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(p,q) type integral operators and generalized Mittag-Leffler function

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Abstract

Recently, a large number of beta type integral operators and their extensions have been developed and explored. This activity has been prompted by the significance of these operators as well as the possible uses they may have in a range of study domains. We start a new class of extended (p,q) type integral operators employing the generalized Mittag-Leffler function that was described by Khan et al. [6]. In addition, our findings are coherent in character and may be interpreted as fundamental equations, from which we have also derived a number of special cases.

Keywords: Generalized Wright function, Fox-Wright function, generalized hypergeometric function, Euler type integral, extended beta function, Mittag-Leffler function

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1. Introduction

The special functions of the mathematical sciences have been discovered to be of great assistance in solving the initial value problems (IVP) and the boundary value problems (BVP) that are connected to partial differential equations and fractional differential equations, respectively. Applications of special functions may be found in a variety of engineering sub fields. The Mittag-Leffler function (MLF) [9], was first described in connection with the technique of summation of some divergent series. The Mittag-Leffler function (MLF) is a mathematical construct that emerges as a solution to equations of fractional order, either differential or integral. In recent years, a number of researchers, including Choi et al. [3], Khan and Ahmed [6], and Khan et al. [7], have developed some interesting and useful integral operators involving various types of special functions. These integral operators are useful in a variety of fields of physics and engineering sciences. We develop a new class of (p,q) beta type integral operator that include the generalized Mittag-Leffler function. We have mentioned a number of previously discovered as well as newly discovered results as a major findings of our main result. We derived a mixed generating function involving the product of generalized Mittag-Leffler function, which are partly unilateral and partly bilateral. For the sake of this investigation, let's begin by reviewing the definitions of a few well-known functions, which are provided in the following section.

The Mittag-Leffler function [9] is defined as

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$$E_{\sigma}(x)=\sum_{r=0}^{\infty}\frac{x^r}{\Gamma(\sigma r+1)};\qquad (x\in\mathbb{C}) \tag{1.1}$$

where $\Gamma(.)$ is the usual Gamma function and $\sigma\in\mathbb{C}$, $\mathbb{R}(\sigma)\geq 0$.

The following is a novel generalization of $E_{\sigma}(x)$ that was created by A. Wiman [19] in 1905:

$$E_{\sigma,\mu}(x)=\sum_{r=0}^{\infty}\frac{x^r}{\Gamma(\sigma r+\mu)}, \tag{1.2}$$

where $\sigma,\mu,x\in\mathbb{C};\ \mathbb{R}(\sigma)>0, \mathbb{R}(\mu)>0$, which is known as Mittag-Leffler function or Wiman's function.

In 1971, Prabhakar [12] developed a new generalization of $E^{\gamma}_{\sigma,\mu}(x)$ in the form as

$$E^{\gamma}_{\sigma,\mu}(x)=\sum_{r=0}^{\infty}\frac{(\gamma)_r}{\Gamma(\sigma r+\mu)}\frac{x^r}{r!};\qquad (\gamma,\mu,\sigma,x\in\mathbb{C};\ \mathbb{R}(\sigma)>0),\ \mathbb{R}(\gamma)>0,\ \mathbb{R}(\mu)>0, \tag{1.3}$$

where $(\gamma)_r$ is known as Pochhammer symbol (*cf.* [14]).

In 2007, Shukla and Prajapati [18] provided a generalization of the Mittag-Leffler Function (MLF), as follows

$$E^{\gamma,b}_{\sigma,\mu}(x)=\sum_{r=0}^{\infty}\frac{(\gamma)_{br}}{\Gamma(\sigma r+\mu)}\frac{x^r}{r!}, \tag{1.4}$$

where $\gamma,\ \delta,\ \sigma,\ \mu,\ x\ \in\mathbb{C};\ \mathbb{R}(\sigma)>0, \mathbb{R}(\mu)>0, \mathbb{R}(\gamma)>0$ and $b\in(0,1)\cup\mathbb{N}$.

The function $E^{\gamma,b}_{\sigma,\mu}(x)$ introduced by Salim [16] in 2009, defined as:

$$E^{\gamma,\delta}_{\sigma,\mu}(x)=\sum_{r=0}^{\infty}\frac{(\gamma)_r\ x^r}{\Gamma(\sigma r+\mu)(\delta)_r}, \tag{1.5}$$

where $\sigma,\ \mu,\ \gamma,\ \delta,\ x\in\mathbb{C};\ \min\{\mathbb{R}(\sigma),\mathbb{R}(\mu),\mathbb{R}(\delta)\}>0$.

