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A symmetric expansion of an analytical function in terms of quadratic functions involving the fractional derivative and some applications to special functions

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Abstract

Motivated by the generalization of Taylor's series of $f(z)$ obtained by Osler (Taylor's series generalized for fractional derivatives and applications, SIAM J. Math. Anal. 2, 37–48, 1971), Tremblay and Fugère presented in 2007 a new expansion of an analytic function $f(z)$ in R in terms of a power series $\theta(t) = tq(t)$, where $q(t)$ is any regular function and t is equal to the quadratic function $[(z-z_1)(z-z_2)]$, $z_1 \neq z_2$, where z_1 and z_2 are two points in R (The use of fractional derivatives to expand analytical functions in terms of quadratic functions with applications to special functions, Appl. Math. Comput. 187 (1), 507–529, 2007).

To illustrate the concept, if $q(t) = 1$, the coefficient of $[(z-z_1)(z-z_2)]^n$ in the power series of the function $(z-z_1)^{\alpha}(z-z_2)^{\beta} f(z)$ is

$$D^{-\alpha+n}_{z_1-z_2}[f(z_1)(z_1-z_2)^{\beta-n-1}(z_1-z_2+z-w)]|_{w=z_1}/\Gamma(1-\alpha+n).$$

In this paper, using the transformation formula for fractional derivatives obtained by Tremblay, Gaboury and Fugère (A new transformation formula for fractional derivatives with applications, Integral Transforms Spec. Funct. 24 (3), 172–186, 2013)

$$D^{\alpha}_{z-b}(z-b)^{\beta} f(z) = \frac{\Gamma(1+\beta)}{\Gamma(-\alpha)} D^{-\beta-1}_{z-b}\{(z-b)^{-\alpha-1} f(w-z)\}\Big|_{w=z},$$

we present a new form of expansion in terms of quadratic functions symmetric with respect to parameters α and β as well as applications to special functions. Finally, several definite integers of Fourier type are evaluated, in particular an integral due to Ramanujan is generalized.

Keywords: Fractional derivatives, Taylor's theorem, power series, special functions, Ramanujan integral, Fourier's integral

2020 MSC: 26A33, 33C05, 33C20, 33C65, 33E20

1. Introduction

The fractional derivative of the function $f(z)$ with respect to $g(z)$ of arbitrary order α, $D^{\alpha}_{g(z)} f(z)$, is a generalization of the familiar derivative $d^n f(z)/(dg(z))^n$ to non integral values of α. This concept has been introduced in many different ways by generalizing the classical definitions of the nth derivative where the order n is replaced by an arbitrary α (*cf.* [2, 6, 15, 16] [18]-[21], [33, 40]). We can find many surveys and discussions on several of these approaches in texts [17, 20, 35, 36].

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Many of these familiar elementary formulas the calculation turned out to be special cases of expressions involving fractional derivatives. The Taylor's and Laurent's series [5, 7, 25, 28], the Leibniz rule [13, 22, 24, 26, 27], the chain rule [23] and the Lagrange's expansion [25] are such examples. In addition, knowledge of the properties and rules of manipulation of the fractional derivative has made it possible to deduce several important properties of the transcendental functions [21]-[27], [38, 39, 41] and extensions of operators to fractional order [41, 45, 46].

In this paper, the approach considered is the Cauchy loop-integral that can be extended to many-valued integrands with an appropriate choice of the contour of integration relative to the branch-cuts, this approach provides definitions of derivative of complex order for analytic functions containing isolated singularities (Nekrassov [18], Osler [22, 23], Lavoie, Osler and Tremblay [13, 14], Campos [2]-[5]).

In 1971, using the integral

$$I = \frac{\theta(z)^{\gamma}}{2\pi i}\int_{C_\epsilon - C_\delta} \frac{\theta(\xi)^{\alpha-\gamma-1}\theta'(\xi)f(\xi)}{\theta(\xi)^a - \theta(z)^a} d\xi,$$

where C_x denotes the curve $C(|\theta(b + x(b - z_0))|)$, Osler obtained with the Cauchy integral formula for fractional derivatives, the interesting following generalization for the Taylor's series involving the fractional derivatives (*cf.* [25, Eq. (1.2), p. 37])

$$\sum_{k\in K} a^{-1}\omega^{-\gamma k} f(\theta^{-1}(\theta(z)\omega^k)) = \sum_{n=-\infty}^{\infty} \frac{D^{an+\gamma}_{z-b}}{\Gamma(an+\gamma+1)} \left[f(z)\theta^{'}(z)\left[(z-z_0)/\theta(z)\right]^{an+\gamma+1}\right]\Big|_{z=z_0} \theta(z)^{an+\gamma}, \tag{1.1}$$

where a is a positive real number, $b \neq z_0$, $\omega = exp(2\pi i/a)$, α and γ are arbitrary complex numbers, $f(z)$ is an analytic function in a simply connected region R and $\theta(z) = (z - z_0)q(z)$ with $q(z)$ is a regular and univalent function without zeros in R and $K = \{0, 1, \ldots, [a]\}$, $[a]$ being the largest integer not greater than a. If $0 < a \leq 1$ and $\theta(z) = z - z_0$, we have $K = \{0\}$ and the formula (1.1) reduces to

$$f(z) = a\sum_{n=-\infty}^{\infty} \frac{D^{an+\gamma}_{z_0-b} f(z_0)}{\Gamma(an+\gamma+1)}(z-z_0)^{an+\gamma},$$

usually called the Taylor-Riemann formula. The case $a = 1$ was considered formally by Riemann [32] in 1847 in a manuscript probably never intended for publication. Nevertheless this structure suggests a definition for the fractional differentiation. Moreover, if $a \to 0^+$ we obtain the analog form

$$f(z) = \int_{-\infty}^{\infty} \frac{D^{\omega+\gamma}_{z_0-b} f(z_0)}{\Gamma(\omega+\gamma+1)}(z-z_0)^{\omega+\gamma} d\omega.$$

This last formula is usually called the Taylor-Riemann formula and has been studied in several papers (*cf.* [21, Chapter 3], [7, 25, 28, 44]). But none considered a more general expansion of $f(z)$ into series of an arbitrary quadratic, cubic or higher degrees functions. In 2007, Tremblay and Fugère [47] obtained the power series of an analytic function $f(z)$ in terms of arbitrary function $(z - z_1)(z - z_2)$ where z_1 and z_2 are two arbitrary points inside the analyticity region R of $f(z)$ by using the Pochhamme's contour representation for fractional derivatives. Explicitly, they found (3.1) (See Theorem 3.1 showing the conditions imposed). Later, in 2013, Tremblay, Gaboury and Fugère [43] explore the properties of symmetry of Pochhammer's integral around the singular points z and b related to the four loops contour. This simple remark leads to the following new transformation formula for fractional derivatives (see Theorem 3.2)

$$D^{\alpha}_{z-b}(z-b)^p f(z) = \frac{\Gamma(1+p)}{\Gamma(-\alpha)} D^{-p-1}_{z-b}\left\{(z-b)^{-\alpha-1} f(w-z)\right\}\Big|_{w=z}. \tag{1.2}$$

We can also put in evidence the symmetry of the result (4.1) with respect to z_1 and z_2. In Theorem 4.1, we give the proof of the following formula

$$
\begin{aligned}
&2\sum_{k\in K} a^{-1}\omega^{-\gamma k}\left[f\left(\frac{z_1+z_2+\sqrt{\Delta_k}}{2}\right)+f\left(\frac{z_1+z_2-\sqrt{\Delta_k}}{2}\right)\right] \\
&\quad\times\Big[\sin[(\beta+a-\gamma)\pi]\Big(\frac{z_2-z_1+\sqrt{\Delta_k}}{2}\Big)^{\alpha}\Big(\frac{z_1-z_2+\sqrt{\Delta_k}}{2}\Big)^{\beta} \\
&\quad\quad -e^{i\pi(\alpha-\beta)}\sin[(\alpha+a-\gamma)\pi]\Big(\frac{z_2-z_1-\sqrt{\Delta_k}}{2}\Big)^{\alpha}\Big(\frac{z_1-z_2-\sqrt{\Delta_k}}{2}\Big)^{\beta}\Big] \\
&= \sum_{-\infty}^{\infty}\theta(z)^{an+\gamma}e^{-i\pi a(n+1)} \\
&\quad\times\Big\{\sin[(\beta-an-\gamma)\pi]\frac{D_{z-z_2}^{-\alpha+an+\gamma}}{\Gamma(1-\alpha+an+\gamma)}\Big[(z-z_2)^{\beta-an-\gamma-1}\Big(\frac{\theta(z)}{(z-z_2)(z-z_1)}\Big)^{-an-\gamma-1}\theta'(z)(f(z)+f(z_1+z_2-z))\Big]\Big|_{z=z_1} \\
&\quad -\sin[(\alpha-an-\gamma)\pi]\frac{D_{z-z_2}^{-\beta+an+\gamma}}{\Gamma(1-\beta+an+\gamma)}\Big[(z-z_2)^{\alpha-an-\gamma-1}\Big(\frac{\theta(z)}{(z-z_2)(z-z_1)}\Big)^{-an-\gamma-1}\theta'(z)(f(z)+f(z_1+z_2-z))\Big]\Big|_{z=z_1}\Big\}
\end{aligned}
\tag{1.3}
$$

and by Theorem 3.2, (1.3) is equivalent to

$$
\begin{aligned}
&\sum_{k\in K} a^{-1}\omega^{-\gamma k}\left[f\left(\frac{z_1+z_2+\sqrt{\Delta_k}}{2}\right)+f\left(\frac{z_1+z_2-\sqrt{\Delta_k}}{2}\right)\right]\Big[\Big(\frac{z_2-z_1+\sqrt{\Delta_k}}{2}\Big)^{\alpha}\Big(\frac{z_1-z_2+\sqrt{\Delta_k}}{2}\Big)^{\beta} \\
&\quad -e^{i\pi(\alpha-\beta)}\frac{\sin(\alpha+a-\gamma)\pi}{\sin(\beta+a-\gamma)\pi}\Big(\frac{z_2-z_1-\sqrt{\Delta_k}}{2}\Big)^{\alpha}\Big(\frac{z_1-z_2-\sqrt{\Delta_k}}{2}\Big)^{\beta}\Big] \\
&= 2\sum_{-\infty}^{\infty}\frac{\sin(\beta-an-\gamma)\pi}{\sin(\beta+a-\gamma)\pi}e^{-i\pi a(n+1)}\theta(z)^{an+\gamma}\frac{D_{z-z_2}^{-\alpha+an+\gamma}}{\Gamma(1-\alpha+an+\gamma)} \\
&\quad\times\Big[(z-z_2)^{\beta-an-\gamma-1}\Big(\frac{\theta(z)}{(z-z_2)(z-z_1)}\Big)^{-an-\gamma-1}\theta'(z)(f(z)+f(z_1+z_2-z)\Big]\Big|_{z=z_1}
\end{aligned}
\tag{1.4}
$$

which are valid for z on C_1 or C_2 or for the set $\left\{z\middle|\left|\theta((z-z_1)(z-z_2))\right|=\left|\theta((z_1-z_2)^2/4)\right|\right\}$.
These are the main results of this paper.

The document is divided as follows. In Section 2, we recall the representation of fractional derivative $D_z^{\alpha}z^p f(z)$ using a Pochhammer's contour outline introduced for the first time by Tremblay in [40]. This representation is symmetrical with respect to parameters α and p. This symmetry made it possible to obtain the transformation formula (3.10). Section 3 recalls two already published theorems which are used to demonstrate the main result (4.1) in Theorem 4.1. Section 3 exhibits several special cases of formulas (3.1) and (3.10) thus showing their effectiveness in discovering new results. Section 4 is devoted to the main result and the demonstration of its symmetry properties. We give some examples of developments in terms of quadratic functions and new integrals are obtained. Note that proof of the main result (4.1) is similar to that appearing in [47] for the quadratic case. Section 5 gives an application for the evaluation of integrals of the same type as those offered by Ramanujan. Finally, in Section 6, we use the Fourier transformation to assess new integrals.

2. Pochhammer integral definition for the fractional derivative

Convention 2.1.

(i) $\mathfrak{R}$ is an open, simply-connected set in the complex plane containing $\xi = g^{-1}(0)$.

(ii) $f(z)$ is an analytic function for $z \in \mathfrak{R}$.

(iii) The notations

$$\int_{z_0}^{(z+)} g(\xi)d\xi = \int_{C(z_0,z+;g_1,g_2)} g(\xi)d\xi$$

denote integrals along the path of integration beginning at $\xi = z_0$, where the integrand takes the value $g(z_0) = g_1$, encloses the point $\xi = z$ once in the positive direction, and returns to $\xi = z_0$, where now the integrand takes the value $g(z_0) = g_2$ (Figure 1). We assume that the contour remains in the region $\mathfrak{R}$ and the integrand $g(\xi)$ varies 'continuously' as we traverse the contour.

(iv) The integrand will contain multiple-valued factors such as ξ^p, $ln(\xi)$, $(\xi - z)^q$, $ln(\xi - z)$, etc. The branch cut for these functions always passes through the beginning and ending point of the contour of integration, but never cuts the contour anywhere else. Unless otherwise stated, these functions denote the principal branch, which is continuous for $arg(\xi)$ (or $arg(\xi - z)$) is zero when ξ (or $(\xi - z)$) is real and positive. In the event that the branch line is $arg(\xi) = 0$ (or $arg(\xi - z) = 0$), then we define the principal branch by $-2\pi < arg(\xi)$ (or $arg(\xi - z)) \leq 0$.

