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Higher order complex Szász-Kantorovich operators

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Abstract

This paper introduces a novel type of generalized complex Szász-Kantorovich operators. Subsequently, we establish quantitative estimates within Voronovskajaís theorem, determining precise orders for approximating analytic functions. Notably, our approach eliminates the need for imposing exponential growth conditions on compact disks when considering these operators.

Keywords: Szasz-Kantorovich operator, Voronovskaja type theorem, exact estimates

2020 MSC: 30E10, 41A25, 41A28, 30C45

1. Introduction

Complex approximation refers to the process of estimating a complex function by using simpler or more manageable functions. Complex approximation is often employed when dealing with complex numbers or functions defined in the complex plane. The goal is to find simpler functions that closely mimic the behavior of the complex function, making it easier to analyze or compute. Various methods and operators, such as Szász–Mirakjan operators or generalized complex Szasz-Kantorovich operators, may be used in complex approximation to achieve the desired level of accuracy in representing complex functions. This field is particularly relevant in complex analysis, where understanding the behavior of complex functions and finding effective ways to approximate them is a key aspect of mathematical research and applications.

For $\psi : [0, \infty) \to \mathbb{R}$ and $\eta \in \mathbb{N}$, the Szász–Mirakjan operators of real variable is defined as

$$S_\eta(\psi; x) = e^{-\eta x} \sum_{j=0}^{\infty} \frac{(\eta x)^j}{j!} \psi\left(\frac{j}{\eta}\right), \tag{1.1}$$

where the convergence of $S_\eta(\psi; x)$ to $\psi(x)$, under the exponential growth condition on ψ, that is $|\psi(x)| \le C \exp(Bx)$, for all $x \in [0, \infty)$, with $C, B > 0$, was established in [1]. In [10], Totik proved that $\left|S_\eta(\psi; x) - \psi(x)\right| \le \frac{C}{\eta}$ for all $x \in [0, \infty)$ and $\eta \in \mathbb{N}$, by adding some additional assumptions to ψ. In [3], Gal acquired quantitative estimates for the convergence within compact disks, for complex Szász-Mirakjan operators attached to analytic functions that satisfying some suitable exponential-type growth condition. Two years later, Gal [4] obtained a quantitative estimate in Voronovskaja's theorem and determined the exact orders for the approximation of analytic functions within a disk

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of radius R and center at 0, all without imposing exponential growth conditions. In [7], authors, introduced Szász-Kantorovich operatos in a compact disks and investigated the order of approximation and quantitative estimates for these operators. Approximation properties of the q-Szász-Kantorovich operators are studied in [5]. Moreover, all the findings in this current paper are derived without imposing exponential growth conditions on the analytic function ψ within the disk. In this paper, we introduce the new type generalized complex Szász-Kantorovich operators, defined as follows (see for real variable in [9])

$$K_{\eta}^{l}(\psi;\varsigma)=\sum_{k=0}^{\infty}p_{\eta,k}(\varsigma)\int_{0}^{1}\dots\int_{0}^{1}\psi\left(\frac{k+t_1+\dots+t_l}{\eta+l}\right)dt_1\dots dt_l, \tag{1.2}$$

where $l\in\mathbb{Z}$.

If ψ is bounded on $[0,\infty)$ then it is clear that $K_{\eta}^{l}(\psi;\varsigma)$ are well defined for all $\varsigma\in\mathbb{C}$.

Let $\mathbb{D}_R$ be a disc $\mathbb{D}_R:=\{\varsigma\in\mathbb{C}:|\varsigma|<R\}$ in the complex plane $\mathbb{C}$. The space of all analytic functions on $\mathbb{D}_R$ is denoted by $H(\mathbb{D}_R)$. For $\psi\in H(\mathbb{D}_R)$ we assume that $\psi(\varsigma)=\sum_{\beta=0}^{\infty}a_{\beta}\varsigma^{\beta}$.

The rest of this study is structured as follows. In Section 3, Section 4, and Section 5, we derive auxiliary results, quantitative estimates of convergence, and Voronovskaja-type outcomes for new operators associated with analytic functions on compact disks, respectively.

It's worth mentioning that certain findings from this paper were showcased during the 7th International Conference COIA [6].

2. Auxiliary results

To establish the main theorems in the subsequent sections, it is necessary to utilize the following auxiliary lemmas in the proofs.

Lemma 2.1. *Let $l\in\mathbb{Z}^+$. For all $\eta\in\mathbb{N}$, $\beta\in\mathbb{N}\cup\{0\}$ and $\varsigma\in\mathbb{C}$, we have*

$$K_{\eta}^{l}\left(e_{\beta};\varsigma\right)=\sum_{j_0+\dots+j_l=\beta}\binom{\beta}{j_0,\dots,j_l}\frac{\eta^{j_0}}{(\eta+l)^{\beta}(j_1+1)\dots(j_l+1)}S_{\eta}(e_{j_0};\varsigma). \tag{2.1}$$

Proof. By direct calculation,

$$\begin{aligned}
K_{\eta}^{l}\left(e_{\beta};\varsigma\right) &= e^{-\eta\varsigma}\sum_{k=0}^{\infty}\frac{(\eta\varsigma)^k}{k!}\int_{0}^{1}\dots\int_{0}^{1}\left(\frac{k+t_1+\dots+t_l}{\eta+l}\right)^{\beta}dt_1\dots dt_l\\
&= e^{-\eta\varsigma}\sum_{k=0}^{\infty}\frac{(\eta\varsigma)^j}{j!}\sum_{j_0+\dots+j_l=\beta}\binom{\beta}{j_0,\dots,j_l}\int_{0}^{1}\dots\int_{0}^{1}\frac{k^{j_0}t_1^{j_1}\dots t_l^{j_l}}{(\eta+l)^{\beta}}dt\\
&= \sum_{j_0+\dots+j_l=\beta}\binom{\beta}{j_0,\dots,j_l}\frac{\eta^{j_0}}{(\eta+l)^{\beta}(j_1+1)\dots(j_l+1)}e^{-\eta\varsigma}\sum_{k=0}^{\infty}\frac{(\eta\varsigma)^k}{k!}\left(\frac{k}{\eta}\right)^{j_0}\\
&= \sum_{j_0+\dots+j_l=\beta}\binom{\beta}{j_0,\dots,j_l}\frac{\eta^{j_0}}{(\eta+l)^{\beta}}S_{\eta}(e_{j_0};\varsigma).
\end{aligned}$$

□

Lemma 2.2. *Let $l\in\mathbb{Z}^+$. For all $\varsigma\in\mathbb{C}$, we have*

$$\left|K_{\eta}^{l}\left(e_{\beta};\varsigma\right)\right|\le(2r)^{\beta},\quad \beta\in\mathbb{N}.$$