Salim and Faraj [17] provided a new generalization of the Mittag-Leffler function (MLF) in 2012 as follows

$$E^{\gamma,\delta,b}_{\sigma,\mu,a}(x)=\sum_{r=0}^{\infty}\frac{(\gamma)_{br}\ x^r}{\Gamma(\sigma r+\mu)(\delta)_{ar}}, \tag{1.6}$$

where $\sigma,\ \mu,\ \gamma,\ \delta,\ x\in\mathbb{C};\ \min\{\mathbb{R}(\sigma),\mathbb{R}(\mu),\mathbb{R}(\gamma),\mathbb{R}(\delta)\}>0; a,b>0, b\leq\mathbb{R}(\sigma)+a$.

In 2013, Khan and Ahmed [6] further introduced a new generalization of Mittag-Leffler function

$$E^{\gamma,\delta,b}_{\sigma,\mu}(x)=\sum_{r=0}^{\infty}\frac{(\gamma)_{br}\ x^r}{\Gamma(\sigma r+\mu)\ (\delta)_r}, \tag{1.7}$$

where $\sigma,\ \mu,\ \gamma,\ \delta\in\mathbb{C};\ \mathbb{R}(\sigma)>0, \mathbb{R}(\mu)>0, \mathbb{R}(\gamma)>0, \mathbb{R}(\delta)>0; b\in(0,1)\cup\mathbb{N}$.

Khan and Ahmed [6] in (2013) further generalized (1.7) in the following form

$$E^{\xi,\lambda,\gamma,b}_{\sigma,\mu,\nu,\phi,\delta,a}(x)=\sum_{r=0}^{\infty}\frac{(\xi)_{\lambda r}\ (\gamma)_{br}\ x^r}{\Gamma(\sigma r+\mu)(\nu)_{\phi r}(\delta)_{ar}}, \tag{1.8}$$

where $\xi,\lambda,\gamma,\sigma,\mu,\nu,\phi,\delta,x\in\mathbb{C}$; $\min\{\mathbb{R}(\sigma),\mathbb{R}(\mu),\mathbb{R}(\nu),\mathbb{R}(\phi),\mathbb{R}(\delta),\mathbb{R}(\xi),\mathbb{R}(\lambda),\mathbb{R}(\gamma)\}>0$; $a,b>0$ and $b\leq\mathbb{R}(\sigma)+a$.

The equation (1.8) is the most generalized of all the above formalization in (1.1)-(1.7). We have to put some particular values in (1.8) to obtain the following well known results:

1. On setting $a=1$ and $\xi=\nu$, $\lambda=\phi$ in (1.8), it becomes to (1.7) established by Khan and Ahmad [6].

2. On setting $\lambda=\phi$ and $\xi=\nu$ in (1.8), it becomes a special case (1.6) established by Salim and Faraj [17].

3. On setting $a = b = 1$ and $\xi = \nu$, $\lambda = \phi$ in (1.8), it becomes (1.5) established by Salim [16].

4. On setting $a = \delta = 1$ and $\xi = \nu$, $\lambda = \phi$ in (1.8), a special case (1.4) established by A. K. Shukla and Prajapati [18], in addition if $b = 1$, it becomes a special case (1.3) given by T. R. Prabhakar [12] .

5. On setting $a = b = \gamma = \delta = 1$ and $\xi = \nu$, $\lambda = \phi$ in (1.8), it becomes (1.2) established by A. Wiman [19]. Moreover if $\mu = 1$ we get Mittag-Leffler function (MLF) $E_\sigma(x)$ defined in (1.1).

Finally, on setting $a = b = \delta = 1$ in (1.8), we establish a new definition of generalized Mittag-Leffler function in the form

$$E^{\xi,\lambda,\gamma}_{\sigma,\mu,\nu,\phi}(x) = \sum_{r=0}^{\infty} \frac{(\xi)_{\lambda r}\,(\gamma)_r\, x^r}{\Gamma(\sigma r + \mu)(\nu)_{\phi r}\, r!}, \tag{1.9}$$

where $\xi, \lambda, \gamma, \sigma, \mu, \nu, \phi, x \in \mathbb{C}$; $\mathbb{R}(\sigma) > 0,\ \mathbb{R}(\mu) > 0,\ \mathbb{R}(\nu) > 0,\ \mathbb{R}(\phi) > 0,\ \mathbb{R}(\xi) > 0,\ \mathbb{R}(\lambda) > 0,\ \mathbb{R}(\gamma) > 0.$