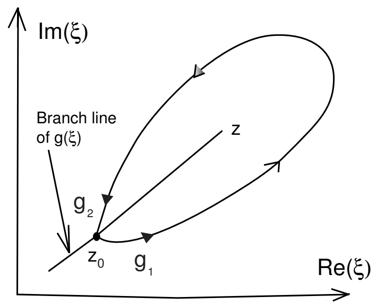

Figure 1. Single-loop contour

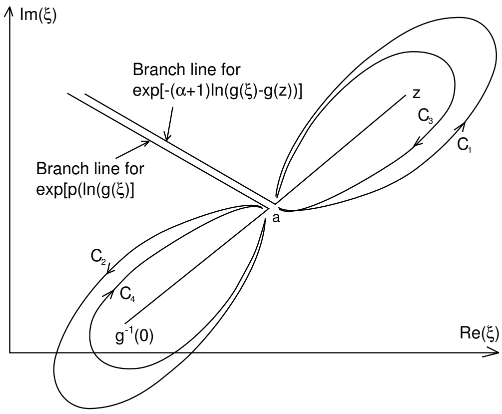

Figure 2. Pochhammer's contour

Definition 2.2. Let $f(z)$ be analytic in a simply connected region R. Let $g(z)$ be regular and univalent on R and let $g^{(-1)}(0)$ be an interior point of $\mathfrak{R}$ then if α is not a negative integer, p is not an integer, and $z \in R - \{g^{(-1)}(0)\}$, one defines the fractional derivative of order α of $(g(z))^p f(z)$ with respect to $g(z)$ ($\delta = 0$ or 1)

$$D^{\alpha}_{g(z)} g(z)^p (ln(g(z))^{\delta}) f(z) = \frac{e^{-i\pi p}\Gamma(1+\alpha)}{4\pi sin(\pi p)} \int_{C(z+,g^{-1}(0)+,z-,g^{-1}(0)-;F(a),F(a))} \frac{f(\xi) ln(g(\xi)) g(\xi)^p g'(\xi)}{(g(\xi) - g(z))^{\alpha+1}}\, d\xi \tag{2.1}$$
$$-\delta \frac{\Gamma(1+\alpha) e^{i\pi(\alpha+1)}}{4\pi \sin^2(\pi p)} \int_{C(z+,g^{-1}(0)+,z-,g^{-1}(0)-;F(a),F(a))} \frac{f(\xi)(g(\xi))^p g'(\xi)}{(g(z) - g(\xi))^{-\alpha-1}} d\xi.$$

If $g(z) = z - z_0$, (2.1) becomes

$$D^{\alpha}_{z-z_0}(z-z_0)^p\{\ln(z-z_0)\}^{\delta} f(z) = \frac{\Gamma(1+\alpha)e^{-i\pi(p-\alpha-1)}}{4\pi\sin(\pi p)}\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{-\alpha-1}\{\ln(\xi-z_0)\}^{\delta}d\xi \quad (2.2)$$
$$-\delta\frac{\Gamma(1+\alpha)e^{i\pi(\alpha+1)}}{4\pi\sin^2(\pi p)}\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{-\alpha-1}d\xi.$$

For non-integers α and p, the functions $(g(\xi))^p$, $\ln(g(z))$ and $(g(\xi)-g(z))^{-\alpha-1}$ in the integrand have two branch lines which begin, respectively, at $\xi = z$ and $\xi = g^{(-1)}(0)$, and both pass through the point $\xi = a$ without crossing the Pochhammer contour $P(a) = \{C_1 \cup C_2 \cup C_3 \cup C_4\}$ at any other point as shown in Figure 2. $F(a)$ denotes the principal value of the integrand in (2.1) at the beginning and ending point of the Pochhammer contour $P(a)$ which is closed on the Riemann surface of the multiple-valued function $F(\xi)$.

Remark 2.3. In [13], the authors review several representations of $D^{\alpha}_z z^p f(z)$ that have appeared in the literature, and compare them with the definition using a Pochhammer contour integral (2.2). Of course, no single representation for $D^{\alpha}_z z^p f(z)$ is obviously superior in all applications. However, the use of the integral of the Pochhammer contour is often the most effective to prove a general theorem on fractional differentiation.

Remark 2.4. In the same paper [13], the Pochhammer contour integral is used to explore the analyticity of both $D^{\alpha}_z z^p f(z)$ and $D^{\alpha}_z z^p \ln z f(z)$ (where $f(z)$) is analytic in a region R containing the point $z = 0$) with respect to the three variables z, α and p. We have: i) for fixed p and α, these are multiple valued analytic functions of the variable z having a branch point at $z = 0$, and a Riemann surface covering the domain $z \in R - \{0\}$; ii) for fixed z and α, these are meromorphic functions of p whose singularities are poles where p is a negative integer or a subset thereof (simple poles for $D^{\alpha}_z z^p f(z)$ and simple or double poles for $D^{\alpha}_z z^p \ln z f(z)$; iii) for fixed z and p, these are entire function of the variable α. Note, however, that the less elegant forms $D^{\alpha}_z \frac{z^p}{\Gamma(1+\alpha)\Gamma(1+p)} f(z)$ and $D^{\alpha}_z \frac{z^p}{\Gamma(1+\alpha)\Gamma(1+p)^2}\ln z f(z)$ do not have the restrictions on p and α.

3. Reminders of two important theorems involving the fractional derivative

We first recall two theorems: Theorem 3.1 published in 2007 concerning the development of an analytical function in series of quadratic functions of the form $\sum a_n[(z-z_1)(z-z_2)]^n$ where z_1 and z_2 are fixed [47] and Theorem 3.2 published in 2013 (3.10) [43] concerning the transformation formula $D^{\alpha}_z z^p f(z) = \Gamma(1+p)/\Gamma(-\alpha) D^{-p-1}_z z^{-\alpha-1} f(w-z)|_{w=z}$. This transformation formula will be used to demonstrate the symmetry with respect to the variables z and $z_1 + z_2 - z$ and parameters α and β of a new formula for series expansion of quadratic functions using the fractional derivative (4.1). These two series in terms of $[(z-z_1)(z-z_2)]^n$ generalize (1.2) obtained by Osler in 1971. We present several examples of application of these formulas to demonstrate the effectiveness of the fractional derivative in obtaining new results involving special functions. We present several examples of the implication of these theorems to obtain formulas involving special functions which seem new to the author. All these results are made possible thanks to the use of the Pochhammer integral symmetric with respect to parameters α and β to represent the fractional derivative first introduced by Tremblay in 1974 [40].

Theorem 3.1. *(i) Let a be real and positive and let* $\omega = e^{2\pi i a}$.

(ii) Let $f(z)$ be analytic in the simply connected region R with z_1 and z_2 being interior point of R.

(iii) Let $\theta(z) = ((z-z_1(z-z_2))q((z-z_1(z-z_2))$ be a given function such that $q(z)$ is regular and univalent for $z \in R$.

(iv) Let the set of curves $C(t)|0 < t \leq r$, $C(t) \subset R$, defined by

$$C(t) = C_1(t) \cup C_2(t) = \left\{ z \middle| |\lambda_t(z_1, z_2; z)| = |\lambda_t(z_1, z_2; (z_1+z_2)/2| \right\},$$

where

$$\lambda_t(z_1, z_2; z) = [(z-(z_1+z_2)/2 + t(z_1-z_2)/2)(z-(z_1+z_2)/2 - t(z_1+z_2)/2)]$$

which are lemniscates of Bernoulli type with the center located at $(z_1+z_2)/2$ *and with double loops; one loop* $C_1(t)$ *leads around the focus point* $(z_1+z_2)/2+t(z_1-z_2)/2$ *and the other loop* $C_2(t)$ *encircles the focus point* $(z_1+z_2)/2-t(z_1-z_2)/2$, *for each* t *such that* $0<t\leq r$.

(v) *Let* $\theta^{\lambda}((z-z_1)(z-z_2)) = \exp\{\lambda\ln(\theta((z-z_1)(z-z_2)))\}$ *denote the principal branch of that function which is continuous and inside* $C(r)$, *cut by the respective two branch lines* $L_{\pm}$ *defined by*

$$L_{\pm}=\begin{cases}\left\{z|z=\dfrac{z_1+z_2}{2}\pm t\dfrac{z_1-z_2}{2}\right\}, & \text{for } 0\leq t\leq 1\\[2ex] \left\{z|z=\dfrac{z_1+z_2}{2}\pm i\,t\dfrac{z_1-z_2}{2}\right\}, & \text{for } t<0\end{cases}$$

such that $\ln((z-z_1)(z-z_2))$ *is real, where* $(z-z_1)(z-z_2)>0$.

(vi) *Let* $f(z)$ *satisfy the conditions of Representation* (2.2) *for the existence of the fractional derivative of* $(z-z_2)^p f(z)$ *of order* α *for* $z\in R-\{L_+\bigcup L_-\}$, *noticed by* $D^{\alpha}_{z-z_2}(z-z_2)^p f(z)$, *where* α *and* p *are real or complex numbers.* *(vi) Let* $K=\{k\in N|\arg(\theta(\lambda_1(z_1,z_2,(z_1+z_2)/2)))<\arg(\theta(\lambda_1(z_1,z_2,(z_1+z_2)/2)))+2\pi k/a\leq\arg(\theta(\lambda_1(z_1,z_2,(z_1+z_2)/2)))+2\pi\}$.

Then, for arbitrary complex numbers α, β *and* γ *(with* $e^{2\pi i\gamma}=1$*) for* z *on* $C_1(1)$ *as shown in Figure* 2, *we have*

$$\begin{aligned}&\sum_{k\in K}a^{-1}\omega^{-\gamma k}\Bigg[f\left(\frac{z_1+z_2+\sqrt{\Delta_k}}{2}\right)\left(\frac{z_2-z_1+\sqrt{\Delta_k}}{2}\right)^{\alpha}\left(\frac{z_1-z_2+\sqrt{\Delta_k}}{2}\right)^{\beta}-e^{i\pi(\alpha-\beta)}\frac{\sin[(\alpha+a-\gamma)\pi]}{\sin[(\beta+a-\gamma)\pi]}\\&\times f\left(\frac{z_1+z_2-\sqrt{\Delta_k}}{2}\right)\left(\frac{z_2-z_1-\sqrt{\Delta_k}}{2}\right)^{\alpha}\left(\frac{z_1-z_2-\sqrt{\Delta_k}}{2}\right)^{\beta}\Bigg]\\=&\sum_{-\infty}^{\infty}\frac{\sin[(\beta-an-\gamma)\pi]}{\sin[(\beta+a-\gamma)\pi]}e^{-i\pi a(n+1)}\theta(z)^{an+\gamma}\frac{D^{-\alpha+an+\gamma}_{z-z_2}}{\Gamma(1-\alpha+an+\gamma)}\left[(z-z_2)^{\beta-an-\gamma-1}\left(\frac{\theta(z)}{(z-z_2)(z-z_1)}\right)^{-an-\gamma-1}\theta'(z)f(z)\right]\Bigg|_{z=z_1}\end{aligned}\tag{3.1}$$

where

$$\Delta_k=(z_1-z_2)^2+4V(\theta(z)\omega^k)$$

and

$$V(z)=\sum_{r=1}^{\infty}D^{r-1}_{z}(q(z)^{-r})\Big|_{z=0}\,z^r/r!.$$

3.1. Some special cases of formula (3.1)

Case 1. In [47, Eq. (4.38), p. 46], we find

$$\begin{aligned}(z-z_1)^{\alpha}(z-z_2)^{\beta}f(z) &= \sum_{-\infty}^{+\infty}\frac{D^{-\alpha+n}_{z_1-z_2}}{\Gamma(1-\alpha+n)}\Big[(z_1-z_2)^{\beta-n}f(z_1)\Big][(z-z_1)(z-z_2)]^n\\&+(z-z_1)\sum_{-\infty}^{+\infty}\frac{D^{-\alpha+n}_{z_1-z_2}}{\Gamma(1-\alpha+n)}\Big[(z_1-z_2)^{\beta-n-1}f(z_1)\Big][(z-z_1)(z-z_2)]^n.\end{aligned}$$

With $f(z)=e^z$ and $f(z)=(1-z+z_2)^{-\gamma}$, we obtain the tow following formulas

$$\begin{aligned}\left(\frac{z-z_1}{z_1-z_2}\right)^{\alpha}\left(\frac{z-z_2}{z_1-z_2}\right)^{\beta}e^{z-z_2} &= \sum_{-\infty}^{\infty}\binom{\beta-n}{\alpha+\beta-2n}\,{}_1F_1\left[\begin{matrix}\beta+1-n\\ \alpha+\beta+1-2n\end{matrix}\Bigg|z_1-z_2\right]\left(\frac{(z-z_1)(z-z_2)}{(z_1-z_2)^2}\right)^n\\&+\frac{(z-z_1)}{(z_1-z_2)}\sum_{-\infty}^{\infty}\binom{\beta-1-n}{\alpha+\beta-1-2n}\,{}_1F_1\left[\begin{matrix}\beta-n\\ \alpha+\beta-2n\end{matrix}\Bigg|z_1-z_2\right]\left(\frac{(z-z_1)(z-z_2)}{(z_1-z_2)^2}\right)^n\end{aligned}\tag{3.2}$$

and

$$\left(\frac{z-z_1}{z_1-z_2}\right)^{\alpha}\left(\frac{z-z_2}{z_1-z_2}\right)^{\beta}(1-z_1+z_2)^{-\gamma} = \sum_{-\infty}^{\infty}\binom{\beta-n}{\alpha+\beta-2n}\,{}_2F_1\left[\left.\begin{matrix}1+\beta-n,\gamma\\ 1+\alpha+\beta-2n\end{matrix}\right|z_1-z_2\right]\left(\frac{(z-z_1)(z-z_2)}{(z_1-z_2)^2}\right)^{n} \tag{3.3}$$
$$+\left(\frac{z-z_1}{z_1-z_2}\right)\sum_{-\infty}^{\infty}\binom{\beta-1-n}{\alpha+\beta-1-2n}\,{}_2F_1\left[\left.\begin{matrix}\beta-n,\gamma\\ \alpha+\beta-2n\end{matrix}\right|z_1-z_2\right]\left(\frac{(z-z_1)(z-z_2)}{(z_1-z_2)^2}\right)^{n}.$$

Putting $\alpha = 0$ in (3.2) and (3.3), with suitable changing of variables, we deduce two presume new formulas

$$(1+t)^{\alpha}e^{-zt} = \sum_{n=0}^{\infty} L_n^{(\alpha-2n)}(z)[t(1+t)]^n + t\sum_{n=0}^{\infty} L_n^{(\alpha-1-2n)}(z)[t(1+t)]^n \tag{3.4}$$

and

$$(1-t)^{\alpha}\left(1+\frac{t(1-z)}{1+z}\right)^{\beta} = \sum_{n=0}^{\infty} P_n^{(\alpha-2n,\beta-n)}(z)\left[-\frac{2t(1-t)}{1+z}\right]^n \tag{3.5}$$
$$-t\sum_{n=0}^{\infty} P_n^{(\alpha-1-2n,\beta-n)}(z)\left[-\frac{2t(1-t)}{1+z}\right]^n.$$

The last special cases (3.4) and (3.5) involving classical polynomials seems new and we can multiply formulas of the same type.

