i(a)) + i\gamma \}$ (cf. [3]).

Let e be the smallest positive integer satisfying $e\gamma \in G_{v_a}$ and $h(x) \in K[x]$ satisfying $\deg h < \deg f$, $v_a(h(a)) = e\gamma$ where v_a is the restriction of \bar{v} to the field $K(a)$. Let $r = \frac{f^e}{h} \in K(x)$ satisfying $w_{(a,\delta)}(r) = 0$. So r^* is transcendental over k_v and also $k_{w_{(a,\delta)}} = k_{v_a}(r^*) = k_{v_a}(p_w(r))$ is the residue field of $w = w_{(a, \delta)}$ (cf. [9]).

Let $g(r^*) \in k_{v_a}[r^*]$. $G(x) \in K[x]$ is a lifting of $g(r^*)$ with respect to w if the followings are satisfied:

- i) $\deg G = e \cdot \deg g \cdot \deg f$
- ii) $w(G) = e \cdot \deg g \cdot \gamma$
- iii) $p_w(G/h^{\deg g}) = (G/h^{\deg g})^* = g$ (cf. [9]).

If k_w/k_v is an algebraic extension and G_w/G_v is a torsion group, w is called residual algebraic torsion(r.a.t.) extension of v . Then we have $G_v \subseteq G_w \subseteq G_{\bar{v}}$. If k_w/k_v is an algebraic extension and G_w/G_v is not a torsion group, w is called residual algebraic free(r.a.f.) extension of v . In this case $rank w = rank v + 1$ and $w = w_1 \circ w_2$ where w_1 is a valuation of $K(x)$ and w_2 is a valuation of k_{w_1} (cf. [4]).

1.1. Preliminaries

In this section basic definitions and theories are given.

Let $v = w \circ u$ be a composite valuation of a field K where w is a discrete real valuation of K with value group $s\mathbb{Z}$ and s is a positive real number. Let u be a valuation of the residue field k_w with value group G and π be a uniformizer of the valuation w with $w(\pi) = s$. For $\beta \in K, \beta \neq 0$, if β^* is the residue of $\beta \Big|_{\pi^{w(\beta)s^{-1}}}$ in the residue field k_w , then

$$v(\beta) = (w(\beta), u(\beta^*))$$

determines a valuation on the field K with value group $s\mathbb{Z} \times G$ (cf. [6]). Let v be a Henselian valuation on a field K , (θ, α) be a d.p. of elements of \bar{K} . The followings hold: $(\alpha, \delta_K(\theta))$ is a minimal pair, $w_{(\alpha, \delta_K(\theta))}(G(x)) = \bar{v}(G(\theta))$ for any polynomial $G(x) \in K[x]$ of degree less than the degree of θ over K (cf. [1]). Let v be a Henselian valuation on a field K and $p(x)$ and $q(x)$ be two monic irreducible polynomials over K of degrees m and n respectively satisfying $p(\theta) = 0$ and $q(\beta) = 0$. Then $\bar{v}(p(\beta)) = \frac{m}{n} \bar{v}(q(\theta))$ (cf. [2]).

Let $p(x) \in K[x]$ be a prime polynomial. The valuation w , defined as $w\left(\frac{f(x)}{g(x)}\right) = n$, $w(0) = \infty$ for each element $\frac{f(x)}{g(x)} = p(x)^n \frac{m(x)}{n(x)}$ of the field $K(x)$, is known as $p(x)$ -adic valuation of $K(x)$ where, $m(x), n(x) \in K[x]$, $(m(x), n(x)) = 1$. Let v be a Henselian valuation on a field K . If (θ, θ_1) and (θ_1, θ_2) are two d.p. of elements of \bar{K} then $\delta_K(\theta) > \delta_K(\theta_1) = \bar{v}(\theta - \theta_2)$ (cf. [5]).

Let v be a Henselian valuation on a field K . Let $a, b \in \bar{K}$ such that $\bar{v}(a - b) > \bar{v}(a - c)$ for every $c \in \bar{K}$ such that $[K(c) : K] < [K(a) : K]$. Let u and w be extensions of the valuation v to the fields $K(a)$ and $K(b)$ respectively. Then the followings are satisfied:

$$G_u \subseteq G_w, \quad k_u \subseteq k_w \quad \text{and} \quad [K(a) : K] \mid [K(b) : K],$$

(cf. [9]).