Now, we retrieve the classical beta function denoted by $\mathrm{B}(\zeta, \eta)$ (see [8, 14]) and it is defined by

$$B(\zeta,\eta) = \frac{\Gamma(\zeta)\,\Gamma(\eta)}{\Gamma(\zeta+\eta)} = \int_0^1 t^{\zeta-1}(1-t)^{\eta-1}dt; \qquad (\mathbb{R}(\zeta) > 0,\ \mathbb{R}(\eta) > 0). \tag{1.10}$$

The underlying extension of Euler's beta function established by Chaudhary et al. [2] in 1997 as

$$B_p(\zeta,\eta) = \int_0^1 t^{\zeta-1}(1-t)^{\eta-1}e^{\left[\frac{-p}{t(1-t)}\right]}dt; \qquad (\mathbb{R}(p) > 0, \mathbb{R}(\zeta) > 0, \mathbb{R}(\eta) > 0). \tag{1.11}$$

For $p = 0$, the extended beta function reduces to the classical beta function (1.10).

The Gauss hypergeometric function, represented by $F(\zeta, \eta; \rho; x)$, is defined as (see [10, 13]):

$$F(\zeta,\eta;\rho;x) = \frac{1}{B(\eta,\rho-\eta)}\int_0^1 t^{\eta-1}(1-t)^{\rho-\eta-1}(1-tx)^{-\zeta}dt; \qquad (\mathbb{R}(\rho) > 0,\ \mathbb{R}(\eta) > 0,\ \mid \arg(1-x)\mid < \pi). \tag{1.12}$$

By using the series expansion of $(1 - xt)^{-\zeta}$ in (1.12), we obtain

$$F(\zeta,\eta;\rho;x) = \sum_{r=0}^{\infty} \frac{(\zeta)_r\, B(\eta+r,\rho-\eta)}{B(\eta,\rho-\eta)}\frac{x^r}{r!}; \qquad (\mid x\mid < 1,\ \mathbb{R}(\rho) > \mathbb{R}(\eta) > 0). \tag{1.13}$$

In 2004, Chaudhary et al. [1] used beta function $B(\zeta; \eta; p)$ to extend the hypergeometric function as:

$$F_p(\zeta,\eta;\rho;x) = \sum_{r=0}^{\infty} \frac{(\zeta)_r\, B_p(\eta+r,\rho-\eta)}{B(\eta,\rho-\eta)}\frac{x^r}{r!}; \qquad (p \ge 0;\ \mid x\mid < 1,\ \mathbb{R}(\rho) > \mathbb{R}(\eta) > 0) \tag{1.14}$$

and gave their Euler's type integral representation as:

$$F_p(\zeta,\eta;\rho;x) = \frac{1}{B(\eta,\rho-\eta)}\int_0^1 t^{\zeta-1}(1-t)^{\rho-\eta-1}(1-xt)^{-\zeta}\, e^{\left[\frac{-p}{t(1-t)}\right]}dt, \tag{1.15}$$

where $(p \ge 0;\ \mid \arg(1-x)\mid < \pi;\ \mathbb{R}(\rho) > \mathbb{R}(\eta) > 0)$.

The Mittag-Leffler function $E^{\gamma}_{\sigma,\mu}(x)$ have the under mentioned connection with Wright hypergeometric function ${}_p\Psi_q$(x) (*cf.* [9]):

$$E^{\gamma}_{\sigma,\mu}(x) = \frac{1}{\Gamma(\gamma)}\,{}_1\Psi_1\left[\begin{array}{c}(\gamma,1)\ ;\\ \\ (\mu,\sigma);\end{array}\ x\right], \tag{1.16}$$

where the Fox-Wright function ${}_r\Psi_s[x]$ (see [20]) is defined by

$${}_r\Psi_s[x] = {}_r\Psi_s\left[\begin{array}{c}(\lambda_1,\acute{\lambda}_1),\ldots,(\lambda_r,\acute{\lambda}_r);\\ \\ (l_1,\acute{l}_1),\ldots,(l_s,\acute{l}_s);\end{array}\ x\right]$$

$$= \sum_{m=0}^{\infty} \frac{\Gamma(\lambda_1 + \acute{\lambda}_1 m) \ldots \Gamma(\lambda_r + \acute{\lambda}_r m)}{\Gamma(l_1 + \acute{l}_1 m) \ldots \Gamma(l_s + \acute{l}_s m)} \frac{x^m}{m!}. \tag{1.17}$$