Next, other special forms are examined and new identities involving special functions and integrals are obtained in [47].

Case 2. Putting $g(z) = -4\frac{(z-z_1)(z-z_2)}{(z_1-z_2)^2}$, we have $g^{-1}(z) = \frac{z_1+z_2 \pm (z_1-z_2)\sqrt{1-z}}{2}$. With this change of variables, the equations (4.28) and (4.40) in [47] take the following form

$${}_2H_2\left[\left.\begin{matrix}1/2-\alpha/2-\beta/2, 1-\alpha/2-\beta/2\\ 1-\alpha, 1-\beta\end{matrix}\right|z\right] \tag{3.6}$$
$$= \frac{\Gamma(1-\alpha)\Gamma(\alpha+\beta)}{\Gamma(\beta)2^{\alpha+\beta}}\left[\frac{\left(-1+\sqrt{1-z}\right)^{\alpha}\left(1+\sqrt{1-z}\right)^{\beta}+\frac{\sin\alpha\pi}{\sin\beta\pi}\left(1+\sqrt{1-z}\right)^{\alpha}\left(-1+\sqrt{1-z}\right)^{\beta}}{\sqrt{1-z}}\right]$$

and

$${}_2H_2\left[\left.\begin{matrix}-\alpha/2-\beta/2, 1/2-\alpha/2-\beta/2\\ 1-\alpha, 1-\beta\end{matrix}\right|z\right] \tag{3.7}$$
$$= \frac{\Gamma(1-\alpha)\Gamma(1+\alpha+\beta)}{\Gamma(\beta)2^{\alpha+\beta}}\left[\frac{\left(-1+\sqrt{1-z}\right)^{\alpha}\left(1+\sqrt{1-z}\right)^{\beta}-\frac{\sin\alpha\pi}{\sin\beta\pi}\left(1+\sqrt{1-z}\right)^{\alpha}\left(-1+\sqrt{1-z}\right)^{\beta}}{\beta-\alpha}\right].$$

valid if z is on $C(1)$.

The bilateral series

$${}_MH_N\left[\left.\begin{matrix}a_1, a_2, \dots, a_M\\ b_1, b_2, \dots, b_N\end{matrix}\right|z\right] = \sum_{n=-\infty}^{+\infty}\frac{(a_1)_n(a_2)_n\dots(a_M)_n}{(b_1)_n(b_2)_n\dots(b_N)_n}z^n$$

is called "H-function" or Dirichlet-Laurent series (*cf.* [1], [37, Eq. (6.1.1.2), p. 180]) (See the convergence conditions in [37, p. 181]).

By subtraction, we obtain from (3.6) and (3.7):

$$\sqrt{1-z}\ {}_2H_2\left[\begin{matrix}1/2-\alpha/2-\beta/2, 1-\alpha/2-\beta/2\\ 1-\alpha, 1-\beta\end{matrix}\middle| z\right] - \frac{\alpha-\beta}{\alpha+\beta}\ {}_2H_2\left[\begin{matrix}-\alpha/2-\beta/2, 1/2-\alpha/2-\beta/2\\ 1-\alpha, 1-\beta\end{matrix}\middle| z\right] \tag{3.8}$$
$$= \frac{\Gamma(1-\alpha)\Gamma(\alpha+\beta)}{\Gamma(\beta)2^{\alpha+\beta-1}}(-1+\sqrt{1-z})^{\alpha}(1+\sqrt{1-z})^{\beta}.$$

Multiplying formulas (3.6) and (3.7) gives

$$\sqrt{1-z}\ {}_2H_2\left[\begin{matrix}1/2-\alpha/2-\beta/2, 1-\alpha/2-\beta/2\\ 1-\alpha, 1-\beta\end{matrix}\middle| z\right]{}_2H_2\left[\begin{matrix}-\alpha/2-\beta/2, 1/2-\alpha/2-\beta/2\\ 1-\alpha, 1-\beta\end{matrix}\middle| z\right] \tag{3.9}$$
$$= \left[\frac{\Gamma(1-\alpha)}{\Gamma(\beta)2^{\alpha+\beta}}\right]^2\frac{\Gamma(\alpha+\beta)\Gamma(1+\alpha+\beta)}{\beta-\alpha}$$
$$\times\left[(-1+\sqrt{1-z})^{2\alpha}(1+\sqrt{1-z})^{2\beta} - \left[\frac{1-\cos 2\alpha\pi}{1-\cos 2\beta\pi}\right](1+\sqrt{1-z})^{2\alpha}(-1+\sqrt{1-z})^{2\beta}\right].$$

With $\beta = \alpha$, (3.6) become

$${}_1H_1\left[\begin{matrix}1/2-\alpha/2\\ 1-\alpha\end{matrix}\middle| z\right] = \frac{\Gamma(1-\alpha)\Gamma(1/2+\alpha)(e^{i\pi}z)^{\alpha}}{\sqrt{\pi}\sqrt{1-z}}.$$

If $\beta \to \alpha$ in (3.7) and (3.9), we find

$${}_2H_2\left[\begin{matrix}-\alpha, 1/2-\alpha/2\\ 1-\alpha, 1-\alpha\end{matrix}\middle| z\right] = \frac{\Gamma(1-\alpha)\Gamma(1+2\alpha)}{\Gamma(\alpha)2^{2\alpha}}(z\,e^{i\pi})^{\alpha}\left\{\ln\left(\frac{\sqrt{1-z}+1}{\sqrt{1-z}-1}\right)+\pi\cot(\alpha\pi)\right\}.$$

If $z = 1$ in (3.7), we obtain after simplifications

$${}_2H_2\left[\begin{matrix}-\alpha/2-\beta/2, \frac{1}{2}-\alpha/2-\beta/2\\ 1-\alpha, 1-\beta\end{matrix}\middle| 1\right] = \frac{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(1+\alpha+\beta)}{2^{\alpha+\beta}\Gamma(1+\beta-\alpha)\Gamma(1+\alpha-\beta)}$$

a special case of the bilateral series summation theorem (*cf.* [37, Eq. (III.28), p. 245]).

Theorem 3.2. *Let $f(z)$ be a function that satisfies the conditions for the existence of the fractional derivative $D^{\alpha}_{g(z)}(g(z))^p f(z))$ listed in Definition 2.2 with $g(z) = z - b$ and using a Pochhammer contour $P(a) = C_1 \cup C_2 \cup (-C_1) \cup (-C_2)$ laid out around the points $g^{(-1)}(0)$ and z (Figure 2). If $f(g^{(-1)}(0)) \neq 0$ and $p \neq -1, -2, \ldots$ then we have*

$$D^{\alpha}_{g(z)}(g(z))^p f(z) = \frac{\Gamma(1+p)}{\Gamma(-\alpha)}D^{-p-1}_{g(z)}\{(g(z))^{-\alpha-1}f(g^{-1}(g(w)-g(z)))\}\big|_{w=z} \tag{3.10}$$

for $z \in R - \{g^{-1}(0)\}$. Note that we must have $w \to b$ in the right side of (6) after the evaluation of the fractional derivative, the point w must be near the point z inside the loop C_1. If $g(z) = z - b$ then $g^{-1}(g(w) - g(z)) = w + b - z$ and we obtain the following reduced form

$$D^{\alpha}_{z-b}(z-b)^p f(z) = \frac{\Gamma(1+p)}{\Gamma(-\alpha)}D^{-p-1}_{z-b}\{(z-b)^{-\alpha-1}f(w+b-z)\}\big|_{w=z}. \tag{3.11}$$

In [43], we find many interesting results obtained with the transformation (1.2). Let's give some examples.

Case 3. From the Abel's binomial theorem [9, Eq. (1b), p. 137]

$$\frac{(x+y)^n}{x} = \sum_{k=0}^{n}\binom{n}{k}(x-kz)^{k-1}(y+kz)^{n-k}, \tag{3.12}$$

we find a similar newton's binomial formula (*cf.* [43])

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}F_1\left(\alpha,-k+1,-n+k;\beta;\frac{kz}{x},-\frac{kz}{y}\right) \tag{3.13}$$

valid if $\beta \neq 0,-1,\dots$ and $xy \neq 0$. The function denoted F_1 is the first Appell's function defined by

$$F_1(a,b,b';c;x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a)_{i+j}(b)_i(b')_j}{(c)_{i+j}i!j!}x^i y^j.$$

If we put $y=-x$ in (3.13), by generating the additional parameters a_i and b_j using the Laplace transform and the inverse Laplace transform, we easily obtain that

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}{}_{p+1}F_q\left[\begin{matrix}-n+1,A_P\\ B_Q\end{matrix}\middle| kz\right]=0, \tag{3.14}$$

where $\Omega_\Phi = \{\omega_1,\omega_2,\dots,\omega_\varphi\}$.

Case 4. From the same identity (3.12), we can obtain the following formula (*cf.* [43, Eqs. (51) and (52), p. 184])

$$\left\{\frac{(\beta-\alpha)_n}{(\beta)_n}-(-1)^n\frac{(\alpha)_n}{(\beta)_n}\right\}z^n = \sum_{k=1}^{n}\binom{n}{k}z(z-kt)^{k-1}(kt)^{n-k}{}_2F_1\left[\begin{matrix}-n+k,\alpha\\ \beta\end{matrix}\middle|\frac{z}{kt}\right]. \tag{3.15}$$

Using the Euler transformation (*cf.* [29, Eq. (5), Theorem 21, p. 60]) in (3.15) and puting $\beta=\alpha-n+1$, we obtain after simplifications

$$z^{n-1}\frac{(\alpha)_n}{(-\alpha)_n} = \sum_{k=1}^{n}\binom{n}{k}(-1)^k(kt)^{n-1}{}_2F_1\left[\begin{matrix}-n+1,\alpha+1-k\\ \alpha+1-n\end{matrix}\middle|\frac{(-1)^{r+s}z}{kt}\right].$$

As in the previous case, we can generate hypergeometric parameters using the Laplace transform and its inverse to obtain the following formula involving the generalized hypergeometric function

$$z^{n-1}\frac{(\alpha)_n}{(-\alpha)_n}\frac{\prod_{i=1}^{p}(a_i)_{n-1}\prod_{u=1}^{r}(c_u)_{n-1}}{\prod_{j=1}^{q}(b_j)_{n-1}\prod_{v=1}^{s}(d_v)_{n-1}} = \sum_{k=1}^{n}\binom{n}{k}(-1)^k(kt)^{n-1}{}_{p+r+2}F_{q+s+1}\left[\begin{matrix}-n+1,\alpha+1-k,A_P,2-C_R-n\\ \alpha+1-n,B_Q,2-D_S-n\end{matrix}\middle|\frac{(-1)^{r+s}z}{kt}\right],$$

where $2-\Omega_\Phi-n=\{2-\omega_1-n,2-\omega_2-n,\dots,2-\omega_\varphi-n\}$.

Case 5. In the same way as in [43], we get from the following form of Abel's identity (*cf.* [34])

$$(x+y)(x+y-an)^{n-1} = \sum_{k=0}^{n}\binom{n}{k}xy(x-ak)^{k-1}(y-a(n-k))^{n-k-1} \tag{3.16}$$

the following formula

$$\begin{aligned}&(an)^{n-1}\left\{\frac{\beta-\alpha}{\beta}{}_2F_1\left[\begin{matrix}-n+1,\beta-\alpha+1\\ \beta+1\end{matrix}\middle|\frac{z}{an}\right]+\frac{\alpha}{\beta}{}_2F_1\left[\begin{matrix}-n+1,\alpha+1\\ \beta+1\end{matrix}\middle|-\frac{z}{an}\right]\right\} \\ =\ &(-1)^{n-1}(z-an)^{n-1}-\frac{\alpha}{\beta}z\sum_{k=1}^{n-1}\binom{n}{k}(z-ak)^{k-1}(-1)^k(a(n-k))^{n-k-1}{}_2F_1\left[\begin{matrix}-n+k+1,\alpha+1\\ \beta+1\end{matrix}\middle|-\frac{z}{a(n-k)}\right]\end{aligned} \tag{3.17}$$

for $n=1,2,\dots,\beta\neq 0$ and $a\neq 0$. If $\beta=\alpha$ in (3.17), we obtain after simplifications

$$(an+z)^{n-1}-(an-z)^{n-1} = z\sum_{k=1}^{n-1}\binom{n}{k}(a(n-k)+z)^{n-k-1}(ak-z)^{k-1}. \tag{3.18}$$

Moreover, applying the operator $z^{1-\beta}D_z^{\alpha-\beta}z^{\alpha-1}$ on the both sides of the equation (3.18), we obtain

$$(an)^{n-1}\left\{{}_2F_1\left[\begin{matrix}-n+k+1,\alpha\\ \beta\end{matrix}\middle| -\frac{z}{an}\right]\right\}$$
$$= \frac{\alpha}{\beta}z\sum_{k=1}^{n-1}\binom{n}{k}(a(n-k))^{n-k-1}(ak)^{k-1}F_1(\alpha+1;-n+k+1,-k+1;\beta+1;-z/(a(n-k)),z/(ak).$$

Several other examples are possible.

The following section is devoted to the proof of a new expansion formula (4.1) in quadratic terms $[(z-z_1)(z-z_2)]^n$ and to the consequences of the symmetry of this expansion formula.

4. Main results

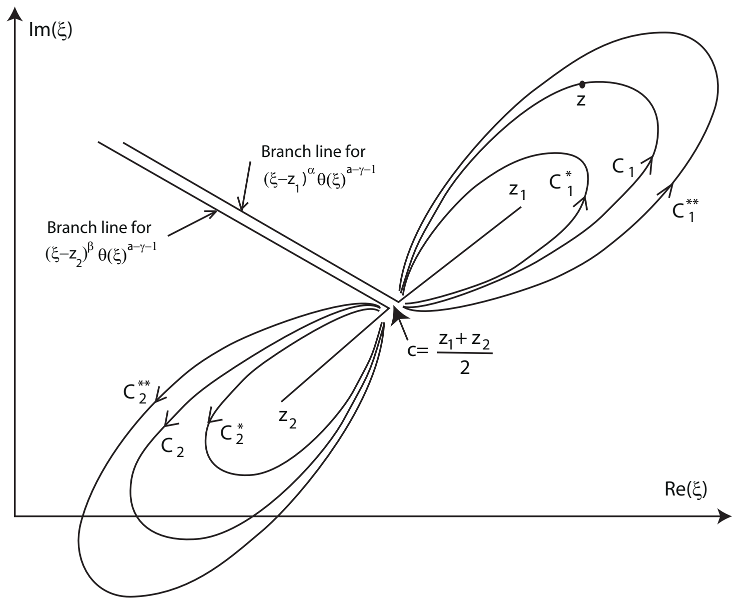

Figure 3. Integration contours

Theorem 4.1. *(i) Let a be real and positive, and let* $\omega = exp(2\pi i/a)$.