Let v be a Henselian valuation on a field K , (θ, α) be a d.p. and $f(x)$ be minimal polynomial of α over the field K . Let u and w be the extensions of v to the fields $K(\theta)$ and $K(\alpha)$ respectively. Then the followings are satisfied:

- i) $G_u = G_w + \bar{v}(f(\theta))\mathbb{Z}$.
- ii) $k_u = k_w \left(\left(\frac{f(\theta)^e}{h(\alpha)} \right)^* \right)$ where $h(x) \in K[x]$, $\deg h(x) < \deg \alpha$ and e is the smallest positive integer satisfying $e\bar{v}(f(\theta)) = \bar{v}(h(\alpha)) \in G_w$ (cf. [5]).

2. Main results

Certain relations between two lifting polynomials and their roots are given in the below:

Theorem 2.1. *Let v be a Henselian valuation on K and w be an extension of v to the field $K(x)$ defined by minimal pair (a, δ) . Let $G \in k_{v_a}[r^*]$ be a prime, monic polynomial and w_1 be G -adic valuation of the residue field $k_{v_a}(r^*)$. The following statements are provided, where $u = w_1 \circ w_2$ is a valuation of $K(x)$ with $\text{rank} u = 2$:*

- i) *Let g and g_1 are the liftings of the polynomial G and b and b_1 are one of the roots of the polynomials g and g_1 respectively such that $\bar{v}(b - a) = \bar{v}(b_1 - a) = \delta$. If (β, b) and (β, b_1) are two distinguished pairs then $\bar{v}(g(\beta)) = \bar{v}(g_1(\beta))$.*
- ii) *$\bar{v}(\varphi(\beta)) = \bar{v}(\varphi(\beta_1))$ holds, where $\text{Irr}(\beta, K) = \varphi(x)$.*

Proof. i) Let $t(x) \in K[x]$ be a uniformizer of the valuation w and $f(x) = \text{Irr}(\beta, K)$. A valuation of the field $K(x)$ with $\text{rank} u = 2$ is defined as

$$u(F) = (w(F), w_1((F/t^{w(F)}))^*).$$

Also valuation u is defined as $u(F) = \inf_i (w(F_i) + iw(g), i)$, where $F \in K[x]$ and $F = \sum_i F_i g^i$ is the g -expansion of F . In the same way using other lifting polynomial g_1 , for $F = \sum_j A_j g_1^j \in K[x]$, $u(F) = \inf_j (w(A_j) + jw(g_1), j)$ is written. Since $\deg g = \deg g_1$ for at least one $i = j$ we obtained that $w(F_i) + iw(g) = w(A_j) + jw(g_1)$ using the above expressions. It is seen that $w(F_i) = \bar{v}(F_i(\beta))$ and $w(A_i) = \bar{v}(A_i(\beta))$ by the assumption for β, b and $b_1 \in \bar{K}$. The equality $w(F_i) = \bar{v}(F_i(\beta)) = w(A_i) = \bar{v}(A_i(\beta))$ is obtained keeping in view of the definition of lifting polynomial. $F(\beta) = \sum_i F_i(\beta)(g(\beta))^i = \sum_i A_i(\beta)(g_1(\beta))^i$ is written from g and g_1 -expansions of F . Hence it is seen that $\bar{v}(F(\beta)) = \inf_i (\bar{v}(F_i(\beta)) + i\bar{v}(g(\beta))) = \inf_i (\bar{v}(A_i(\beta)) + i\bar{v}(g_1(\beta)))$. Then the equality $\bar{v}(g(\beta)) = \bar{v}(g_1(\beta))$ is obtained.

- ii) Let $\text{Irr}(\beta, K) = \varphi(x)$ and $\deg \varphi = s$. Then $s\bar{v}(g_1(\beta)) = \deg g_1 \bar{v}(\varphi(b_1))$ and $s\bar{v}(g(\beta)) = \deg g \bar{v}(\varphi(b))$. So it seen seen that $\bar{v}(\varphi(\beta)) = \bar{v}(\varphi(b_1))$. □

Theorem 2.2. *Let v be a Henselian valuation on K and $a = a_0, a_1, \dots, a_n$ be a c.d.c. for $a \in \bar{K} \setminus K$. Let $f_i = \text{Irr}(a_i, K)$, $\deg f_i = m_i$ and v_{a_i} be the extension of the valuation v to the field $K(a_i)$ for $i = 0, \dots, n$. Then the followings are satisfied:*

- i) $G_{v_a} = G_v + \{m_1 \delta_K(a) + m_2 \delta_K(a_1) + \dots + m_n \delta_K(a_{n-1})\} \mathbb{Z}$.
- ii) $k_{v_a} = k_v \left(\left(\left(\frac{f_1(a)}{f_0(a_1)^{m_1/m_0}} \right)^{e_1} \right)^*, \dots, \left(\left(\frac{f_n(a_{n-1})}{f_{n-1}(a_n)^{m_n/m_{n-1}}} \right)^{e_n} \right)^* \right)$.