Choi et al. [3] introduced the extended Beta and extended Gauss hypergeometric functions given as:

$$B_{p,q}(\zeta,\eta) = \int_0^1 t^{\zeta-1}(1-t)^{\eta-1} e^{\left[\frac{-p}{t} - \frac{q}{(1-t)}\right]} dt; \qquad (\mathbb{R}(p) > 0,\ \mathbb{R}(q) > 0) \tag{1.18}$$

and

$$F_{p,q}(\zeta,\eta;\rho;x) = \sum_{r=0}^{\infty} \frac{(\zeta)_r\,(\eta)_r}{(\rho)_r} \frac{x^r}{r!} = \sum_{r=0}^{\infty} \frac{(\zeta)_r\ B_{p,q}(\eta + r, \rho - \eta)}{B(\eta, \rho - \eta)} \frac{x^r}{r!}; \qquad (\mathbb{R}(\rho) > 0,\ \mathbb{R}(\eta) > 0,\ p,q \geq 0). \tag{1.19}$$

Now, we introduce the extension of Beta function $B(\zeta,\eta)$ defined for any ζ, $\eta > 0$ and $p, q \geq 0$.

$$B^m_{p,q}(\zeta,\eta) = \int_0^1 t^{\zeta-1}(1-t)^{\eta-1}\ e^{\left[\frac{-p}{t^m} - \frac{q}{(1-t)^m}\right]} dt; \qquad (\mathbb{R}(p) > 0,\ \mathbb{R}(q) > 0). \tag{1.20}$$

2. (p, q) type integral operators involving Mittag-Leffler function

Theorem 2.1. *For $\xi, \lambda, \gamma, \sigma, \mu, \nu, \phi \in \mathbb{C}$; $\mathbb{R}(\xi) > 0$, $\mathbb{R}(\lambda) > 0$, $\mathbb{R}(\gamma) > 0$, $\mathbb{R}(\sigma) > 0$, $\mathbb{R}(\mu) > 0$, $\mathbb{R}(\nu) > 0$, $\mathbb{R}(\phi) > 0$, $\mathbb{R}(p) > 0$, $\mathbb{R}(q) > 0$, the underlying result holds true:*

$$\int_0^1 t^{\zeta-1}(1-t)^{\eta-1}\ e^{\left[\frac{-p}{t^m} - \frac{q}{(1-t)^m}\right]}\ E^{\xi,\lambda,\gamma}_{\sigma,\mu,\nu,\phi}(xt^{\sigma})dt = \sum_{r=0}^{\infty} \frac{(\xi)_{\lambda r}(\gamma)_r\ x^r}{\Gamma(\sigma r + \mu)\ (\nu)_{\phi r}\ r!}\ \ B^m_{p,q}(\sigma r + \zeta, \eta), \tag{2.1}$$

where $E^{\xi,\lambda,\gamma}_{\sigma,\mu,\nu,\phi}(x)$ is known as generalized Mittag-Leffler function given in (1.9).

Proof. In order to obtain our result, we indicate the left hand side of (2.1) by I, expanding $E^{\xi,\lambda,\gamma}_{\sigma,\mu,\nu,\phi}(xt^{\sigma})$ in its summation formula in the integrand with the help of equation (1.9), we have

$$\text{I} = \int_0^1 t^{\zeta-1}(1-t)^{\eta-1} e^{\left[\frac{-p}{t^m} - \frac{q}{(1-t)^m}\right]} \sum_{r=0}^{\infty} \frac{(\xi)_{\lambda r}(\gamma)_r\ x^r\ t^{\sigma r}}{\Gamma(\sigma r + \mu)\ (\nu)_{\phi r}\ r!}\ dt. \tag{2.2}$$

Now, we simplify the above equation, we get

$$\text{I} = \sum_{r=0}^{\infty} \frac{(\xi)_{\lambda r}(\gamma)_r}{\Gamma(\sigma r + \mu)\ (\nu)_{\phi r}}.\frac{x^r}{r!} \int_0^1 t^{\sigma r + \zeta - 1}(1-t)^{\eta-1} e^{\left[\frac{-p}{(t)^m} - \frac{q}{(1-t)^m}\right]} dt. \tag{2.3}$$

Finally, applying the result (1.20) and after some simplifications we can obtain our main result (2.1). □