Proof. Using the inequality $\left|S_\eta\left(e_j;\varsigma\right)\right| \leq (2r)^j$ (*cf.* [2, p. 115]), we get the following result:

$$\left|K_\eta\left(e_\beta;\varsigma\right)\right| \leq \sum_{j_0+\cdots+j_l=\beta}\binom{\beta}{j_0,\ldots,j_l}\frac{\eta^{j_0}}{(\eta+l)^\beta\,(j_1+1)\ldots(j_l+1)}\left|S_\eta(e_{j_0};\varsigma)\right|$$
$$\leq \left(\frac{\eta+\overbrace{1+\cdots+1}^{l}}{\eta+l}\right)^\beta \left|S_\eta(e_\beta;\varsigma)\right| \leq (2r)^\beta .$$

□

Lemma 2.3. *We have*

$$\begin{aligned}
K_\eta^l\,(e_0;\varsigma) &= 1,\\
K_\eta^l\,(e_1;\varsigma) &= \frac{1}{2\,(\eta+l)}+\frac{\eta}{\eta+l}\varsigma,\\
K_\eta^l\,(e_2;\varsigma) &= \frac{l\,(3l+1)}{12\,(\eta+1)^2}+\frac{\eta\,(l+1)}{(\eta+1)^2}\varsigma+\frac{\eta^2}{(\eta+l)^2}\varsigma^2,\\
K_\eta^l\left((e_1-xe_0)^2;\varsigma\right) &= \frac{3l^2+l}{12(\eta+l)^2}+\frac{\eta-l^2}{(\eta+l)^2}\varsigma+\frac{l^2}{(\eta+l)^2}\varsigma^2.
\end{aligned}$$

Lemma 2.4. *Let $l \in \mathbb{Z}^+$. For all $\eta,\beta \in \mathbb{N}$, $\varsigma \in \mathbb{C}$, we have*

$$\begin{aligned}
K_\eta^l\left(e_{\beta+1};\varsigma\right) &= \frac{\varsigma}{\eta}K_\eta^{'l}\left(e_\beta;\varsigma\right)+\varsigma K_\eta^l\left(e_\beta;\varsigma\right)+\frac{1}{(\eta+l)^\beta}\\
&\times \sum_{\substack{j_0+\cdots+j_l=\beta+1\\ 0\leq j_0\leq\beta+1}}\binom{\beta+1}{j_0,\ldots,j_l}\frac{\eta^{j_0}}{(j_1+1)\ldots(j_l+1)}\left\{\frac{1}{\eta+l}-\frac{j_0}{(\beta+1)\,\eta}\right\}S_\eta\left(e_{j_0};\varsigma\right). \quad (2.2)
\end{aligned}$$

Proof. Taking the derivative of $K_\eta^l\left(e_\beta;\varsigma\right)$ and by the substitution $S_\eta^{'}\left(e_{j_0};\varsigma\right) = -\eta S_\eta\left(e_{j_0};\varsigma\right)+\frac{\eta}{\varsigma}S_\eta\left(e_{j_0+1};\varsigma\right)$ (see [2, p. 115])

$$\begin{aligned}
K_\eta^{'l}\left(e_\beta;\varsigma\right) &= \frac{1}{(\eta+l)^\beta}\sum_{j_0+\cdots+j_l=\beta}\binom{\beta}{j_0,\ldots,j_l}\frac{\eta^{j_0}}{(j_1+1)\ldots(j_l+1)}\left(-\eta S_\eta\left(e_{j_0};\varsigma\right)+\frac{\eta}{\varsigma}S_\eta\left(e_{j_0+1};\varsigma\right)\right)\\
&= \frac{\eta}{\varsigma}\frac{1}{(\eta+l)^\beta}\sum_{j_0+\cdots+j_l=\beta}\binom{\beta}{j_0,\ldots,j_l}\frac{\eta^{j_0}}{(j_1+1)\ldots(j_l+1)}S_\eta\left(e_{j_{0+1}};\varsigma\right)\\
&\quad -\eta\frac{1}{(\eta+l)^\beta}\sum_{j_0+\cdots+j_l=\beta}\binom{\beta}{j_0,\ldots,j_l}\frac{\eta^{j_0}}{(j_1+1)\ldots(j_l+1)}S_\eta\left(e_{j_0};\varsigma\right)\\
\frac{\varsigma}{\eta}K_\eta^{'l}\left(e_\beta;\varsigma\right) &= \frac{1}{(\eta+l)^\beta}\sum_{\substack{j_0+\cdots+j_l=\beta+1\\ 1\leq j_0\leq\beta+1}}\binom{\beta}{j_0-1,j_1,\ldots,j_l}\frac{\eta^{j_0-1}}{(j_1+1)\ldots(j_l+1)}S_\eta\left(e_{j_0};\varsigma\right)-\varsigma K_\eta(e_\beta;\varsigma).
\end{aligned}$$

If $K_\eta^l\left(e_{\beta+1};\varsigma\right)$ is added to both sides in the above equality, the following result is obtained

$$\begin{aligned} K_\eta^l\left(e_{\beta+1};\varsigma\right) &= \frac{\varsigma}{\eta}K_\eta^{'}\left(e_\beta;\varsigma\right)+\varsigma K_\eta\left(e_\beta;\varsigma\right)+\frac{1}{(\eta+l)^{\beta+1}}\sum_{j_0+\cdots+j_l=\beta+1}\binom{\beta+1}{j_0,\ldots,j_l}\frac{\eta^{j_0}}{(j_1+1)\ldots(j_l+1)}S_\eta\left(e_{j_0};\varsigma\right) \\ &\quad -\frac{1}{(\eta+l)^\beta}\sum_{\substack{j_0+\cdots+j_l=\beta+1\\ 1\le j_0\le\beta+1}}\binom{\beta}{j_0-1,\ldots,j_l}\frac{\eta^{j_0-1}}{(j_1+1)\ldots(j_l+1)}S_\eta\left(e_{j_0};\varsigma\right). \end{aligned}$$