Proof. i) We have the equality

$$G_{v_{a_i}} = G_{v_{a_{i+1}}} + \bar{v}(f_{i+1}(a_i))\mathbb{Z}$$

for $i = 0, \dots, n - 1$ from [2]. By using $\deg f_i = m_i$ the equality $\bar{v}(f_i(a)) = m_i \bar{v}(a - a_i)$ is obtained. Since (a, a_1) and (a_1, a_2) are two d.p. it is seen that

$$\bar{v}(a - a_2) = \bar{v}(a_1 - a_2) = \delta_K(a_1).$$

In addition to it is seen that

$$\bar{v}(a - a_i) = \bar{v}(a_{i-1} - a_i) = \delta_K(a_{i-1})$$

for $i = 3, \dots, n$ by continuing in the same way. Considering the above equations we can write

$$G_{v_a} = G_v + \{m_1\bar{v}(a - a_1) + m_2\bar{v}(a - a_2) + \dots + m_n\bar{v}(a - a_n)\} \mathbb{Z} \tag{2.1}$$

$$= G_v + \{m_1\delta_K(a) + m_2\delta_K(a_1) + \dots + m_n\delta_K(a_{n-1})\} \mathbb{Z}. \tag{2.2}$$

ii) Since $k_{v_{a_{i-1}}} = k_{v_{a_i}} \left(\left(\frac{f_i(a_{i-1})^{e_i}}{h_i(a_i)} \right)^* \right)$, $\deg h_i < \deg a_i$ and e_i is the smallest positive integer such that $e_i\bar{v}(f_i(a_{i-1})) = \bar{v}(h_i(a_i)) \in G_{v_{a_i}}$ for $i = 1, \dots, n - 1$, it is seen that $\left(\frac{f_i(a_{i-1})^{e_i}}{h_i(a_i)} \right)^* = \left(\left(\frac{f_{i-1}(a_{i-1})}{f_{i-1}(a_i)} \right)^{e_i} \right)^*$. Thus we have the following equality

$$k_{v_a} = k_v \left(\left(\left(\frac{f_1(a)}{f_0(a_1)} \right)^{e_1} \right)^*, \dots, \left(\left(\frac{f_n(a_{n-1})}{f_{n-1}(a_n)} \right)^{e_n} \right)^* \right).$$

□

Theorem 2.3. *v* be a Henselian valuation on K . If $a = a_0, a_1, \dots, a_n$ and $b = b_0, b_1, \dots, b_n$ are c.d.c. for $a \in \bar{K} \setminus K$ and $b \in \bar{K} \setminus K$ respectively then $G_{v_{a_i b_i}} \subseteq G_{v_{a_{i-1} b_{i-1}}}$ and $k_{v_{a_i b_i}} \subseteq k_{v_{a_{i-1} b_{i-1}}}$, $k_{v_{b_i}} \subseteq k_{v_{b_{i-1}}}$.

Proof. From [2] it is seen that;

$G_{v_{a_i}} \subseteq G_{v_{a_{i-1}}}$, $G_{v_{b_i}} \subseteq G_{v_{b_{i-1}}}$ and $k_{v_{a_i}} \subseteq k_{v_{a_{i-1}}}$, $k_{v_{b_i}} \subseteq k_{v_{b_{i-1}}}$ for $i = 1, \dots, n$. For completing the proof it is enough to show the images of the elements a_i and b_i are in the value groups $G_{v_{a_{i-1} b_{i-1}}}$ and the residues a_i^* and b_i^* are in the residue fields $k_{v_{a_{i-1} b_{i-1}}}$ as the elements of the field $K(a_i, b_i)$ are written in terms of a_i and b_i .

Considering the above inclusions we have $\bar{v}(b_i) \in G_{v_{b_i}} \subseteq G_{v_{b_{i-1}}} \subseteq G_{v_{a_{i-1} b_{i-1}}}$, $\bar{v}(a_i) \in G_{v_{a_i}} \subseteq G_{v_{a_{i-1}}} \subseteq G_{v_{a_{i-1} b_{i-1}}}$ and $b_i^* \in k_{v_{b_i}} \subseteq k_{v_{b_{i-1}}} \subseteq k_{v_{a_{i-1} b_{i-1}}}$, $a_i^* \in k_{v_{a_i}} \subseteq k_{v_{a_{i-1}}} \subseteq k_{v_{a_{i-1} b_{i-1}}}$ as desired. □

3. Conclusion

In this study basic properties of d.p. and c.d.c. are given and certain new relations between their value groups and residue fields are obtained.

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