Corollary 2.2. *On setting $p = q = 0$ in* (2.1)*, we obtain the underlying result as:*

$$\int_0^1 t^{\zeta-1}(1-t)^{\eta-1}\ E^{\xi,\lambda,\gamma}_{\sigma,\mu,\nu,\phi}(xt^{\sigma})dt = E^{\xi,\lambda,\gamma}_{\sigma,\mu,\nu,\phi}(x)\ \ B(\sigma r + \zeta, \eta). \tag{2.4}$$

Theorem 2.3. *For $\xi, \lambda, \gamma, \sigma, \mu, \nu, \phi, \rho_1, \rho_2, \delta \in \mathbb{C}$; $\mathbb{R}(\xi) > 0$, $\mathbb{R}(\lambda) > 0$, $\mathbb{R}(\gamma) > 0$, $\mathbb{R}(\sigma) > 0$, $\mathbb{R}(\mu) > 0$, $\mathbb{R}(\nu) > 0$, $\mathbb{R}(\phi) > 0$, $\mathbb{R}(\rho_1) > 0$, $\mathbb{R}(\rho_2) > 0$, $\mathbb{R}(\delta) > 0$, $\mathbb{R}(p) > 0$, $\mathbb{R}(q) > 0$, $\left|\arg\left(\frac{\zeta_2\tau_1+\tau_2}{\zeta_1\tau_1+\tau_2}\right)\right| < \pi$, the following result holds true:*

$$\begin{aligned}
&\int_{\zeta_1}^{\zeta_2} (t-\zeta_1)^{\rho_1-1}(\zeta_2-t)^{\rho_2-1}(\tau_1 t+\tau_2)^{\delta} e^{\left[\frac{-p}{(t-\zeta_1)^m} - \frac{q}{(\zeta_2-t)^m}\right]} E^{\xi,\lambda,\gamma}_{\sigma,\mu,\nu,\phi}\left[x(\zeta_2-t)^f\right] dt \\
&= (\zeta_1\tau_1+\tau_2)^{\delta} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-p)^s}{s!}\frac{(-q)^k}{k!}\ \frac{(\xi)_{\lambda r}(\gamma)_r}{\Gamma(\sigma r+\mu)\ (\nu)_{\phi r}} \frac{x^r}{r!} \\
&\times(\rho_1 - ms, \rho_2 + fr - mk)\left[(\zeta_2-\zeta_1)^{\rho_1+\rho_2+fr-ms-mk-1}\right] {}_2F_1\left[\rho_1 - ms, -\delta;\ \rho_1+\rho_2+fr-ms-mk;\ \frac{-(\zeta_2-\zeta_1)\tau_1}{\zeta_1\tau_1+\tau_2}\right].
\end{aligned} \tag{2.5}$$

Proof. On taking LHS of (2.5), expanding the exponential function, applying the definition of generalized Mittag-Leffler function in (1.9), and then changing the order of summation and integration, we have

$$\int_{\zeta_1}^{\zeta_2}(t-\zeta_1)^{\rho_1-1}(\zeta_2-t)^{\rho_2-1}(\tau_1 t+\tau_2)^{\delta}e^{\left[\frac{-p}{(t-\zeta_1)^m}-\frac{q}{(\zeta_2-t)^m}\right]}E^{\xi,\lambda,\gamma}_{\sigma,\mu,\nu,\phi}\Big[x(\zeta_2-t)^f\Big]\,dt$$
$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-p)^s(-q)^k\,(\xi)_{\lambda r}\,(\gamma)_r}{\Gamma(\sigma r+\mu)\,(\nu)_{\phi r}\,k!s!}\,\frac{x^r}{r!}$$
$$\times\int_{\zeta_1}^{\zeta_2}\frac{(t-\zeta_1)^{\rho_1-1}(\zeta_2-t)^{\rho_2-1}(\tau_1 t+\tau_2)^{\delta}(\zeta_2-t)^{fr}}{(t-\zeta_1)^{ms}\,(\zeta_2-t)^{mk}}dt$$
$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-p)^s(-q)^k\,(\xi)_{\lambda r}\,(\gamma)_r}{\Gamma(\sigma r+\mu)\,(\nu)_{\phi r}\,k!s!}\,\frac{x^r}{r!}$$
$$\times\int_{\zeta_1}^{\zeta_2}(t-\zeta_1)^{\rho_1-ms-1}(\zeta_2-t)^{\rho_2+fr-mk-1}(\tau_1 t+\tau_2)^{\delta}dt \tag{2.6}$$