(ii) Let $f(z)$ *be analytic in the simply connected region R, with* z_1 *and* z_2 *being interior points of R.*

(iii) Let $\theta(z) = zq(z)$ *be a given function such that* $q(z)$ *is regular and univalent for* $z \in R$.

(iv) Let $C^{**} = \{C_1^{**}\cup C_2^{**}\cup(-C_1^{**})\cup(-C_2^{**})\}$ *and* $C^{*} = \{C_1^{*}\cup C_2^{*})\cup(-C_1^{*})\cup(-C_2^{*})\}$ *be two four loops contours (called Pochhammer's contours) running around the branch* z_1 *and* z_2 *where* $C_1^{**}\cup C_2^{**} = C(1+\epsilon)$, $0<\epsilon\le r-1$ *and* $C_1^{*}\cup C_2^{*} = C(1-\delta)$, $0<\delta\le 1-\frac{\sqrt{2}}{2}$, $C(t)$ *being defined by* $|\theta(\lambda_t[z_1,z_2;\xi])| = |\theta(\lambda_t[z_1,z_2;(z_1+z_2)/2])|$, *with* $\lambda_t(z_1,z_2;\xi) = [\xi-(z_1+z_2)/2+t(z_2-z_1)][\xi-(z_1+z_2)/2+t(z_2-z_1)]$, z_1 *and* z_2 *be chosen such that the set of curves* $\{C(t)|\ 0<t\le r\}\subset R$.

(v) Let $\{\theta(z)\}^\delta = exp(\delta\ ln(\theta(z)) = exp(\delta[ln(z-z_1)+ln(z-z_2)+ln(q((z-z_1)(z-z_2)))]$ *denote the branch of the function which is continuous and single valued on the region inside* $C(r)$ *cut by two broken branch lines* $L_1 = \{z = z_2+s(z_1-z_2)\}\cup\{z=(z_1+z_2)/2+t\,e^{i\mu}\}$ *and* $L_2 = \{z = z_1+(1-s)(z_2-z_1)\}\cup\{z=(z_1+z_2)/2+t\,e^{i\mu}\}$, $0<s<1/2$, $t\ge 0$ *and* $arg(z_1-z_2)<\mu<arg(z_1-z_2)+\pi/4$, *such that* $ln(\theta(z))$ *is real where* $\theta(z)>0$.

(vi) Let K denotes the set of integers k defined by

$$arg(\theta\{(z_1+z_2)/2)^2\}) < arg(\theta\{(z-z_1)(z-z_2)\}) + 2\pi/a < \theta\{(z_1+z_2)/2)^2\}) + 2\pi.$$

(vii) Let $f(z)$ satisfy the conditions of Definition 2.2 for the existence of the fractional derivative of $(z-z_2)^p f(z)$ of order α for $z \in R - \{L_1 \cup L_2\}$, denoted by $D^{\alpha}_{z-z_2}(z-z_2)^p f(z)$ where α and p are real or complex numbers. Then for arbitrary complex numbers α, β, γ, and for $z\ (\neq (z_1+z_2)/2)$ on C_1, we have

$$2\sum_{k\in K} a^{-1}\omega^{-\gamma k}\left[f\left(\frac{z_1+z_2+\sqrt{\Delta_k}}{2}\right)+f\left(\frac{z_1+z_2-\sqrt{\Delta_k}}{2}\right)\right]\left[\sin[(\beta+a-\gamma)\pi]\left(\frac{z_2-z_1+\sqrt{\Delta_k}}{2}\right)^{\alpha}\left(\frac{z_1-z_2+\sqrt{\Delta_k}}{2}\right)^{\beta}\right. \tag{4.1}$$

$$\left.-e^{i\pi(\alpha-\beta)}\sin[(\alpha+a-\gamma)\pi]\left(\frac{z_2-z_1-\sqrt{\Delta_k}}{2}\right)^{\alpha}\left(\frac{z_1-z_2-\sqrt{\Delta_k}}{2}\right)^{\beta}\right]$$

$$=\sum_{-\infty}^{\infty}\theta(z)^{an+\gamma}e^{-i\pi a(n+1)}$$

$$\times\left\{\sin[(\beta-an-\gamma)\pi]\frac{D^{-\alpha+an+\gamma}_{z-z_2}}{\Gamma(1-\alpha+an+\gamma)}\left[(z-z_2)^{\beta-an-\gamma-1}\left(\frac{\theta(z)}{(z-z_2)(z-z_1)}\right)^{-an-\gamma-1}\theta'(z)(f(z)+f(z_1+z_2-z))\right]\Bigg|_{z=z_1}\right.$$

$$\left.-\sin[(\alpha-an-\gamma)\pi]\frac{D^{-\beta+an+\gamma}_{z-z_2}}{\Gamma(1-\beta+an+\gamma)}\left[(z-z_2)^{\alpha-an-\gamma-1}\left(\frac{\theta(z)}{(z-z_2)(z-z_1)}\right)^{-an-\gamma-1}\theta'(z)(f(z)+f(z_1+z_2-z))\right]\Bigg|_{z=z_1}\right\},$$

where

$$\Delta_k = (z_1-z_2)^2 + 4V(\theta(z)\omega^k)$$

and

$$V(z) = \sum_{r=1}^{\infty} D_z^{r-1}(q(z)^{-r})\Big|_{z=0}\, z^r/r!.$$

Proof. Consider the integral

$$I = \int_{C^{**}-C^{*}} F(\xi)\frac{\theta(\xi)^{a-\gamma-1}\theta'(\xi)}{\theta(\xi)^a-\theta(z)^a}d\xi$$

with

$$F(\xi) = [f(\xi)+f(z_1+z_2-\xi]\left\{(\xi-z_1)^{\alpha}(\xi-z_2)^{\beta}-(z_1-\xi)^{\beta}(z_2-\xi)^{\alpha}\right\}$$

and $C^{**}-C^{*} = C_1^{**}\cup C_2^{**}\cup -C_1^{**}\cup -C_2^{**}\cup C_2^{*}\cup C_1^{*}\cup -C_2^{*}\cup -C_1^{*}$ as shown in Figure 2. Note that the Maximum Modulus Theorem insures that the set of closed curves $C(t)$, $t>0$ ($q(z)$ being a regular function) are such that $C(t_1)$ is contained inside in $C(t_2)$ for $t_1<t_2$. On the one hand, we have

$$I = \int_{C_1^{**}\cup C_2^{**}\cup -C_1^{**}\cup -C_2^{**}} - \int_{C_1^{*}\cup C_2^{*}\cup -C_1^{*}\cup -C_2^{*}} = 2\int_{C_1^{**}\cup C_2^{**}\cup -C_1^{**}\cup -C_2^{**}}.$$

Also we have

$$\begin{aligned} I &= \int_{C(1+\varepsilon)\cup C(1+\varepsilon)} F(\xi)\frac{\theta(\xi)^{-\gamma-1}\theta'(\xi)}{\{1-(\frac{\theta(z)}{\theta(\xi)})^a\}}d\xi + \int_{C(1-\delta)\cup C(1-\delta)} F(\xi)\frac{\theta(\xi)^{a-\gamma-1}\theta'(\xi)}{\theta(z)^a\{1-(\frac{\theta(\xi)}{\theta(z)})^a\}}d\xi \\ &= \left[\sum_{n=0}^{N}\int_{C(1+\varepsilon)\cup C(1+\varepsilon)}\theta(\xi)^{-an-\gamma-1}F(\xi)\theta'(\xi)\right]\theta(z)^{an} + R_{1+\varepsilon}(N) \\ &\quad + \left[\sum_{n=-N}^{-1}\int_{C(1-\delta)\cup C(1-\delta)}\theta(\xi)^{-an-\gamma-1}F(\xi)\theta'(\xi)\right]\theta(z)^{an} + R_{1-\delta}(N), \end{aligned} \tag{4.2}$$

where

$$R_{1+\varepsilon}(N) = \int_{C(1+\varepsilon)\cup -C(1+\varepsilon)} \frac{F(\xi)\theta'(\xi)\theta(\xi)^{-\gamma-1}\left(\frac{\theta(z)}{\theta(\xi)}\right)^{aN+a}}{1-\left(\frac{\theta(z)}{\theta(\xi)}\right)^{a}} d\xi$$

and

$$R_{1-\delta}(N) = \int_{C(1-\delta)\cup -C(1-\delta)} \frac{F(\xi)\theta'(\xi)\theta(\xi)^{-\gamma-1}\left(\frac{\theta(\xi)}{\theta(z)}\right)^{aN+a}}{1-\left(\frac{\theta(\xi)}{\theta(z)}\right)^{a}} d\xi.$$

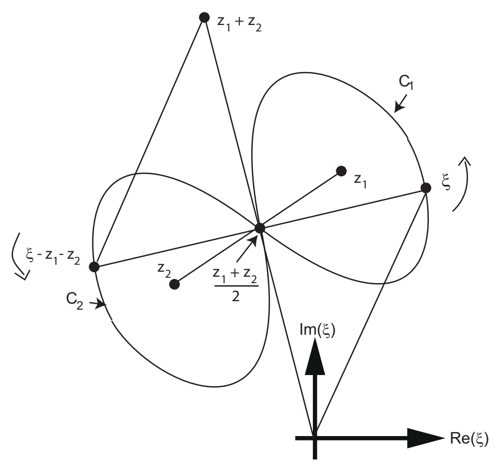

Figure 4. Change of variables ξ to $z_1 + z_2 - \xi$

The regularity of $\theta(z)$ and $f(z) + f(z_1 + z_2 - z)$ allow us to deform the contours of integration $C(1+\varepsilon)\cup C(1+\varepsilon)$ and $C(1-\delta)\cup C(1-\delta)$ in (4.2) provided the contours start and end at $\xi = (z_1+z_2)/2$ and do not cross the branch lines for $\theta(z)^{-a-\gamma}[(z-z_2)^{\beta}(z-z_1)^{\alpha} - (z_1-z)^{\beta}(z_2-z)^{\alpha}]$ defined by $L_1 = \{z = z_2 + s(z_1 - z_2)\} \cup \{z = (z_1+z_2)/2 + t\, e^{i\mu}\}$ and $L_2 = \{z = z_1 + (1-s)(z_2 - z_1)\} \cup \{z = (z_1+z_2)/2 + t\, e^{i\mu}\}$, $0 < s < 1/2$, $t \geq 0$ and $arg(z_1 - z_2) < \mu < arg(z_1 - z_2) + \pi/4$. If z is on $C(1)$, for all $\xi \neq (z_1 = z_2)/2$, we have by the Maximum Modulus Principle, $|\frac{\theta(z)}{\theta(\xi)}| < 1$ on $C(1+\varepsilon)$ and $|\frac{\theta(\xi)}{\theta(z)}| < 1$ on $C(1-\delta)$. Also, if $\varepsilon \to 0$ and $\delta \to 0$ and large N, $R_{1+\varepsilon}(N)$ and $R_{1-\delta}(N)$ can be made arbitrary small. Consequently we have

$$\begin{aligned} I &= \sum_{n=-\infty}^{\infty} \int_P \left[\theta(\xi)^{-an-\gamma-1} F(\xi)\theta'(\xi) d\xi\right] \theta(z)^{an} \qquad (4.3) \\ &= \sum_{n=-\infty}^{\infty} \left\{ \int_P [f(\xi) + f(z_1+z_2-\xi)]\,(\xi - z_2)^{\beta}(\xi - z_1)^{\alpha}\theta(\xi)^{-an-\gamma-1}\theta'(\xi) d\xi \right. \\ &\quad \left. -e^{i\pi(\beta-\alpha)} \int_P [f(\xi) + f(z_1+z_2-\xi)](\xi - z_2)^{\alpha}(\xi - z_1)^{\beta}\theta(\xi)^{-an-\gamma-1}\theta'(\xi) d\xi \right\} \theta(z)^{an}. \end{aligned}$$

Using the definition of the fractional derivative, the first integral in (4.3) becomes

$$\begin{aligned}
&\int_P [f(\xi)+f(z_1+z_2-\xi)](\xi-z_2)^{\beta}(\xi-z_1)^{\alpha}\theta(\xi)^{-an-\gamma-1}\theta'(\xi)d\xi \\
=\;& \frac{4\pi\sin[(\beta-an-\gamma)\pi]e^{i\pi(\beta-an-\gamma)}}{\Gamma(1-\alpha+an+\gamma)} \\
& D_{z-z_2}^{-\alpha+an+\gamma}\Big\{(z-z_2)^{\beta-an-\gamma-1}[f(z)+f(z_1+z_2-z)]\theta'(z)\Big[\frac{\theta(z)}{(z-z_1)(z-z_2)}\Big]^{-an-\gamma-1}\Big\}\Big|_{z=z_1}.
\end{aligned}$$

The second integral in (4.3) gives the same result by changing the roles of α and β. By substitution, we find

$$\begin{aligned}
I \;=\;& 4\pi\sum_{n=-\infty}^{\infty} e^{i\pi(\beta-an-\gamma)}\theta(z)^{an}\Bigg[\sin[(\beta-an-\gamma)\pi]\frac{D_{z-z_2}^{-\alpha+an+\gamma}}{\Gamma(1-\alpha+an+\gamma)} \\
&\times\Big\{(z-z_2)^{\beta-an-\gamma-1}[f(z)+f(z_1+z_2-z)]\theta'(z)\Big[\frac{\theta(z)}{(z-z_1)(z-z_2)}\Big]^{-an-\gamma-1}\Big\}\Big|_{z=z_1} \\
&-e^{i\pi(\alpha-\beta)}\sin[(\alpha-an-\gamma)\pi]\frac{D_{z-z_2}^{-\beta+an+\gamma}}{\Gamma(1-\beta+an+\gamma)} \\
&\times\Big\{(z-z_2)^{\alpha-an-\gamma-1}[f(z)+f(z_1+z_2-z)]\theta'(z)\Big[\frac{\theta(z)}{(z-z_1)(z-z_2)}\Big]^{-an-\gamma-1}\Big\}\Big|_{z=z_1}\Bigg].
\end{aligned} \tag{4.4}$$

On the other hand, we have

$$\begin{aligned}
I =& \int_{C_1^{**}\cup C_2^{**}\cup -C_1^{**}\cup -C_2^{**}\cup C_2^{*}\cup C_1^{*}\cup -C_2^{*}\cup -C_1^{*}} \frac{\theta(\xi)^{a-\gamma-1}\theta'(\xi)}{\theta(\xi)^a-\theta(z)^a}G_1(\xi)d\xi \\
&+\int_{C_1^{*}\cup C_2^{*}\cup -C_1^{*}\cup -C_2^{*}\cup C_2^{**}\cup C_1^{**}\cup -C_2^{**}\cup -C_1^{**}} \frac{\theta(\xi)^{a-\gamma-1}\theta'(\xi)}{\theta(\xi)^a-\theta(z)^a}G_2(\xi)d\xi
\end{aligned} \tag{4.5}$$

with

$$G_1(\xi) = G_2(z_1+z_2-\xi) = [f(\xi)+f(z_1+z_2-\xi)](\xi-z_2)^{\beta}(\xi-z_1)^{\alpha}.$$

Note that the function $F_1(\xi) = F_2(z_1+z_2-\xi) = G_1(\xi)\theta(\xi)^{a-\gamma-1}\theta'(\xi)/(\theta(\xi)^a-\theta(z)^a)$ is analytic on $R\backslash\{z\}$.