$$\begin{aligned} K_\eta^l\left(e_{\beta+1};\varsigma\right) &= \frac{\varsigma}{\eta}K_\eta^{'l}\left(e_\beta;\varsigma\right)+\varsigma K_\eta^l\left(e_\beta;\varsigma\right) \\ &\quad +\frac{1}{(\eta+l)^{\beta+1}}\sum_{\substack{j_0+\cdots+j_l=\beta+1\\ j_0=0}}\binom{\beta+1}{j_0,\ldots,j_l}\frac{\eta^{j_0}}{(j_1+1)\ldots(j_l+1)}S_\eta\left(e_{j_0};\varsigma\right) \\ &\quad +\frac{1}{(\eta+l)^{\beta}}\sum_{\substack{j_0+\cdots+j_l=\beta+1\\ 1\le j_0\le\beta+1}}\binom{\beta+1}{j_0,\ldots,j_l}\frac{\eta^{j_0}}{(j_1+1)\ldots(j_l+1)}\left\{\frac{1}{\eta+l}-\frac{j_0}{(\beta+1)\eta}\right\}S_\eta\left(e_{j_0};\varsigma\right) \\ &= \frac{\varsigma}{\eta}K_\eta^{'l}\left(e_\beta;\varsigma\right)+\varsigma K_\eta^l\left(e_\beta;\varsigma\right) \\ &\quad +\frac{1}{(\eta+l)^{\beta}}\sum_{\substack{j_0+\cdots+j_l=\beta+1\\ 0\le j_0\le\beta+1}}\binom{\beta+1}{j_0,\ldots,j_l}\frac{\eta^{j_0}}{(j_1+1)\ldots(j_l+1)}\left\{\frac{1}{\eta+l}-\frac{j_0}{(\beta+1)\eta}\right\}S_\eta\left(e_{j_0};\varsigma\right). \end{aligned}$$

Also, for the above result, we need the following identity

$$\binom{\beta}{j_0-1,j_1,\ldots,j_l}=\binom{\beta+1}{j_0,j_1,\ldots,j_l}\frac{j_0}{(\beta+1)}.$$

□

Define

$$E_{\eta,\beta}(\varsigma):=K_\eta^l\left(e_\beta;\varsigma\right)-e_\beta(\varsigma)-\frac{\left(\beta^2+\beta(l-1)-2l\beta\varsigma\right)\varsigma^{\beta-1}}{2(\eta+l)}.$$

Lemma 2.5. *Let $l\in\mathbb{Z}^+$. For all $\eta,\beta\in\mathbb{N}$, we have*

$$\begin{aligned} E_{\eta,\beta}(\varsigma) &= \frac{\varsigma}{\eta}\left\{\left(K_\eta^{'l}\left(e_{\beta-1};\varsigma\right)-e_{\beta-1};(\varsigma)\right)^{'}+\left(\varsigma^{\beta-1}\right)^{'}\right\}+\varsigma E_{\eta,\beta-1}(\varsigma)+\frac{l(\beta-1)}{\eta(\eta+l)}\varsigma^{\beta-1}-\frac{l}{2(\eta+l)}\varsigma^{\beta-1}+\frac{l}{(\eta+l)}\varsigma^\beta \\ &\quad +\frac{1}{(\eta+l)^\beta}\sum_{\substack{j_0+\cdots+j_l=\beta\\ 0\le j_0\le\beta}}\binom{\beta}{j_0,\ldots,j_l}\frac{\eta^{j_0}}{(j_1+1)\ldots(j_l+1)}\left\{1-\frac{j_0}{\beta}-l\frac{j_0}{\beta\eta}\right\}S_\eta\left(e_{j_0};\varsigma\right). \end{aligned}$$

Proof. It is obvious that $E_{\eta,\beta}(\varsigma)$ is a polynomial and $\deg\left(E_{\eta,\beta}(\varsigma)\right)\le\beta$. Additionally, $E_{\eta,0}(\varsigma)=0$.

Using the formula (2.2), we get

$$\begin{aligned}
E_{\eta,\beta}(\varsigma) &= \frac{\varsigma}{\eta}K_{\eta}^{'l}\left(e_{\beta-1};\varsigma\right)+\varsigma K_{\eta}^{l}\left(e_{\beta-1};\varsigma\right) \\
&\quad+\frac{1}{(\eta+l)^{\beta-1}}\sum_{\substack{j_0+\cdots+j_l=\beta \\ 0\leq j_0\leq\beta}}\binom{\beta}{j_0,\ldots,j_l}\frac{\eta^{j_0}}{(j_1+1)\ldots(j_l+1)}\left\{\frac{1}{\eta+l}-\frac{j_0}{\beta\eta}\right\}S_{\eta}\left(e_{j_0};\varsigma\right) \\
&\quad-\varsigma^{\beta}-\frac{\left(\beta^2+\beta(l-1)-2l\beta\varsigma\right)}{2(\eta+l)}\varsigma^{\beta-1} \\
&= \frac{\varsigma}{\eta}\left\{\left(K_{\eta}^{'l}\left(e_{\beta-1};\varsigma\right)-e_{\beta-1};(\varsigma)\right)'+\left(\varsigma^{\beta-1}\right)'\right\}+\varsigma E_{\eta,\beta-1}(\varsigma)+\frac{\left((\beta-1)^2+(\beta-1)(l-1)-2(\beta-1)l\varsigma\right)\varsigma^{\beta-1}}{2(\eta+l)} \\
&\quad+\frac{1}{(\eta+l)^{\beta-1}}\sum_{\substack{j_0+\cdots+j_l=\beta \\ 0\leq j_0\leq\beta}}\binom{\beta}{j_0,\ldots,j_l}\frac{\eta^{j_0}}{(j_1+1)\ldots(j_l+1)}\left\{\frac{1}{\eta+l}-\frac{j_0}{\beta\eta}\right\}S_{\eta}\left(e_{j_0};\varsigma\right) \\
&\quad-\frac{\left(\beta^2+\beta(l-1)-2l\beta\varsigma\right)}{2(\eta+l)}\varsigma^{\beta-1}.
\end{aligned}$$

$$\begin{aligned}
E_{\eta,\beta}(\varsigma) &= \frac{\varsigma}{\eta}\left\{\left(K_{\eta}^{'l}\left(e_{\beta-1};\varsigma\right)-e_{\beta-1};(\varsigma)\right)'\right\}+\varsigma E_{\eta,\beta-1}(\varsigma)+\frac{\beta-1}{\eta}\varsigma^{\beta-1} \\
&\quad+\frac{\left(\left((\beta-1)^2+(\beta-1)(l-1)-2(\beta-1)l\varsigma\right)-\left(\beta^2+\beta(l-1)-2l\beta\varsigma\right)\right)\varsigma^{\beta-1}}{2(\eta+l)} \\
&\quad+\frac{1}{(\eta+l)^{\beta-1}}\sum_{\substack{j_0+\cdots+j_l=\beta \\ 0\leq j_0\leq\beta}}\binom{\beta}{j_0,\ldots,j_l}\frac{\eta^{j_0}}{(j_1+1)\ldots(j_l+1)}\left\{\frac{1}{\eta+l}-\frac{j_0}{\beta\eta}\right\}S_{\eta}\left(e_{j_0};\varsigma\right) \\
&= \frac{\varsigma}{\eta}\left(K_{\eta}^{'l}\left(e_{\beta-1};\varsigma\right)-e_{\beta-1};(\varsigma)\right)'+\varsigma E_{\eta,\beta-1}(\varsigma)+\frac{\beta-1}{\eta}\varsigma^{\beta-1}+\frac{(\beta-1)^2-\beta^2-(l-1)}{2(\eta+l)}\varsigma^{\beta-1}+\frac{l}{(\eta+l)}\varsigma^{\beta} \\
&= \frac{\varsigma}{\eta}\left\{\left(K_{\eta}^{'l}\left(e_{\beta-1};\varsigma\right)-e_{\beta-1};(\varsigma)\right)'+\left(\varsigma^{\beta-1}\right)'\right\}+\varsigma E_{\eta,\beta-1}(\varsigma)+\frac{2l\beta-\eta l-2l}{2\eta(\eta+l)}\varsigma^{\beta-1}+\frac{l}{(\eta+l)}\varsigma^{\beta} \\
&\quad+\frac{1}{(\eta+l)^{\beta-1}}\sum_{\substack{j_0+\cdots+j_l=\beta \\ 0\leq j_0\leq\beta}}\binom{\beta}{j_0,\ldots,j_l}\frac{\eta^{j_0}}{(j_1+1)\ldots(j_l+1)}\left\{\frac{1}{\eta+l}-\frac{j_0}{\beta\eta}\right\}S_{\eta}\left(e_{j_0};\varsigma\right) \\
&= \frac{\varsigma}{\eta}\left\{\left(K_{\eta}^{'l}\left(e_{\beta-1};\varsigma\right)-e_{\beta-1};(\varsigma)\right)'\right\}+\varsigma E_{\eta,\beta-1}(\varsigma)+\frac{l(\beta-1)}{\eta(\eta+l)}\varsigma^{\beta-1}-\frac{l}{2(\eta+l)}\varsigma^{\beta-1}+\frac{l}{(\eta+l)}\varsigma^{\beta} \\
&\quad+\frac{1}{(\eta+l)^{\beta}}\sum_{\substack{j_0+\cdots+j_l=\beta \\ 0\leq j_0\leq\beta}}\binom{\beta}{j_0,\ldots,j_l}\frac{\eta^{j_0}}{(j_1+1)\ldots(j_l+1)}\left\{1-\frac{j_0}{\beta}-l\frac{j_0}{\beta\eta}\right\}S_{\eta}\left(e_{j_0};\varsigma\right).
\end{aligned}$$

which is the desired recurrence formula. □

3. Upper estimates

In the following theorem, we examine the quantitative estimates of convergence for K_{η}^{l} associated with an analytic function within a disk of radius $R > 1$ and centered at 0.

Theorem 3.1. *Let $\psi \in H(\mathbb{D}_R)$, $l \in \mathbb{Z}^+$ and $\psi : [R,\infty) \cup \overline{\mathbb{D}_R} \to \mathbb{C}$ be bounded on $[0,\infty)$. If $r \in \left[1, \frac{R}{2}\right)$, then for all $|\varsigma| \le r$ and $\eta \in \mathbb{N}$ we have*

$$\left|K_\eta(\psi;\varsigma) - \psi(\varsigma)\right| \le \frac{3}{2\eta}\sum_{\beta=1}^{\infty}\left|c_\beta\right|\beta(\beta+l)(2r)^\beta.$$

Proof. Using the recurrence formula (2.2), we get:

$$\begin{aligned} K_\eta^l\left(e_\beta;\varsigma\right) - e_\beta(\varsigma) &= \frac{\varsigma}{\eta}K_\eta^{\prime l}\left(e_{\beta-1};\varsigma\right) + \varsigma\left(K_\eta^l\left(e_{\beta-1};\varsigma\right) - e_{\beta-1}(\varsigma)\right) \\ &+ \frac{1}{(\eta+l)^\beta}\sum_{\substack{j_0+\cdots+j_l=\beta \\ 0\le j_0\le\beta}}\binom{\beta}{j_0,\ldots,j_l}\frac{\eta^{j_0}}{(j_1+1)\ldots(j_l+1)}\left\{1-\frac{j_0}{\beta}-l\frac{j_0}{\beta\eta}\right\}S_\eta\left(e_{j_0};\varsigma\right). \end{aligned} \tag{3.1}$$

We can easily guess what is the sum in the above formula

$$\begin{aligned} &\left|\frac{1}{(\eta+l)^\beta}\sum_{\substack{j_0+\cdots+j_l=\beta \\ 0\le j_0\le\beta}}\binom{\beta}{j_0,\ldots,j_l}\frac{\eta^{j_0}}{(j_1+1)\ldots(j_l+1)}\left\{1-\frac{j_0}{\beta}-l\frac{j_0}{\beta\eta}\right\}S_\eta\left(e_{j_0};\varsigma\right)\right| \\ &\le \frac{1}{(\eta+1)^\beta}\left\{\sum_{\substack{j_0+\cdots+j_l=\beta-1 \\ 0\le j_0\le\beta-1}}\binom{\beta-1}{j_0,\ldots,j_l}\beta\frac{\eta^{j_0}}{(j_1+1)\ldots(j_l+1)}\left|1-\frac{j_0}{\beta}-l\frac{j_0}{\beta\eta}\right|\right\}\left|S_\eta\left(e_j;\varsigma\right)\right| \\ &\quad + \frac{\eta^{\beta-1}l}{(\eta+l)^\beta}S_\eta\left(e_\beta;\varsigma\right) \\ &\le \frac{2\beta(\eta+l)^{\beta-1}+\eta^{\beta-1}l}{(\eta+l)^\beta}(2r)^\beta \\ &\le \frac{2\beta+l}{\eta+l}(2r)^\beta. \end{aligned} \tag{3.2}$$

It is a recognized fact that through a linear transformation, the Bernstein inequality within the closed unit disk undergoes a modification.