Table 1 shows the values of the integrant at the start and end of each of the loops of the two double Pochhammer's contours of the integrals in (4.5) with $r=\alpha+a-\gamma$ and $s=\beta+a-\gamma$.

	Integrals with F_1	At the beginning	At the end	*Integrals with* F_2	
A	C_1^{**}	F_0	$F_0\, e^{2\pi ir}$	C_2^{**}	A'
B	C_2^{**}	$F_0\, e^{2\pi ir}$	$F_0\, e^{2\pi i(r+s)}$	C_1^{**}	B'
C	$-C_1^{**}$	$F_0\, e^{2\pi i(r+s)}$	$F_0\, e^{2\pi is}$	$-C_2^{**}$	C'
D	$-C_2^{**}$	$F_0\, e^{2\pi is}$	F_0	$-C_1^{**}$	D'
E	C_2^{*}	F_0	$F_0\, e^{2\pi is}$	C_1^{*}	E'
F	C_1^{*}	$F_0\, e^{2\pi is}$	$F_0\, e^{2\pi i(r+s)}$	C_2^{*}	F'
G	$-C_2^{*}$	$F_0\, e^{2\pi i(r+s)}$	$F_0\, e^{2\pi ir}$	$-C_1^{*}$	G'
H	$-C_1^{*}$	$F_0\, e^{2\pi ir}$	F_0	$-C_2^{*}$	H'

Table 1. Value of the integrant at the beginning and at the end of each loop of two double Pochhammer's contour of integrals with F_1 and F_2, $F_0 = F_1((z_1+z_2)/2)$ or $F_0 = F_2((z_1+z_2)/2)$ depending on the case

Table 2 gives the value of each of the pairs of loops composing the two double Pochhammer's contours having as simple poles of the functions $F_1(\xi)$ and $F_2(\xi)$.

$A \cup H \to 2\pi i \sum_i Res_{\xi_{1,i}}(F_1(\xi))$	$A' \cup H' \to 2\pi i \sum_i Res_{\xi_{2,i}}(F_2(\xi))$
$B \cup G \to 2\pi i e^{2\pi i r} \sum_i Res_{\xi_{2,i}}(F_1(\xi))$	$B' \cup G' \to 2\pi i e^{2\pi i r} \sum_i Res_{\xi_{1,i}}(F_2(\xi))$
$C \cup F \to -2\pi i e^{2\pi i s} \sum_i Res_{\xi_{1,i}}(F_1(\xi))$	$C' \cup F' \to -2\pi i e^{2\pi i s} \sum_i Res_{\xi_{2,i}}(F_2(\xi))$
$D \cup E \to -2\pi i \sum_i Res_{\xi_{2,i}}(F_1(\xi))$	$D' \cup E' \to -2\pi i \sum_i Res_{\xi_{1,i}}(F_2(\xi))$
$I_1 \to 2\pi i(1 - e^{2\pi i s}) \sum_i Res_{\xi_{1,i}}(F_1(\xi)) + 2\pi i(e^{2\pi i r} - 1) \sum_i Res_{\xi_{2,i}}(F_1(\xi))$	$I_2 \to 2\pi i(1 - e^{2\pi i s}) \sum_i Res_{\xi_{2,i}}(F_2(\xi)) + 2\pi i(e^{2\pi i r} - 1) \sum_i Res_{\xi_{1,i}}(F_2(\xi))$

Table 2. Evaluation of integrals

For example, consider the loops C' and F' of Table 1. We have

$$\begin{aligned} C' \cup F' &= \int_{-C_2^{**}((z_1+z_2)/2, z_1^-:F_0 e^{2\pi(r+s)}, F_0 e^{2\pi i s})} F(\psi)d\psi \\ &= e^{2\pi i s}\left\{\int_{-C_2^{**}((z_1+z_2)/2, z_1^+:F_0 e^{2\pi i r}, F_0)} F(\psi)d\psi + \int_{C_2^{*}((z_1+z_2)/2, z_1^-:F_0, F_0 e^{2\pi i r})} F(\psi)d\psi\right\} \\ &= -e^{2\pi i s}\oint^{(C_2+)} F(\psi)d\psi = -2\pi i e^{2\pi i s}\sum_i Res_{\xi_{2,i}}\left(F_2(\xi)\frac{\theta(\xi)^{a-\gamma-1}\theta'(\xi)}{\theta(\xi)^a - \theta(z)^a}\right). \end{aligned}$$

The poles of integrals in (4.5) are given by the equation $\theta(\xi)^a - \theta(z)^a = 0$ or $\theta(\xi) = \theta(z)\omega^j$ (j is an integer) where $\omega = e^{2\pi i/a}$ with $0 \le j < a$. By the Lagrange's theorem, we find

$$(\xi - z_1)(\xi - z_2) = \sum_{s=1}^{\infty} D_x^{s-1}(q(x))^{-s}\Big|_{x=0}\frac{(\theta(z)\omega^i)^s}{s!} = V\left(\theta(z)\omega^i\right)$$

giving two sets $\{\xi_{1,j}\}$ and $\{\xi_{2,j}\}$ respectively on C_1 and C_2.

The value of residues at $\xi = \xi_{1,i}$ and $\xi = \xi_{2,i}$ are

$$\begin{aligned} Res_{\xi_{1,i}}(F_1(\xi)) &= Res_{\xi_{2,i}}(F_2(\xi)) \\ &= \frac{1}{a}[f(\xi_{1,i}) + f(z_1 + z_2 - \xi_{1,i})](\xi_{1,i} - z_2)^\beta(\xi_{1,i} - z_1)^\alpha\theta(\xi_{1,i})^{-\gamma} \\ &= \frac{1}{a}\left[f\left(\frac{z_1 + z_2 + \sqrt{\Delta_i}}{2}\right) + f\left(\frac{z_1 + z_2 - \sqrt{\Delta_i}}{2}\right)\right]\left(\frac{z_1 - z_2 + \sqrt{\Delta_i}}{2}\right)^\beta\left(\frac{z_2 - z_1 + \sqrt{\Delta_i}}{2}\right)^\alpha\theta(z)^{-\gamma}\omega^{-i\gamma}. \end{aligned} \tag{4.6}$$

Taking into account the equality (4.6), and the fact that $I = I_1 + I_2$, we obtain by substitution

$$\begin{aligned} I &= 8\pi e^{i\pi(\beta+a-\gamma)}\sin[(\beta + a - \gamma)\pi]\left\{\sum_i \frac{1}{a}\theta(z)^{-\gamma}\omega^{-i\gamma}\left[f\left(\frac{z_1 + z_2 + \sqrt{\Delta_i}}{2}\right) + f\left(\frac{z_1 + z_2 - \sqrt{\Delta_i}}{2}\right)\right]\right. \\ &\left.\times\left[\left(\frac{z_1 - z_2 + \sqrt{\Delta_i}}{2}\right)^\beta\left(\frac{z_2 - z_1 + \sqrt{\Delta_i}}{2}\right)^\alpha - e^{i\pi(\alpha-\beta)}\frac{\sin[(\alpha + a - \gamma)\pi]}{\sin[(\beta + a - \gamma)\pi]}\left(\frac{z_1 - z_2 - \sqrt{\Delta_i}}{2}\right)^\beta\left(\frac{z_2 - z_1 - \sqrt{\Delta_i}}{2}\right)^\alpha\right]\right\}. \end{aligned} \tag{4.7}$$

Bringing together the equalities (4.4) and (4.7), we obtain after simplifications (4.1), which completes the proof. □

Remark 4.2. With the transformation (3.11) in Theorem 3.2, we have

$$\begin{aligned} &D_{z-z_2}^{-\beta+an+\gamma}(z - z_2)^{\alpha-an-\gamma-1}\left\{[f(z) + f(z_1 + z_2 - z)]\theta'(z)\left[\frac{\theta(z)}{(z - z_1(z - z_2)}\right]^{-an-\gamma-1}\right\}\Bigg|_{z=z_1} \\ &= -\frac{\Gamma(\alpha - an - \gamma)}{\Gamma(\beta - an - \gamma)}D_{z-z_2}^{-\alpha+an+\gamma}(z - z_2)^{\beta-an-\gamma-1}\left\{[f(z) + f(z_1 + z_2 - z)]\theta'(z)\Big[\frac{\theta(z)}{(z - z_1)(z - z_2)}\Big]^{-an-\gamma-1}\right\} \end{aligned} \tag{4.8}$$

since $\theta(z_1+z_2-z)=\theta(z)$ and $\theta'(z_1+z_2-z)=-\theta'(z)$. Using (4.7), we obtain an equivalent form of the formula (4.1)

$$\begin{aligned}&\sum_{k\in K} a^{-1}\omega^{-\gamma k}\left[f\left(\frac{z_1+z_2+\sqrt{\Delta_k}}{2}\right)+f\left(\frac{z_1+z_2-\sqrt{\Delta_k}}{2}\right)\right]\\ &\times\left[\left(\frac{z_2-z_1+\sqrt{\Delta_k}}{2}\right)^{\alpha}\left(\frac{z_1-z_2+\sqrt{\Delta_k}}{2}\right)^{\beta}-e^{i\pi(\alpha-\beta)}\frac{\sin[(\alpha+a-\gamma)\pi]}{\sin[(\beta+a-\gamma)\pi]}\left(\frac{z_2-z_1-\sqrt{\Delta_k}}{2}\right)^{\alpha}\left(\frac{z_1-z_2-\sqrt{\Delta_k}}{2}\right)^{\beta}\right]\\ =\ &\sum_{-\infty}^{\infty}\frac{\sin[(\beta-an-\gamma)\pi]}{\sin[(\beta+a-\gamma)\pi]}e^{-i\pi a(n+1)}\theta(z)^{an+\gamma}\\ &\times\frac{D_{z-z_2}^{-\alpha+an+\gamma}}{\Gamma(1-\alpha+an+\gamma)}\left[(z-z_2)^{\beta-an-\gamma-1}\left(\frac{\theta(z)}{(z-z_2)(z-z_1)}\right)^{-an-\gamma-1}\theta'(z)(f(z)+f(z_1+z_2-z))\right]\Bigg|_{z=z_1}.\end{aligned}$$

4.1. *Symmetry of formula* (4.1)

We can show the symmetry of (4.1) using the transformation formula (3.11). Puting $g(z)=z-z_2$ then

$$\begin{gathered}g^{-1}(z)=z+z_2,\\ g^{-1}(g(w)-g(z))=g(w)-g(z)+z_2,\\ f(g^{-1}(g(w)-g(z_1)))=f(w-z+z_2),\\ f(w+z_2-g^{-1}(g(w)-g(z)))=f(z),\\ \theta'(z)|_{z=z_1}=2z_1-w-z_2|_{z=z_1},\\ \theta'(g^{-1}(g(w)-g(z_1)))|_{z=z_1}=w-2z_1+z_2|_{z=z_1}=-\theta'(z_1),\\ (g^{-1}(g(w)-g(z_1))-z_1)(g^{-1}(g(w)-g(z_1))-z_2)|_{z=z_1}=(z_2-z)(z_1-z)|_{z=z_1},\\ \theta(g^{-1}(g(w)-g(z)))|_{z=z_1}=(z_2-z)(z_1-z)q((z_2-z)(z_1-z))=\theta(z_1),\\ \Delta_k=(z_1-z_2)^2+4V((z_2-z)(z_1-z)q((z_2-z)(z_1-z))\omega^k).\end{gathered}$$

After all these substitutions, we find

$$\begin{aligned}&\frac{D_{z_1-z_2}^{-\alpha+an+\gamma}}{\Gamma(1-\alpha+an+\gamma)}\left[(z_1-z_2)^{\beta-an-\gamma-1}\left(\frac{\theta(z_1)}{(z_1-w)(z_1-z_2)}\right)^{-an-\gamma-1}\theta'(z_1)(f(z_1)+f(w++z_2-z_1))\right]\Bigg|_{z=z_1}\\ =\ &-\frac{\Gamma(\beta-an-\gamma)}{\Gamma(\alpha-an-\gamma)\Gamma(1-\alpha+an+\gamma)}D_{z-z_2}^{-\beta+an+\gamma}\Big[(z-z_1)^{\alpha-an-\gamma-1}\\ &\times\left(\frac{\theta(z_1)}{(z_1-w)(z_1-z_2)}\right)^{-an-\gamma-1}\theta'(w)(f(w-z_1+z_2)+f(z_1))\Big]\Big|_{w=z=z_1}.\end{aligned}$$

By multiplying the left side of (4.1) by $-\dfrac{\sin\pi(\beta+a-\gamma)}{\sin\pi(\alpha-a-\gamma)}$, we get after simplifications

$$\begin{aligned}&e^{i\pi(\alpha-\beta)}\left(1-e^{i\pi(\beta-\alpha)}\frac{\sin(\beta+a-\gamma)\pi}{\sin(\alpha+a-\gamma)\pi}\right)\sum_{k\in K}a^{-1}\omega^{-\gamma k}\left[f\left(\frac{z_1+z_2+\sqrt{\Delta_k}}{2}\right)+f\left(\frac{z_1+z_2-\sqrt{\Delta_k}}{2}\right)\right]\\ &\times\left[\left(\frac{z_2-z_1+\sqrt{\Delta_k}}{2}\right)^{\alpha}\left(\frac{z_1-z_2+\sqrt{\Delta_k}}{2}\right)^{\beta}+\left(\frac{z_2-z_1-\sqrt{\Delta_k}}{2}\right)^{\alpha}\left(\frac{z_1-z_2-\sqrt{\Delta_k}}{2}\right)^{\beta}\right].\end{aligned}$$

Finally we get an identical formula (4.1) where α and β are interchanged. Consequently, the left sides of (4.1) vanishes if $\alpha=\beta$ which is confirmed by the right side. However, it is not obvious that the right side of formula (3.1) vanishes if $\alpha=\beta$. In corollary 4.3, we prove this for any analytic function of the form $f(z)+f(z_1+z_2-z)$.

Corollary 4.3. *Consider an analytical function $f(z)$ in R. If $\theta(z)=(z-z_1)(z-z_2)q\{(z-z_1)(z-z_2)\}$ with a regular function $q(z)$ analytic on R and $\{z_1,z_2\}\in R$. Suppose that* $F(z)=\left[\frac{\theta(z)}{(z-z_2)(z-z_1)}\right]^{-an-\gamma-1}\theta'(z)(f(z)+f(z_1+z_2-z))$

satisfies the conditions for the existence of the fractional derivative $D^{\alpha}_{z-z_2}(z-z_2)^p F(z)\Big|_{z=z_1}$ *listed in Definition* 2.2 *with* $g(z_1) = z_1 - z_2$ *using a Pochhammer contour* $P(a) = C_1 \cup C_2 \cup -C_1 \cup -C_2$ *laid out around the points* $g^{-1}(0) = z_2$ *and* z_1 *(see Figure* 1*). If* $G(z) = f(z) + f(z_1 + z_2 - z)$ *and* $\beta = \alpha$ *then we have*

$$D^{-\alpha+an+\gamma}_{z-z_2}\left[(z-z_2)^{\beta-an-\gamma-1}\left[\frac{\theta(z)}{(z-z_2)(z-z_1)}\right]^{-an-\gamma-1}\theta'(z)G(z)\right]\Bigg|_{z=z_1} = 0. \tag{4.9}$$

Proof. In (3.1), z_1 and z_2 are two fixed points in the z-plane such that $\left|\theta((z-z_1)(z-z_2))\right| = \left|\theta((z_1-z_2)^2/4)\right|$ defines a double-loop curve $C = C_1 \cup C_2$ (lemniscate's type) on which the series converges with $z_l \neq z_2$ and $\Delta_k = (z_1 - z_2)^2 + 4V(\theta(z)\omega^k)$. If we put $(z-z_1)(z-z_2) = \frac{(z_1-z_2)^2}{4}u$, then $f(z) = f\left(\frac{z_1+z_2\pm\sqrt{1-u}}{2}\right)$ and $f(z_1+z_2-z) = f\left(\frac{z_1+z_2\mp\sqrt{1-u}}{2}\right)$. Consequently, if $f(z)$ is analytic, the function $G(z)$ can be written as a power series of $(z-z_1)(z-z_2)$. Indeed, we have

$$G(z_1, z_2; z) = f(z) + f(z_1 + z_2 - z) = \sum_i f_i\left\{z^i + (z_1 + z_2 - z)^i\right\}$$

and we can easily demonstrate by mathematical induction on coefficients $a_{i,k}$ that

$$z^i + (z_1 + z_2 - z)^i = \sum_{k=0}^{[\frac{i}{2}]} a_{i,k}(z_1, z_2)(z-z_1)^k(z-z_2)^k$$

with

$$a_{i,k} = \frac{1}{i!}\frac{1}{k!}\frac{\partial^i}{\partial x^i}\frac{\partial^i}{\partial y^k}\left\{\frac{2 - x(z_1+z_2)}{(1-xz_1)(1-xz_2) - x^2 y}\right\}\Bigg|_{(x,y)=(0,0)}.$$

Also the function $\theta(z) = (z-z_1)(z-z_2)q\{(z-z_1)(z-z_2)\}$ with $q(0) \neq 0$ is analytic. Using the fact that $\theta'(z) = (2z - z_1 - z_2)\{q\{(z-z_1)(z-z_2)\} + (z-z_1)(z-z_2)q'\{(z-z_1)(z-z_2)\}\}$, we can find a series expansion in the following form

$$\left[\frac{\theta(z)}{(z-z_2)(z-z_1)}\right]^{-an-\gamma-1}\theta'(z)G(\theta(z)) = (2z - z_1 - z_2)\sum_s c_s(z-z_1)^s(z-z_2)^s. \tag{4.10}$$

From the definition of the fractional derivative with Pochhammer'contour (3.1), the left side of (4.8) using (4.10) implies

$$\begin{aligned}
&\int_{P(c)} G(\theta(\xi))\left[\frac{\theta(\xi)}{(\xi-z_1)(\xi-z_2)}\right]^{-an-\gamma-1}\theta'(\xi)(\xi-z_1)^{\alpha-an-\gamma-1}(\xi-z_2)^{\beta-an-\gamma-1}d\xi \\
= \;& \sum_s c_s \int_{P(c)} (2\xi - z_1 - z_2)(\xi-z_1)^{\alpha-an-\gamma+s-1}(\xi-z_2)^{\beta-an-\gamma+s-1}d\xi \\
= \;& 4\sum_s c_s \frac{(\beta-\alpha)\pi^2 e^{i\pi(\beta-an-\gamma)}(z_1-z_2)^{\alpha+\beta-2an-2\gamma}(-1)^s(z_1-z_2)^{2s}}{\Gamma(1-\beta+an+\gamma-s)\Gamma(1-\alpha+an+\gamma-s)\Gamma(1+\alpha+\beta-2an-2\gamma+2s)}
\end{aligned}$$

which is zero if $\alpha = \beta$. This completes the proof.

Remark 4.4. The symmetry of formula (4.1) with respect to the parameters α and β and to the variables z and z_1+z_2-z allows interchanging in this formula the poles ξ_1 and ξ_2 for the evaluation of two contour integrals in Table 2. We then obtain a formula similar to (4.1) valid for $z \in C_2$ on C where Δ_k is replaced by $-\Delta_k$. Its addition to the equation (4.1) gives, after simplifications, the formula (3.1) valid for z in $z \in C(1) = C_1 \cup C_2$.

Corollary 4.5. *With the same assumptions of Theorem* 3.2*, we have*

$$
\begin{aligned}
& e^{i(\alpha+a-\gamma)\pi}\sin([\beta-\alpha]\pi)\sum_{k\in K}a^{-1}\omega^{-\gamma k}\left[f\left(\frac{z_1+z_2+\sqrt{\Delta_k}}{2}\right)+f\left(\frac{z_1+z_2-\sqrt{\Delta_k}}{2}\right)\right] \qquad (4.11)\\
& \times\left[\left(\frac{z_2-z_1+\sqrt{\Delta_k}}{2}\right)^{\alpha}\left(\frac{z_1-z_2+\sqrt{\Delta_k}}{2}\right)^{\beta}+\left(\frac{z_2-z_1-\sqrt{\Delta_k}}{2}\right)^{\alpha}\left(\frac{z_1-z_2-\sqrt{\Delta_k}}{2}\right)^{\beta}\right]\\
= & \sum_{-\infty}^{\infty}\theta(z)^{an+\gamma}e^{-i\pi a(n+1)}\\
& \times\left\{\sin([\beta-an-\gamma]\pi)\frac{D_{z-z_2}^{-\alpha+an+\gamma}}{\Gamma(1-\alpha+an+\gamma)}\left[(z-z_2)^{\beta-an-\gamma-1}\left(\frac{\theta(z)}{(z-z_2)(z-z_1)}\right)^{-an-\gamma-1}\theta'(z)[f(z)+f(z_1+z_2-z)]\right]\Bigg|_{z=z_1}\right.\\
& \left.-\sin([\alpha-an-\gamma]\pi)\frac{D_{z-z_2}^{-\beta+an+\gamma}}{\Gamma(1-\beta+an+\gamma)}\left[(z-z_2)^{\alpha-an-\gamma-1}\left(\frac{\theta(z)}{(z-z_2)(z-z_1)}\right)^{-an-\gamma-1}\theta'(z)[f(z)+f(z_1+z_2-z)]\right]\Bigg|_{z=z_1}\right\}
\end{aligned}
$$

valid for z on $C(1)=C_1\cup C_2$ or $\left|(z-z_1)(z-z_2)\right|=\left|(z_1-z_2)^2/4\right|$.

Next, we examine more deeply some important special cases of the formulas (4.1) and (4.11).

5. A generalization of a Ramanujan's formula

If $0<a\leq 1$ and if we put $q(z)=1$ in (4.9), then $K=\{0\}$, $\theta(z)=(z-z_1)(z-z_2)$, $\theta'(z)=2z-z_1-z_2$ and $\Delta_0=(2z-z_1-z_2)^2$. Also

$$
\begin{aligned}
f\left(\frac{z_1+z_2+\sqrt{\Delta_k}}{2}\right)&=f(z), & f\left(\frac{z_1+z_2-\sqrt{\Delta_k}}{2}\right)&=f(z_1+z_2-z)\\
\frac{z_2-z_1+\sqrt{\Delta_k}}{2}&=z-z_1 & \frac{z_2-z_1-\sqrt{\Delta_k}}{2}&=z_2-z\\
\frac{z_1-z_2+\sqrt{\Delta_k}}{2}&=z-z_2 & \frac{z_1-z_2-\sqrt{\Delta_k}}{2}&=z_1-z
\end{aligned}
$$

and we obtain

$$
\begin{aligned}
& e^{i\pi(\alpha+a-\gamma)}\sin[(\beta-\alpha)\pi]\{f(z)+f(z_1+z_2-z)\}\left[(z-z_1)^{\alpha}(z-z_2)^{\beta}+(z_2-z)^{\alpha}(z_1-z)^{\beta}\right]\\
= \quad & a\sum_{-\infty}^{\infty}e^{-i\pi a(n+1)}[(z-z_1)(z-z_2)]^{an+\gamma}\Big\{\sin[(\beta-an-\gamma)\pi]\\
& \times\frac{D_{z-z_2}^{-\alpha+an+\gamma}}{\Gamma(1-\alpha+an+\gamma)}\left[(z-z_2)^{\beta-an-\gamma-1}(2z-z_1-z_2)\{f(z)+f(z_1+z_2-z)\}\right]\Bigg|_{z=z_1}-\sin[(\alpha-an-\gamma)\pi]\\
& \times\frac{D_{z-z_2}^{-\beta+an+\gamma}}{\Gamma(1-\beta+an+\gamma)}\left[(z-z_2)^{\alpha-an-\gamma-1}(2z-z_1-z_2)\{f(z)+f(z_1+z_2-z)\}\right]\Bigg|_{z=z_1}\Big\}
\end{aligned}
$$

valid for $\left|(z-z_1)(z-z_2)\right|=\left|(z_1-z_2)^2/4\right|$.
If $f(z)=1$, using the fact that

$$
\frac{D_{z-z_2}^{-\alpha+an+\gamma}}{\Gamma(1-\alpha+an+\gamma)}\left[(z-z_2)^{\beta-an-\gamma-1}(2z-z_1-z_2)\right]\Bigg|_{z=z_1}=\frac{(z_1-z_2)^{\alpha+\beta-2an-2\gamma}(\beta-\alpha)\Gamma(\beta-an-\gamma)}{\Gamma(1-\alpha+an+\gamma)\Gamma(1+\alpha+\beta-2an-2\gamma)}
$$

and using the well-known formulas for the gamma function (*cf.* [29, Eq. (2), p. 24 and Theorem 8, p. 21])

$$
\sqrt{\pi}\,\Gamma(2z)=2^{2z-1}\Gamma(z)\Gamma(1/2+z),\ \ \Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}
$$

with $\gamma = 0$ (or with the equivalent choose α by $\alpha + \gamma$, β by $\beta + \gamma$), we obtain after some simplifications

$$\begin{aligned} & a \sum_{-\infty}^{\infty} \frac{\left[\frac{-4(z-z_1)(z-z_2)}{(z_1-z_2)^2}\right]^{an}}{\Gamma(1-\alpha+an)\Gamma(1-\beta+an)\Gamma(\frac{1}{2}+\frac{\alpha+\beta}{2}-an)\Gamma(1+\frac{\alpha+\beta}{2}-an)} \\ = \;& \frac{e^{i\pi(\alpha+2a)}\sin(\beta-\alpha)\pi}{2\pi^{3/2}(\beta-\alpha)}\left[\frac{z_1-z_2}{2}\right]^{-\alpha-\beta}\left[(z-z_1)^{\alpha}(z-z_2)^{\beta}+(z_2-z)^{\alpha}(z_1-z)^{\beta}\right] \end{aligned} \tag{5.1}$$

and if $a \to 0^+$, we obtain the integral analog

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{\left[\frac{-4(z-z_1)(z-z_2)}{(z_1-z_2)^2}\right]^{\omega} d\omega}{\Gamma(1-\alpha+\omega)\Gamma(1-\beta+\omega)\Gamma(\frac{1}{2}+\frac{\alpha+\beta}{2}-\omega)\Gamma(1+\frac{\alpha+\beta}{2}-\omega)} \\ = \;& \frac{e^{i\pi\alpha}\sin(\beta-\alpha)\pi}{2\pi^{3/2}(\beta-\alpha)}\left[\frac{z_1-z_2}{2}\right]^{-\alpha-\beta}\left[(z-z_1)^{\alpha}(z-z_2)^{\beta}+(z_2-z)^{\alpha}(z_1-z)^{\beta}\right]. \end{aligned} \tag{5.2}$$