$$\left|P_\beta'(\varsigma)\right| \le \frac{\beta}{r}\left\|P_\beta\right\|_r, \quad \text{for all } |\varsigma|\le r, \ r \ge 1,$$

where $P_\beta(\varsigma)$ is a complex polynomial and $\deg(P_\beta(\varsigma)) \le \beta$. From the formula (3.1), we get

$$\begin{aligned} \left|K_\eta^l\left(e_\beta;\varsigma\right) - e_\beta(\varsigma)\right| &\le \frac{|\varsigma|}{\eta}\left|K_\eta^{\prime l}\left(e_{\beta-1};\varsigma\right)\right| + |\varsigma|\left|K_\eta^l\left(e_\beta;\varsigma\right) - e_{\beta-1}(\varsigma)\right| + \frac{2\beta+l}{\eta+1}(2r)^\beta \\ &\le \frac{r}{\eta}\frac{\beta-1}{r}\left\|K_\eta^l\left(e_{\beta-1};\varsigma\right)\right\|_r + r\left|K_\eta^l\left(e_{\beta-1};\varsigma\right) - e_{\beta-1}(\varsigma)\right| + \frac{2\beta+l}{\eta+1}(2r)^\beta \\ &\le \frac{\beta-1}{\eta}(2r)^{\beta-1} + \frac{2\beta+l}{\eta+l}(2r)^\beta + r\left|K_\eta^l\left(e_{\beta-1};\varsigma\right) - e_{\beta-1}(\varsigma)\right| \\ &\le r\left|K_\eta^l\left(e_{\beta-1};\varsigma\right) - e_{\beta-1}(\varsigma)\right| + \frac{3\beta+l-1}{\eta}(2r)^\beta. \end{aligned}$$

For $\beta = 1, 2, \ldots,$ we get

$$\begin{aligned}\left|K_\eta^l\left(e_\beta;\varsigma\right)-e_\beta\left(\varsigma\right)\right| &\le \frac{(2r)^\beta}{\eta}\left(3\beta+l-1\right)+r\frac{(2r)^{\beta-1}}{\eta}\left(3\left(\beta-1\right)+l-1\right)+r^2\frac{(2r)^{\beta-2}}{\eta}+\cdots+r^{\beta-1}\frac{(2r)}{\eta}\left(3+l-1\right)\\ &= \frac{3\left(\beta+\beta-1+\cdots+1\right)+\beta\left(l-1\right)}{\eta}\left(2r\right)^\beta\\ &\le \frac{3\beta\left(\beta+l\right)}{2\eta}\left(2r\right)^\beta. \end{aligned} \tag{3.3}$$

Since $K_\eta\left(\psi;\varsigma\right)$ is analytic in $\mathbb{D}_R$, we can write

$$K_\eta^l\left(\psi;\varsigma\right)=\sum_{\beta=0}^{\infty}a_\beta K_\eta^l\left(e_\beta;\varsigma\right),\quad \varsigma\in\mathbb{D}_R,$$

which, together with estimation(3.3) immediately implies for all $|\varsigma|\le r$

$$\left|K_\eta^l\left(\psi;\varsigma\right)-\psi\left(\varsigma\right)\right|\le\sum_{\beta=0}^{\infty}\left|a_\beta\right|\left|K_\eta^l\left(e_\beta;\varsigma\right)-e_\beta\left(\varsigma\right)\right|\le\frac{3}{2\eta}\sum_{\beta=1}^{\infty}\left|c_\beta\right|\beta\left(\beta+l\right)\left(2r\right)^\beta.$$

□

4. Voronovskaja's theorem

Voronovskaja's theorem is a result in approximation theory, specifically dealing with the convergence properties of certain approximation operators. The theorem provides an estimate for the rate of convergence of a sequence of approximation operators to a given function. Named after the Soviet mathematician Tamara Voronovskaja, the theorem often involves expressing the difference between the function being approximated and its approximation in terms of a remainder term.

In this section, we study a Voronovskaja type result in a compact disk, for K_η^l attached to analytic function in $\mathbb{D}_R$, $R\in[1,\infty)$ and center 0.

Theorem 4.1. *Let $\psi\in H\left(\mathbb{D}_R\right)$, $l\in\mathbb{Z}^+$and $\psi:[R,\infty)\cup\overline{\mathbb{D}_R}\to\mathbb{C}$ be bounded on $[0,\infty)$. If $r\in\left[1,\frac{R}{2}\right)$, then for all $|\varsigma|\le r$ and $\eta\in\mathbb{N}$ we have*

$$\left|K_\eta^l\left(\psi;\varsigma\right)-\psi\left(\varsigma\right)-\frac{l\left(1-2\varsigma\right)}{2\left(\eta+l\right)}\psi'\left(\varsigma\right)-\frac{\varsigma}{2\left(\eta+l\right)}\psi''\left(\varsigma\right)\right|\le\frac{10}{\eta^2}\sum_{\beta=2}^{\infty}\left|a_\beta\right|10l\left(\beta-1+l\right)\left(\beta-1\right)^3\left(2r\right)^\beta.$$

Proof. If we use the recurrence formula (2.2) after doing a simple calculation, we get the following relationship.

$$\begin{aligned}E_{\eta,\beta}\left(\varsigma\right) &= \frac{\varsigma}{\eta}\left\{\left(K_\eta^{'l}\left(e_{\beta-1};\varsigma\right)-e_{\beta-1};\left(\varsigma\right)\right)'\right\}+\varsigma E_{\eta,\beta-1}\left(\varsigma\right)+\frac{l\left(\beta-1\right)}{\eta(\eta+l)}\varsigma^{\beta-1}-\frac{l}{2(\eta+l)}\varsigma^{\beta-1}+\frac{l}{(\eta+l)}\varsigma^\beta\\ &\quad+\frac{1}{(\eta+l)^\beta}\sum_{\substack{j_0+\cdots+j_l=\beta\\ 0\le j_0\le\beta}}\binom{\beta}{j_0,\ldots,j_l}\frac{\eta^{j_0}}{(j_1+1)\ldots(j_l+1)}\left\{1-\frac{j_0}{\beta}-l\frac{j_0}{\beta\eta}\right\}S_\eta\left(e_{j_0};\varsigma\right)\\ &= \frac{\varsigma}{\eta}\left\{\left(K_\eta^{'l}\left(e_{\beta-1};\varsigma\right)-e_{\beta-1};\left(\varsigma\right)\right)'\right\}+\varsigma E_{\eta,\beta-1}\left(\varsigma\right)+\frac{l\left(\beta-1\right)}{\eta(\eta+l)}\varsigma^{\beta-1}-\frac{l}{2(\eta+l)}\varsigma^{\beta-1}\\ &\quad+\frac{l}{(\eta+l)}\varsigma^\beta-\frac{l}{(\eta+l)^\beta}\eta^{\beta-1}S_\eta\left(e_\beta;\varsigma\right)-\frac{l}{2(\eta+l)^\beta}\left(\beta-1\right)\eta^{\beta-2}S_\eta\left(e_{\beta-1};\varsigma\right)+\frac{l}{2(\eta+l)^\beta}\eta^{\beta-1}S_\eta\left(e_{\beta-1};\varsigma\right)\\ &\quad+\frac{1}{(\eta+l)^\beta}\sum_{\substack{j_0+\cdots+j_l=\beta-2\\ 0\le j_0\le\beta-2}}\binom{\beta}{j_0,\ldots,j_l}\frac{\eta^{j_0}}{(j_1+1)\ldots(j_l+1)}\left\{1-\frac{j_0}{\beta}-l\frac{j_0}{\beta\eta}\right\}S_\eta\left(e_{j_0};\varsigma\right)\end{aligned}$$