We can write (5.2) in the following form

$$\int_{-\infty}^{\infty} \frac{\left[\left(\frac{z_1-z}{z_1-z_2}\right)\left(\frac{z-z_2}{z_1-z_2}\right)\right]^{\omega} d\omega}{\Gamma(1-\alpha+\omega)\Gamma(1-\beta+\omega)\Gamma(1+\alpha+\beta-2\omega)} = \frac{\sin(\beta-\alpha)\pi}{2\pi(\beta-\alpha)}\left[\left(\frac{z_1-z}{z_1-z_2}\right)^{\alpha}\left(\frac{z-z_2}{z_1-z_2}\right)^{\beta}+\left(\frac{z-z_2}{z_1-z_2}\right)^{\alpha}\left(\frac{z_1-z}{z_1-z_2}\right)^{\beta}\right],$$

where $0 < arg\left(\frac{-4(z-z_1)(z-z_2)}{(z_1-z_2)^2}\right) < 2\pi$ with $\left|\frac{-4(z-z_1)(z-z_2)}{(z_1-z_2)^2}\right| = 1$.
These formulas seem to be new to the author.
If we put $-\frac{4(z-z_1)(z-z_2)}{(z_1-z_2)^2} = e^{i\theta}$ in (4.1) with $0 < \theta < 2\pi$, then $z = \frac{(z_1+z_2) \pm (z_1-z_2)\sqrt{1-e^{i\theta}}}{2}$ and (for $\pm$)

$$\begin{aligned} & \left[\frac{z_1-z_2}{2}\right]^{-\alpha-\beta}\left[(z-z_1)^{\alpha}(z-z_2)^{\beta}+(z_2-z)^{\alpha}(z_1-z)^{\beta}\right] \\ = \;& \left(\sqrt{1-e^{i\theta}}-1\right)^{\alpha}\left(\sqrt{1-e^{i\theta}}+1\right)^{\beta}+e^{i\pi(\beta-\alpha)}\left(\sqrt{1-e^{i\theta}}+1\right)^{\alpha}\left(\sqrt{1-e^{i\theta}}-1\right)^{\beta}. \end{aligned} \tag{5.3}$$

The equation (5.2) becomes

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{e^{i\theta\omega} d\omega}{\Gamma(1-\alpha+\omega)\Gamma(1-\beta+\omega)\Gamma\left(\frac{1}{2}+\frac{\alpha+\beta}{2}-\omega\right)\Gamma\left(1+\frac{\alpha+\beta}{2}-\omega\right)} \\ = \;& \frac{e^{i\pi\alpha}\sin(\beta-\alpha)\pi}{2\pi^{3/2}(\beta-\alpha)}\left[\left(\sqrt{1-e^{i\theta}}-1\right)^{\alpha}\left(\sqrt{1-e^{i\theta}}+1\right)^{\beta}+e^{i\pi(\beta-\alpha)}\left(\sqrt{1-e^{i\theta}}+1\right)^{\alpha}\left(\sqrt{1-e^{i\theta}}-1\right)^{\beta}\right], \end{aligned} \tag{5.4}$$

where $0 \leq \theta < 2\pi$.
If $\theta = 0$, using the fact that $e^{-i\alpha} = (-1)^{\alpha}$ and $1^{\alpha} = 1$, we obtain

$$\int_{-\infty}^{\infty} \frac{d\omega}{\Gamma(1-\alpha+\omega)\Gamma(1-\beta+\omega)\Gamma\left(\frac{1}{2}+\frac{\alpha+\beta}{2}-\omega\right)\Gamma\left(1+\frac{\alpha+\beta}{2}-\omega\right)} = \frac{\sin(\beta-\alpha)\pi}{\pi^{3/2}(\beta-\alpha)}.$$

If $\beta \to \alpha$, we obtain

$$\int_{-\infty}^{\infty} \frac{d\omega}{\Gamma(1-\alpha+\omega)\Gamma(1-\alpha+\omega)\Gamma\left(\frac{1}{2}+\alpha-\omega\right)\Gamma(1+\alpha-\omega)} = \frac{1}{\pi^{3/2}}.$$

If $\beta = \alpha + n$ with $n = \pm 1, \pm 2, \ldots$ then

$$\int_{-\infty}^{\infty} \frac{d\omega}{\Gamma(1-\alpha+\omega)\Gamma(1-\alpha-n+\omega)\Gamma\left(\frac{1}{2}+\alpha+\frac{n}{2}-\omega\right)\Gamma\left(1+\alpha+\frac{n}{2}-\omega\right)} = 0.$$

If $\theta = \pi$ in (5.4), we obtain

$$\int_{-\infty}^{\infty} \frac{e^{i\pi\omega} d\omega}{\Gamma(1-\alpha+\omega)\Gamma(1-\beta+\omega)\Gamma\left(\frac{1}{2}+\frac{\alpha+\beta}{2}-\omega\right)\Gamma\left(1+\frac{\alpha+\beta}{2}-\omega\right)} \tag{5.5}$$
$$= \frac{e^{i\pi\alpha}\sin(\beta-\alpha)\pi}{2\pi^{3/2}(\beta-\alpha)}\left[\left(\sqrt{2}-1\right)^{\alpha}\left(\sqrt{2}+1\right)^{\beta} + e^{i\pi(\beta-\alpha)}\left(\sqrt{2}+1\right)^{\alpha}\left(\sqrt{2}-1\right)^{\beta}\right].$$

Using the fact that $\sqrt{(\sqrt{2}+1)} = 2^{\frac{3}{4}}\ \cos\left(\frac{\pi}{8}\right)$ and $\sqrt{(\sqrt{2}-1)} = 2^{\frac{3}{4}}\ \sin\left(\frac{\pi}{8}\right)$, we have

$$\int_{-\infty}^{\infty} \frac{e^{i\pi\omega} d\omega}{\Gamma(1-\alpha+\omega)\Gamma(1-\beta+\omega)\Gamma(\frac{1}{2}+\frac{\alpha+\beta}{2}-\omega)\Gamma(1+\frac{\alpha+\beta}{2}-\omega)}$$
$$= \frac{e^{i\pi\alpha}2^{\frac{3}{2}(\alpha+\beta)-1}\sin(\beta-\alpha)\pi}{\pi^{3/2}(\beta-\alpha)}\left[\left(\sin\left(\frac{\pi}{8}\right)\right)^{2\alpha}\left(\cos\left(\frac{\pi}{8}\right)\right)^{2\beta} + e^{i\pi(\beta-\alpha)}\left(\cos\left(\frac{\pi}{8}\right)\right)^{2\alpha}\left(\sin\left(\frac{\pi}{8}\right)\right)^{2\beta}\right].$$

The integral (5.5) is a generalized version $\left(\text{with } d = b+\frac{1}{2} = 1+\frac{\alpha+\beta}{2}\right)$ of the following integral (*cf.* [10, Eq. (25), p. 301]) due at Ramanujan (*cf.* [31])

$$\int_{-\infty}^{\infty} \frac{\phi(x)dx}{\Gamma(a+x)\Gamma(b-x)\Gamma(c+x)\Gamma(d-x)} = \frac{\int_0^1 \phi(t)\cos\left[\frac{1}{2}\pi(2t+a-b)\right]dt}{\Gamma\left(\frac{a+b}{2}\right)\Gamma\left(\frac{c+d}{2}\right)\Gamma(a+d-1)} \tag{5.6}$$

with $a+d = b+c$, $Re(a+b+c+d) > 2$ and $\phi(x+1) = -\phi(x)$.

Equation (5.6) allows us to verify the special case $\beta = \alpha + \frac{1}{2}$ of (4.7). We then have $Re(a+b+c+d) = \frac{7}{2} > 2$, $a+d = b+c = \frac{7}{4}$, $\frac{\alpha+\beta}{2} = \alpha+\frac{1}{4}$ and (5.5) becomes

$$\int_{-\infty}^{\infty} \frac{e^{i\pi\omega} d\omega}{\Gamma(1-\alpha+\omega)\Gamma(5/4+\alpha-\omega)\Gamma(\frac{1}{2}-\alpha+\omega)\Gamma(3/4+\alpha-\omega)}$$
$$= \frac{e^{i\pi\alpha}2^{3\alpha-\frac{1}{4}}\sin(\pi/2)}{\pi^{3/2}(1/2)}$$
$$\times\left[\left(\sin\left(\frac{\pi}{8}\right)\right)^{2\alpha}\left(\cos\left(\frac{\pi}{8}\right)\right)^{2\alpha+1} + e^{i\pi/2}\left(\cos\left(\frac{\pi}{8}\right)\right)^{2\alpha}\left(\sin\left(\frac{\pi}{8}\right)\right)^{2\alpha+1}\right]$$
$$= \frac{e^{i\pi\alpha}2^{3\alpha+\frac{3}{4}}}{\pi^{3/2}}\left(\sin\left(\frac{\pi}{8}\right)\right)^{2\alpha}\left(\cos\left(\frac{\pi}{8}\right)\right)^{2\alpha}\left[\cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right)\right]$$
$$= \frac{e^{i\pi\alpha}2^{3\alpha+\frac{3}{4}}}{\pi^{3/2}}\left(\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)\right)^{2\alpha} e^{i\frac{\pi}{8}} = \frac{e^{i\pi(\alpha+\frac{1}{8})}2^{3\alpha+\frac{3}{4}}}{\pi^{3/2}}\left(\frac{\sin(\frac{\pi}{4})}{2}\right)^{2\alpha}$$
$$= \frac{e^{i\pi(\alpha+\frac{1}{8})}2^{3\alpha+\frac{3}{4}}}{\pi^{3/2}}\left(\frac{1}{2\sqrt{2}}\right)^{2\alpha} = \frac{e^{i\pi(\alpha+\frac{1}{8})}2^{\frac{3}{4}}}{\pi^{3/2}}.$$

Now, from (5.6), if we put $\phi(t) = e^{i\pi t}$, $a = 1-\alpha$, $b = \frac{5}{4}+\alpha$, $c = \frac{1}{2}-\alpha$ and $d = \frac{3}{4}+\alpha$, we have $a+d = b+c = \frac{7}{4}$ and

$$\int_0^1 \phi(t)\cos\left[\frac{1}{2}\pi(2t-a-b)\right]dt = \int_0^1 e^{i\pi t}\cos\left(t-\alpha-\frac{1}{8}\right)\pi\, dt$$
$$= \cos\left(\alpha+\frac{1}{8}\right)\pi\int_0^1 e^{i\pi t}\cos\pi t\, dt + \sin\left(\alpha+\frac{1}{8}\right)\pi\int_0^1 e^{i\pi t}\sin\pi t\, dt$$
$$= \frac{1}{2}\cos\left(\alpha+\frac{1}{8}\right)\pi + \frac{i}{2}\sin\left(\alpha+\frac{1}{8}\right)\pi = \frac{1}{2}e^{i\pi(\alpha+\frac{1}{8})}$$

and

$$\frac{\int_0^1 \phi(t)\cos\left[\frac{1}{2}\pi(2t+a-b)\right]dt}{\Gamma(\frac{a+b}{2})\Gamma(\frac{c+d}{2})\Gamma(a+d-1)} = \frac{1}{2}e^{i\pi(\alpha+\frac{1}{8})}\frac{1}{\Gamma\left(\frac{9}{8}\right)\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{3}{4}\right)}$$
$$= \frac{e^{i\pi(\alpha+\frac{1}{8})}}{2^{\frac{3}{4}}\sqrt{\pi}\,\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{4}\right)} = \frac{e^{i\pi(\alpha+\frac{1}{8})}}{2^{\frac{1}{4}}\pi\Gamma\left(\frac{3}{2}\right)} = \frac{2^{\frac{3}{4}}e^{i\pi\left(\alpha+\frac{1}{8}\right)}}{\pi^{\frac{3}{2}}}.$$

The equality is proved.
We can add another special case of equation (5.5). Posing $\beta = \alpha - \frac{1}{2}$, we find after simplifications

$$\int_{-\infty}^{\infty}\frac{e^{i\pi\omega}d\omega}{\Gamma(1-\alpha+\omega)\Gamma(1/4+\alpha-\omega)\Gamma(\frac{3}{2}-\alpha+\omega)\Gamma(3/4+\alpha-\omega)} = -ie^{i\pi(\alpha-\frac{3}{8})}2^{\frac{3}{4}}\pi^{-\frac{3}{2}},$$

which is conform to the Ramanujan's result (*cf.* [30, Eq. (7.2), p. 227]). Note that if $\theta \to 0^+$, the term $\sqrt{\sqrt{1-e^{i\theta}}-1} \to -i$ and (5.5) gives with $\beta = \alpha + \frac{1}{2}$

$$\int_{-\infty}^{\infty}\frac{d\omega}{\Gamma(1-\alpha+\gamma+\omega)\Gamma\left(\frac{1}{2}-\alpha+\gamma+\omega\right)\Gamma\left(\frac{3}{4}+\alpha-\gamma-\omega\right)\Gamma\left(\frac{5}{4}+\alpha-\gamma-\omega\right)} = \frac{2}{\pi^{3/2}}$$

which is conform to (*cf.* [10, Eq. (21), p. 300]).
If $\theta \to 2\pi^-$, the term $\sqrt{\sqrt{1-e^{i\theta}}-1} \to i$ and (4.5) gives with $\beta = \alpha + \frac{1}{2}$

$$\int_{-\infty}^{\infty}\frac{e^{2\pi\omega}d\omega}{\Gamma(1-\alpha+\omega)\Gamma\left(\frac{1}{2}-\alpha+\omega\right)\Gamma\left(\frac{3}{4}+\alpha-\omega\right)\Gamma\left(\frac{5}{4}+\alpha-\omega\right)} = 0.$$

The case $\beta = \alpha - \frac{1}{2}$ also gives

$$\int_{-\infty}^{\infty}\frac{e^{i\pi\omega}d\omega}{\Gamma(1-\alpha+\omega)\Gamma(1/4+\alpha-\omega)\Gamma\left(\frac{3}{2}-\alpha+\omega\right)\Gamma(3/4+\alpha-\omega)} = 0$$

which is conform to the Ramanujan's result (*cf.* [31, Eq. (6.2), p. 226]).