$$
\begin{aligned}
E_{\eta,\beta}(\varsigma) &= \frac{\varsigma}{\eta}\left\{\left(K_{\eta}^{'l}\left(e_{\beta-1};\varsigma\right)-e_{\beta-1};(\varsigma)\right)'\right\}+\varsigma E_{\eta,\beta-1}(\varsigma)+\frac{l(\beta-1)}{\eta(\eta+l)}\varsigma^{\beta-1}+\frac{l}{\eta+l}S_{\eta}\left(e_{\beta};\varsigma\right) \\
&-\frac{l}{\eta+l}S_{\eta}\left(e_{\beta};\varsigma\right)-\frac{l}{2(\eta+l)}\varsigma^{\beta-1}+\frac{l}{(\eta+l)}\varsigma^{\beta}-\frac{l}{(\eta+l)^{\beta}}\eta^{\beta-1}S_{\eta}\left(e_{\beta};\varsigma\right) \\
&-\frac{l}{2(\eta+l)^{\beta}}(\beta-1)\eta^{\beta-2}S_{\eta}\left(e_{\beta-1};\varsigma\right)+\frac{l}{2(\eta+l)^{\beta}}\eta^{\beta-1}S_{\eta}\left(e_{\beta-1};\varsigma\right)+\frac{l}{2(\eta+l)}S_{\eta}\left(e_{\beta-1};\varsigma\right)-\frac{l}{2(\eta+l)}S_{\eta}\left(e_{\beta-1};\varsigma\right) \\
&+\frac{1}{(\eta+l)^{\beta}}\sum_{\substack{j_0+\cdots+j_l=\beta-2\\0\le j_0\le\beta-2}}\binom{\beta}{j_0,\ldots,j_l}\frac{\eta^{j_0}}{(j_1+1)\ldots(j_l+1)}\left\{1-\frac{j_0}{\beta}-l\frac{j_0}{\beta\eta}\right\}S_{\eta}\left(e_{j_0};\varsigma\right).
\end{aligned}
$$

$$
\begin{aligned}
E_{\eta,\beta}(\varsigma) &= \frac{\varsigma}{\eta}\left\{\left(K_{\eta}^{'l}\left(e_{\beta-1};\varsigma\right)-e_{\beta-1};(\varsigma)\right)'\right\}+\varsigma E_{\eta,\beta-1}(\varsigma)+\frac{l(\beta-1)}{\eta(\eta+l)}\varsigma^{\beta-1} \\
&+\frac{l}{\eta+l}\left(\varsigma^{\beta}-S_{\eta}\left(e_{\beta};\varsigma\right)\right)+\frac{l}{\eta+l}\left(1-\frac{\eta^{\beta-1}}{(\eta+l)^{\beta-1}}\right)S_{\eta}\left(e_{\beta};\varsigma\right) \\
&-\frac{l}{2(\eta+l)^{\beta}}(\beta-1)\eta^{\beta-2}S_{\eta}\left(e_{\beta-1};\varsigma\right)+\frac{l}{2(\eta+l)}\left(S_{\eta}\left(e_{\beta-1};\varsigma\right)-\varsigma^{\beta-1}\right) \\
&-\frac{l}{2(\eta+l)}\left(1-\frac{\eta^{\beta-1}}{(\eta+l)^{\beta-1}}\right)S_{\eta}\left(e_{\beta-1};\varsigma\right) \\
&+\frac{1}{(\eta+l)^{\beta}}\sum_{\substack{j_0+\cdots+j_l=\beta-2\\0\le j_0\le\beta-2}}\binom{\beta}{j_0,\ldots,j_l}\frac{\eta^{j_0}}{(j_1+1)\ldots(j_l+1)}\left\{1-\frac{j_0}{\beta}-l\frac{j_0}{\beta\eta}\right\}S_{\eta}\left(e_{j_0};\varsigma\right) \\
&:= \sum_{k=1}^{9}I_k,
\end{aligned}
$$

where

$$
\begin{aligned}
I_3 &= \frac{l(\beta-1)}{\eta(\eta+l)}\varsigma^{\beta-1} \\
I_4 &= \frac{l}{\eta+l}\left(\varsigma^{\beta}-S_{\eta}\left(e_{\beta};\varsigma\right)\right) \\
I_5 &= \frac{l}{\eta+l}\left(1-\frac{\eta^{\beta-1}}{(\eta+l)^{\beta-1}}\right)S_{\eta}\left(e_{\beta};\varsigma\right) \\
I_6 &= \frac{l}{2(\eta+l)^{\beta}}(\beta-1)\eta^{\beta-2}S_{\eta}\left(e_{\beta-1};\varsigma\right) \\
I_7 &= \frac{l}{2(\eta+l)}\left(\varsigma^{\beta-1}-S_{\eta}\left(e_{\beta-1};\varsigma\right)\right) \\
I_8 &= \frac{l}{2(\eta+1)}\left(1-\frac{\eta^{\beta-1}}{(\eta+1)^{\beta-1}}\right)S_{\eta}\left(e_{\beta-1};\varsigma\right) \\
I_9 &= \frac{1}{(\eta+l)^{\beta}}\sum_{\substack{j_0+\cdots+j_l=\beta-2\\0\le j_0\le\beta-2}}\binom{\beta}{j_0,\ldots,j_l}\frac{\eta^{j_0}}{(j_1+1)\ldots(j_l+1)}\left\{1-\frac{j_0}{\beta}-l\frac{j_0}{\beta\eta}\right\}S_{\eta}\left(e_{j_0};\varsigma\right).
\end{aligned}
$$

From the proof of Theorem 1.8.4 of [2], we have

$$
\left|\varsigma^{\beta}-S_{\eta}\left(e_{\beta};\varsigma\right)\right|\le\frac{6(\beta-1)}{\eta}(2r)^{\beta-1}.
$$

It follows that

$$|I_4| \leq \frac{l}{\eta+l}\left|\varsigma^{\beta} - S_{\eta}\left(e_{\beta};\varsigma\right)\right| \leq \frac{6l\left(\beta-1\right)\left(2r\right)^{\beta-1}}{\eta\left(\eta+l\right)},$$

$$|I_7| \leq \frac{l}{2\left(\eta+l\right)}\left|\varsigma^{\beta-1} - S_{\eta}\left(e_{\beta-1};\varsigma\right)\right| \leq \frac{3l\left(\beta-2\right)\left(2r\right)^{\beta-2}}{\eta\left(\eta+l\right)}.$$

Applying the inequality

$$1-\prod_{j=1}^{k} x_j \leq \sum_{j=1}^{k}\left(1-x_j\right), \quad 0 \leq x_j \leq 1, \quad j=1,\ldots,k$$

we have

$$|I_5| \leq \frac{l}{\eta+l}\left(1-\frac{\eta^{\beta-1}}{\left(\eta+l\right)^{\beta-1}}\right)\left|S_{\eta}\left(e_{\beta};\varsigma\right)\right| \leq \frac{l\left(\beta-1\right)}{\left(\eta+l\right)^2}\left(2r\right)^{\beta},$$

$$|I_8| \leq \frac{l}{2\left(\eta+1\right)}\left(1-\frac{\eta^{\beta-1}}{\left(\eta+1\right)^{\beta-1}}\right)\left|S_{\eta}\left(e_{\beta-1};\varsigma\right)\right| \leq \frac{l\left(\beta-1\right)}{2\left(\eta+l\right)^2}\left(2r\right)^{\beta-1}.$$

For I_9 we have

$$|I_9| \leq \frac{1}{\left(\eta+l\right)^{\beta}} \sum_{\substack{j_0+\cdots+j_l=\beta-2 \\ 0\leq j_0\leq\beta-2}} \binom{\beta}{j_0,\ldots,j_l}\frac{\eta^{j_0}}{\left(j_1+1\right)\ldots\left(j_l+1\right)}\left\{1-\frac{j_0}{\beta}-l\frac{j_0}{\beta\eta}\right\}S_{\eta}\left(e_{j_0};\varsigma\right)$$

$$\leq \frac{1}{\left(\eta+l\right)^{\beta}} \sum_{\substack{j_0+\cdots+j_l=\beta-2 \\ 0\leq j_0\leq\beta-2}} \binom{\beta-2}{j_0,\ldots,j_l}\frac{\beta(\beta-1)\eta^{j_0}}{\left(j_1+1\right)\ldots\left(j_l+1\right)}\left|1-\frac{j_0}{\beta}-l\frac{j_0}{\beta\eta}\right|S_{\eta}\left(e_{j_0};\varsigma\right)$$

$$\leq \frac{2l\beta(\beta-1)\left(\eta+l\right)^{\beta-2}}{\left(\eta+l\right)^{\beta}}\left(2r\right)^{\beta-2} \leq \frac{2l\beta(\beta-1)}{\left(\eta+l\right)^2}\left(2r\right)^{\beta-2}.$$

Thus

$$\begin{aligned}
\left|E_{\eta,\beta}\left(\varsigma\right)\right| &\leq \frac{r}{\eta}\left|\left(K_{\eta}^{l}\left(e_{\beta-1};\varsigma\right)-e_{\beta-1}\left(\varsigma\right)\right)'\right| + r\left|E_{\eta,\beta-1}\left(\varsigma\right)\right| + \frac{l\left(\beta-1\right)}{\eta(\eta+l)}r^{\beta-1} \\
&\quad + \frac{6l\left(\beta-1\right)\left(2r\right)^{\beta-1}}{\eta\left(\eta+l\right)} + \frac{l\left(\beta-1\right)}{\left(\eta+l\right)^2}\left(2r\right)^{\beta} + \frac{l\left(\beta-1\right)}{2\left(\eta+l\right)^2}\left(2r\right)^{\beta-1} \\
&\quad + \frac{3l\left(\beta-2\right)\left(2r\right)^{\beta-2}}{\eta\left(\eta+l\right)} + \frac{l\left(\beta-1\right)}{2\left(\eta+l\right)^2}\left(2r\right)^{\beta-1} + \frac{2l\beta(\beta-1)}{\left(\eta+l\right)^2}\left(2r\right)^{\beta-2} \\
&\leq \frac{r}{\eta}\frac{\beta-1}{r}\frac{3\left(\beta-1\right)\left(\beta-1+l\right)}{2\eta}\left(2r\right)^{\beta} + r\left|E_{\eta,\beta-1}\left(\varsigma\right)\right| + \frac{8l\beta(\beta-1)}{\left(\eta+l\right)^2}\left(2r\right)^{\beta} \\
&\leq \frac{2l\left(\beta-1\right)^2\left(\beta-1+l\right)}{\eta}\left(2r\right)^{\beta} + \frac{8l(\beta-1)^2\left(\beta-1+l\right)}{\left(\eta+l\right)^2}\left(2r\right)^{\beta} \\
&\leq r\left|E_{\eta,\beta-1}\left(\varsigma\right)\right| + \frac{10l\left(\beta-1+l\right)\left(\beta-1\right)^2}{\eta^2}\left(2r\right)^{\beta}.
\end{aligned}$$

Therefore, we have

$$\left|E_{\eta,\beta}\left(\varsigma\right)\right| \leq r\left|E_{\eta,\beta-1}\left(\varsigma\right)\right| + \frac{10l\left(\beta-1+l\right)\left(\beta-1\right)^2}{\eta^2}\left(2r\right)^{\beta}.$$

As a consequence, we get

$$\left|E_{\eta,\beta}(\varsigma)\right| \leq \frac{10l(\beta-1+l)(\beta-1)^3}{\eta^2}(2r)^\beta .$$

Note that since $\psi^{(4)} = \sum_{\beta=4}^{\infty} a_\beta \beta(\beta-1)(\beta-2)(\beta-3)\varsigma^{\beta-4}$ and the series is absolutely convergent for all $|\varsigma| < R$, it easily follows the finiteness of the involved constants in the statement. □

5. Exact estimates

The following lower estimate holds.

Theorem 5.1. *Let $\psi \in H(\mathbb{D}_R)$ and $\psi : [R,\infty) \cup \overline{\mathbb{D}_R} \to \mathbb{C}$ be bounded on $[0,\infty)$. If $r \in \left[1, \frac{R}{2}\right)$ and if ψ is not a constant function then the estimate*

$$\left\|K_\eta^l(\psi) - \psi\right\|_r \geq \frac{1}{\eta} C_r(\psi), \quad \eta \in \mathbb{N},$$

holds, where the constant $C_r(\psi)$ depends on rand ψ but not depends on η.