6. Some Fourier's integral type

From (4.1) with $0 < a \le 1$, $q(z) = 1$, $\gamma = 0$ then $k = 0$, $\theta(z) = (z-z_1)(z-z_2)$, $\theta'(z) = 2z-z_1-z_2$, $\Delta_0 = (2z-z_1-z_2)^2$ and we obtain after simplifications

$$2\left[f(z)+f(z_1+z_2-z)\right]\left[\sin[(\beta+a)\pi](z-z_1)^{\alpha}(z-z_2)^{\beta} - e^{i\pi(\alpha-\beta)}\sin[(\alpha+a)\pi](z_2-z)^{\alpha}(z_1-z)^{\beta}\right]$$
$$= a\sum_{-\infty}^{\infty}[(z-z_1)(z-z_2)]^{an}e^{-i\pi a(n+1)}\Big\{\sin[(\beta-an)\pi]\frac{D_{z-z_2}^{-\alpha+an}}{\Gamma(1-\alpha+an)}\Big[(z-z_2)^{\beta-an-1}(2z-z_1-z_2)(f(z)+f(z_1+z_2-z))\Big]\Big|_{z=z_1}$$
$$-\sin[(\alpha-an)\pi]\frac{D_{z-z_2}^{-\beta+an}}{\Gamma(1-\beta+an)}\Big[(z-z_2)^{\alpha-an-1}(2z-z_1-z_2)(f(z)+f(z_1+z_2-z))\Big]\Big|_{z=z_1}\Big\}$$

which is valid for $\left|(z-z_1)(z-z_2)\right| = \left|(z_1-z_2)^2/4\right|$. Putting $(z-z_1)(z-z_2) = e^{i\theta}(z_1-z_2)^2/4$ or $z = \dfrac{z_1+z_2+(z_1-z_2)\sqrt{1+e^{i\theta}}}{2}$, we obtain

$$f(z)+f(z_1+z_2-z) = f\left(\frac{z_1+z_2+(z_1-z_2)\sqrt{1+e^{i\theta}}}{2}\right) + f\left(\frac{z_1+z_2-(z_1-z_2)\sqrt{1+e^{i\theta}}}{2}\right) = g\left(e^{i\theta}\right).$$

$$
\begin{aligned}
& 2g\left(e^{i\theta}\right)\left(\frac{z_1-z_1}{2}\right)^{\alpha+\beta}\Big[\sin[(\beta+a)\pi]\left(-1+\sqrt{1+e^{i\theta}}\right)^{\alpha}\left(1+\sqrt{1+e^{i\theta}}\right)^{\beta} \\
& -e^{i\pi(\alpha-\beta)}\sin[(\alpha+a)\pi]\left(-1-\sqrt{1+e^{i\theta}}\right)^{\alpha}\left(1-\sqrt{1+e^{i\theta}}\right)^{\beta}\Big] \\
= \;& ae^{-i\pi a}\sum_{-\infty}^{\infty}\left(\frac{z_1-z_1}{2}\right)^{2an}[(z-z_1)(z-z_2)]^{an}e^{-ian} \\
& \times\Big\{\sin[(\beta-an)\pi]\frac{D_{z-z_2}^{-\alpha+an}}{\Gamma(1-\alpha+an)}\Big[(z-z_2)^{\beta-an-1}(2z-z_1-z_2)(f(z)+f(z_1+z_2-z))\Big]\Big|_{z=z_1} \\
& -\sin[(\alpha-an)\pi]\frac{D_{z-z_2}^{-\beta+an}}{\Gamma(1-\beta+an)}\Big[(z-z_2)^{\alpha-an-1}(2z-z_1-z_2)(f(z)+f(z_1+z_2-z))\Big]\Big|_{z=z_1}\Big\}.
\end{aligned}
$$

If $a\to 0^+$, we obtain the analog form

$$
\begin{aligned}
& 2\left[f\left(\frac{z_1+z_2+(z_1-z_2)\sqrt{1+e^{i\theta}}}{2}\right)+f\left(\frac{z_1+z_2-(z_1-z_2)\sqrt{1+e^{i\theta}}}{2}\right)\right] \\
& \times\left(\frac{z_1-z_1}{2}\right)^{\alpha+\beta}[\sin[\beta\pi]\left(-1+\sqrt{1+e^{i\theta}}\right)^{\alpha}\left(1+\sqrt{1+e^{i\theta}}\right)^{\beta} \\
& -e^{i\pi(\alpha-\beta)}\sin[\alpha\pi]\left(-1-\sqrt{1+e^{i\theta}}\right)^{\alpha}\left(1-\sqrt{1+e^{i\theta}}\right)^{\beta} \\
= \;& \int_{-\infty}^{\infty}\Big\{\sin[(\beta-\omega)\pi]\frac{D_{z-z_2}^{-\alpha+\omega}}{\Gamma(1-\alpha+\omega)}\Big[(z-z_2)^{\beta-\omega-1}(2z-z_1-z_2)(f(z)+f(z_1+z_2-z))\Big]\Big|_{z=z_1} \\
& -\sin[(\alpha-\omega)\pi]\frac{D_{z-z_2}^{-\beta+\omega}}{\Gamma(1-\beta+\omega)}\Big[(z-z_2)^{\alpha-\omega-1}(2z-z_1-z_2)(f(z)+f(z_1+z_2-z))\Big]\Big|_{z=z_1}\Big\}e^{i\theta\omega}d\omega.
\end{aligned}
$$

With the application of the Fourier's theory, we can write

$$
\begin{aligned}
& \left(\frac{z_1-z_1}{2}\right)^{-\alpha-\beta+2\omega}\Big\{\sin[(\beta-\omega)\pi]\frac{D_{z-z_2}^{-\alpha+\omega}}{\Gamma(1-\alpha+\omega)}\Big[(z-z_2)^{\beta-\omega-1}(2z-z_1-z_2)\,g\Big(-4\frac{(z-z_1)(z-z_2)}{(z_1-z_2)^2}\Big)\Big]\Big|_{z=z_1} \\
& -\sin[(\alpha-\omega)\pi]\frac{D_{z-z_2}^{-\beta+\omega}}{\Gamma(1-\beta+\omega)}\Big[(z-z_2)^{\alpha-\omega-1}(2z-z_1-z_2)\,g\Big(-4\frac{(z-z_1)(z-z_2)}{(z_1-z_2)^2}\Big)\Big]\Big|_{z=z_1}\Big\} \\
= \;& \frac{1}{\pi}\int_{-\pi}^{\pi}g(e^{i\theta})\Big\{\sin[\beta\pi]\left(-1+\sqrt{1+e^{i\theta}}\right)^{\alpha}\left(1+\sqrt{1+e^{i\theta}}\right)^{\beta} \\
& -e^{i\pi(\alpha-\beta)}\sin[\alpha\pi]\left(-1-\sqrt{1+e^{i\theta}}\right)^{\alpha}\left(1-\sqrt{1+e^{i\theta}}\right)^{\beta}\Big\}e^{-i\omega\theta}d\theta.
\end{aligned}
\tag{6.1}
$$

If $g(z)=1$ and using the fact that

$$
D_{z-z_2}^{-\alpha+\omega}[(z-z_2)^{\beta-\omega-1}(2z-z_1-z_2)]\Big|_{z=z_1}=\frac{(\beta-\alpha)\Gamma(\beta-\omega)}{\Gamma(1+\alpha+\beta-2\omega)}(z_1-z_2)^{\alpha+\beta-2\omega},
$$

we obtain from (6.1)

$$
\begin{aligned}
& \int_{-\pi}^{\pi}\Big\{\sin[(\beta-\omega)\pi]\left(-1+\sqrt{1+e^{i\theta}}\right)^{\alpha}\left(1+\sqrt{1+e^{i\theta}}\right)^{\beta} \\
& -e^{i\pi(\alpha-\beta)}\sin[(\alpha-\omega)\pi]\left(-1-\sqrt{1+e^{i\theta}}\right)^{\alpha}\left(1-\sqrt{1+e^{i\theta}}\right)^{\beta}\Big\}e^{-i\omega\theta}d\theta \\
= \;& \frac{2^{1+\alpha+\beta-2\omega}\pi^2(\beta-\alpha)}{\Gamma(1-\alpha+\omega)\Gamma(1-\beta+\omega)\Gamma(1+\alpha+\beta-2\omega)}.
\end{aligned}
\tag{6.2}
$$

If $\beta=-\alpha$ in (6.2), we obtain after simplifications the following integral

$$
\int_{-\pi}^{\pi}\left\{\left(\frac{\sqrt{1+e^{i\theta}}-1}{\sqrt{1+e^{i\theta}}+1}\right)^{\alpha}+\left(\frac{\sqrt{1+e^{i\theta}}+1}{\sqrt{1+e^{i\theta}}-1}\right)^{\alpha}\right\}e^{-i\omega\theta}d\theta=\frac{2^{2-2\omega}\Gamma(1+\alpha)\Gamma(1-\alpha)\pi}{\Gamma(1-2\omega)\Gamma(1+\alpha+\omega)\Gamma(1-\alpha+\omega)}.
\tag{6.3}
$$

If ω is a positive integer, the integral (6.3) is zero. If $\omega = -n$ (a negative integer), the integral takes the form

$$\int_{-\pi}^{\pi}\left\{\left(\frac{\sqrt{1+e^{i\theta}}-1}{\sqrt{+e^{i\theta}}+1}\right)^{\alpha}+\left(\frac{\sqrt{1+e^{i\theta}}+1}{\sqrt{1+e^{i\theta}}-1}\right)^{\alpha}\right\}e^{in\theta}d\theta = (\alpha)_n(-\alpha)_n\frac{2^{2n+2}}{(2n)!}\pi.$$

If $\omega = 0$, the integral is constant for all α and we get

$$\int_{-\pi}^{\pi}\left\{\left(\frac{\sqrt{1+e^{i\theta}}-1}{\sqrt{1+e^{i\theta}}+1}\right)^{\alpha}+\left(\frac{\sqrt{1+e^{i\theta}}+1}{\sqrt{1+e^{i\theta}}-1}\right)^{\alpha}\right\}d\theta = 4\pi.$$

In addition, if $\beta \to \alpha$ in (5.1), we find

$$\begin{aligned}
&\int_{-\pi}^{\pi}\Big\{\pi\cos[(\alpha-\omega)\pi]\left(-1+\sqrt{1+e^{i\theta}}\right)^{\alpha}\left(1+\sqrt{1+e^{i\theta}}\right)^{\alpha}\\
&+\sin[(\alpha-\omega)\pi]\left(-1+\sqrt{1+e^{i\theta}}\right)^{\alpha}(1+\sqrt{1+e^{i\theta}})^{\alpha}\ln\left(\frac{\sqrt{1+e^{i\theta}}+1}{\sqrt{1+e^{i\theta}}-1}\right)\Big\}e^{-i\omega\theta}d\theta\\
=\ &\frac{\pi\sin(2(\alpha-\omega)\pi)}{\alpha-\omega}+\sin((\alpha-\omega)\pi)\int_{-\pi}^{\pi}\ln\left(\frac{\sqrt{1+e^{i\theta}}+1}{\sqrt{1+e^{i\theta}}-1}\right)e^{i(\alpha-\omega)\theta}d\theta\\
=\ &\frac{2^{1+2\alpha-2\omega}\pi^2}{\Gamma(1-\alpha+\omega)\Gamma(1-\alpha+\omega)\Gamma(1+2\alpha-2\omega)}
\end{aligned}$$

and if $\omega = 0$, we get

$$\begin{aligned}
&\int_{-\pi}^{\pi}\Big\{\pi\cos[\alpha\pi]\left(-1+\sqrt{1+e^{i\theta}}\right)^{\alpha}\left(1+\sqrt{1+e^{i\theta}}\right)^{\alpha}\\
&+\sin[\alpha\pi]\left(-1+\sqrt{1+e^{i\theta}}\right)^{\alpha}\left(1+\sqrt{1+e^{i\theta}}\right)^{\alpha}\ln\left(\frac{\sqrt{1+e^{i\theta}}+1}{\sqrt{1+e^{i\theta}}-1}\right)\Big\}d\theta\\
=\ &\frac{\pi\sin(2\alpha\pi)}{\alpha}+\sin(\alpha\pi)\int_{-\pi}^{\pi}\ln\left(\frac{\sqrt{1+e^{i\theta}}+1}{\sqrt{1+e^{i\theta}}-1}\right)e^{i\alpha\theta}d\theta\\
=\ &\frac{2^{1+2\alpha}\pi^2}{\Gamma(1-\alpha)\Gamma(1-\alpha)\Gamma(1+2\alpha)}.
\end{aligned}$$

Also, we deduce that

$$\frac{\alpha}{2\pi}\int_{-\pi}^{\pi}\ln\left(\frac{\sqrt{1+e^{i\theta}}+1}{\sqrt{1+e^{i\theta}}-1}\right)e^{i\alpha\theta}d\theta = \frac{\sqrt{\pi}}{\Gamma(1-\alpha)\Gamma\left(\frac{1}{2}+\alpha\right)}-\cos(\alpha\pi),$$

$$\frac{\alpha-\omega}{2\pi}\int_{-\pi}^{\pi}\ln\left(\frac{\sqrt{1+e^{i\theta}}+1}{\sqrt{1+e^{i\theta}}-1}\right)e^{i(\alpha-\omega)\theta}d\theta = \frac{\sqrt{\pi}}{\Gamma(1-\alpha+\omega)\Gamma\left(\frac{1}{2}+\alpha-\omega\right)}-\cos((\alpha-\omega)\pi)$$

which seems to be new. If $\beta = \omega = 0$ in (6.2), we obtain after simplifications

$$\int_{-\pi}^{\pi}\left(1+\sqrt{1+e^{i\theta}}\right)^{\alpha} = 2^{\alpha+1}\pi.$$

Many other interesting special cases do exist.

7. Conclusion

In this article, the usefulness of the fractional derivative represented using a Pochhammer integral has been clearly demonstrated. This type of representation constitutes a powerful tool for studying and obtaining new results. The variety of applications and results obtained confirm this. Most of the results obtained involving special functions are new to the author, in particular Ramanujan and Fourier type integrals. Furthermore, this approach is an efficient way to obtain a large and unified class of results involving classical functions of mathematical physics. In future work, results involving Ramanujan and Fourier types will be studied in more detail.

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