Proof. For all $\varsigma \in \mathbb{D}_R$ and $\eta \in \mathbb{N}$ we get

$$K_\eta^l(\psi;\varsigma) - \psi(\varsigma) = \frac{1}{\eta}\left\{\frac{\eta l(1-2\varsigma)}{2(\eta+l)}\psi'(\varsigma) + \frac{\varsigma}{2}\psi''(\varsigma) + \frac{1}{\eta}\eta^2\left(K_\eta^l(\psi;\varsigma) - \psi(\varsigma) - \frac{l(1-2\varsigma)}{2(\eta+l)}\psi'(\varsigma) - \frac{\varsigma}{2\eta}\psi''(\varsigma)\right)\right\}$$

$$= \frac{1}{\eta}\left\{\frac{l(1-2\varsigma)}{2}\psi'(\varsigma) + \frac{\varsigma}{2}\psi''(\varsigma) + \frac{1}{\eta}\eta^2\left(K_\eta^l(\psi;\varsigma) - \psi(\varsigma) - \frac{l(1-2\varsigma)}{2(\eta+l)}\psi'(\varsigma) - \frac{\varsigma}{2\eta}\psi''(\varsigma) - \frac{l^2(1-2\varsigma)}{2\eta(\eta+l)}\psi'(\varsigma)\right)\right\}.$$

We apply

$$\|F+G\|_r \geq \left|\|F\|_r - \|G\|_r\right| \geq \|F\|_r - \|G\|_r$$

to get

$$\left\|K_\eta^l(\psi) - \psi\right\|_r \geq \frac{1}{\eta}\left\{\left\|\frac{l(1-2e_1)}{2}\psi' + \frac{e_1}{2}\psi''\right\|_r - \frac{1}{\eta}\eta^2\left\|\left(K_\eta^l(\psi) - \psi - \frac{l(1-2e_1)}{2(\eta+l)}\psi' - \frac{e_1}{2\eta}\psi'' - \frac{l^2(1-2\varsigma)}{2\eta(\eta+l)}\psi'\right)\right\|_r\right\}.$$

Given the assumption that ψ is a non-constant polynomial in D_R, we get $\|\frac{l(1-2e_1)}{2}\psi' - \frac{e_1}{2}\psi''\|_r > 0$.

Indeed, supposing the contrary it follows that

$$\frac{l(1-2\varsigma)}{2}\psi'(\varsigma) + \frac{\varsigma}{2}\psi''(\varsigma) = 0 \text{ for all } |\varsigma| \leq r.$$

Denoting $y(\varsigma) = \psi'(\varsigma)$, seeking $y(\varsigma)$ in the form $\sum_{k=0}^{\infty} b_k \varsigma^k$, and replacing in the above differential equation, we easily get $b_k = 0$ for all $k = 0, 1, \ldots$. (We can apply analogous reasoning in this context as in the work of [7]). Thus, we get that $\psi(\varsigma)$ is a constant function, which is a contradiction.

Now, by Theorem 4.1 we have

$$\eta^2\left\|\left(K_\eta^l(\psi;\varsigma) - \psi(\varsigma) - \frac{l(1-2\varsigma)}{2(\eta+l)}\psi'(\varsigma) - \frac{\varsigma}{2\eta}\psi''(\varsigma) - \frac{l^2(1-2\varsigma)}{2\eta(\eta+l)}\psi'(\varsigma)\right)\right\|_r$$

$$\leq \eta^2\left\|K_\eta^l(\psi;\varsigma) - \psi(\varsigma) - \frac{l(1-2\varsigma)}{2(\eta+l)}\psi'(\varsigma) - \frac{\varsigma}{2\eta}\psi''(\varsigma)\right\|_r + \left\|l^2(1-2\varsigma)\psi'\right\|_r$$

$$\leq 10\sum_{\beta=2}^{\infty}\left|a_\beta\right| 10l(\beta-1+l)(\beta-1)^3(2r)^\beta + \left\|l^2(1-2\varsigma)\psi'\right\|_r .$$

Consequently, there exists $\eta_1(\psi, r)$ such that for all $\eta \geq \eta_1$ we have

$$\left\|\frac{l(1-2\varsigma)}{2}\psi^{'}(\varsigma)+\frac{\varsigma}{2}\psi^{''}(\varsigma)\right\|_r-\frac{1}{\eta}.\eta^2\left\|\left(K_\eta^l(\psi)-\psi-\frac{l(1-2e_1)}{2(\eta+l)}\psi^{'}-\frac{e_1}{2\eta}\psi^{''}--\frac{l^2(1-2\varsigma)}{2\eta(\eta+l)}\psi^{'}\right)\right\|_r$$

$$\geq\frac{1}{2}\left\|\frac{l(1-2e_1)}{2}\psi^{'}+\frac{e_1}{2}\psi^{''}\right\|_r,$$

which implies

$$\left\|K_\eta^l(\psi)-\psi\right\|_r\geq\frac{1}{2\eta}\left\|\frac{l(1-2e_1)}{2}\psi^{'}+\frac{e_1}{2}\psi^{''}\right\|_r,\quad\text{for all }\eta\geq\eta_1.$$

For $\eta \in \{\eta_0+1,\ldots,\eta_1\}$, we get $\left\|K_\eta^l(\psi)-\psi\right\|_r \geq \frac{1}{\eta}B_r(\psi)$ with $B_r(\psi) = \eta.\left\|K_\eta^l(\psi)-\psi\right\|_r > 0$, which implies that $\left\|K_\eta^l(\psi)-\psi\right\|_r \geq \frac{C_r(\psi)}{\eta}$ for all $\eta \geq \eta_0$, with

$$C_r(\psi)=\min\left\{B_{r,\eta_0+1}(\psi),\ldots,B_{r,\eta_1}(\psi),\frac{1}{2}\left\|\frac{l(1-2e_1)}{2}\psi^{'}+\frac{e_1}{2}\psi^{''}\right\|_r\right\}$$

which proves the theorem. □

Theorem 5.2. *Let $\psi \in H(\mathbb{D}_R)$ and $\psi : [R,\infty)\cup\overline{\mathbb{D}_R} \to \mathbb{C}$ be bounded on $[0,\infty)$. If $r \in \left[1,\frac{R}{2}\right)$ and ψ is not a polynomial of degree ≤ 1 then for all $\eta \in \mathbb{N}$, we have*

$$\left\|K_\eta^l(\psi)-\psi\right\|_r\sim\frac{1}{\eta},$$

where the constants in the equivalence rely on ψ and r but remain independent of η.

6. Conclusion

This paper introduced a novel category of generalized complex Szasz-Kantorovich operators. Subsequently, we established quantitative estimates within Voronovskaja's theorem, determining precise orders for approximating analytic functions. Notably, our approach eliminated the need for imposing exponential growth conditions on compact disks when considering these operators.

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