then using the following inequality given in (*cf.* [13, p. 301])

$$\int_{\zeta_1}^{\zeta_2}(t-\zeta_1)^{\rho_1-1}(\zeta_2-t)^{\rho_2-1}(ut+v)^{\delta}dt=B(\rho_1,\rho_2)(\zeta_2-\zeta_1)^{\rho_1+\rho_2-1}(\zeta_1 u+v)^{\delta}\,{}_2F_1\left[\rho_1,-\delta;\ \rho_1+\rho_2;\ \frac{-(\zeta_2-\zeta_1)u}{(\zeta_1 u+v)}\right],$$

where $\mathbb{R}(\rho_1)>0$, $\mathbb{R}(\rho_2)>0$; $\left|\arg\left(\frac{\zeta_2 u+v}{\zeta_1 u+v}\right)\right|<\pi$ yields the required result. □

Corollary 2.4. *On setting $p=q=0$ in* (2.5)*, we get the following result:*

$$\int_{\zeta_1}^{\zeta_2}(t-\zeta_1)^{\rho_1-1}(\zeta_2-t)^{\rho_2-1}(\zeta_1\tau_1+\tau_2)^{\delta}\,E^{\xi,\lambda,\gamma}_{\sigma,\mu,\nu,\phi}\Big[x(\zeta_2-t)^f\Big]\,dt$$
$$=(\zeta_1\tau_1+\tau_2)^{\delta}\sum_{r=0}^{\infty}\frac{(\xi)_{\lambda r}(\gamma)_r\,x^r}{\Gamma(\sigma r+\mu)\,(\nu)_{\phi r}\,r!}\,B(\rho_1,\rho_2+fr)\Big[(\zeta_2-\zeta_1)^{\rho_1+\rho_2+fr-1}\Big]$$
$$\times\ {}_2F_1\left[\rho_1,-\delta;\ \rho_1+\rho_2+fr;\ \frac{-(\zeta_2-\zeta_1)\tau_1}{\zeta_1\tau_1+\tau_2}\right]. \tag{2.7}$$

Corollary 2.5. *On setting $\zeta_1=0$, $\zeta_2=1$ in* (2.5)*, we obtain the underlying result:*

$$\int_0^1(t)^{\rho_1-1}(1-t)^{\rho_2-1}(\tau_2)^{\delta}e^{\left[\frac{-p}{(t)^m}-\frac{q}{(1-t)^m}\right]}E^{\xi,\lambda,\gamma}_{\sigma,\mu,\nu,\phi}\Big[x(1-t)^f\Big]\,dt$$
$$=(\tau_2)^{\delta}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-p)^s}{s!}\frac{(-q)^k}{k!}\frac{(\xi)_{\lambda r}(\gamma)_r\,x^r}{\Gamma(\sigma r+\mu)\,(\nu)_{\phi r}\,r!}\,B(\rho_1-ms,\rho_2+fr-mk)$$
$$\times\ {}_2F_1\left[\rho_1-ms,-\delta;\ \rho_1+\rho_2+fr-ms-mk;\ \frac{-\tau_1}{\tau_2}\right]. \tag{2.8}$$

Theorem 2.6. *For $\xi,\lambda,\gamma,\sigma,\mu,\nu,\phi,\rho_1,\rho_2,\eta,\delta\in\mathbb{C}$;* $\mathbb{R}(\xi)>0$, $\mathbb{R}(\lambda)>0$, $\mathbb{R}(\gamma)>0$, $\mathbb{R}(\sigma)>0$, $\mathbb{R}(\mu)>0$, $\mathbb{R}(\nu)>0$, $\mathbb{R}(\phi)>0$, $\mathbb{R}(\rho_1)>0$, $\mathbb{R}(\rho_2)>0$, $\mathbb{R}(\eta)>0$, $\mathbb{R}(\delta)>0$, $\mathbb{R}(p)>0$, $\mathbb{R}(q)>0$*, then the following result holds true:*

$$\int_0^1(t)^{\rho_1-1}(1-t)^{\rho_2-\rho_1-1}(1-ut^{\eta}(1-t)^{\delta})^{-a}e^{\left[\frac{-p}{(t)^m}-\frac{q}{(1-t)^m}\right]}E^{\xi,\lambda,\gamma}_{\sigma,\mu,\nu,\phi}(xt^{\sigma})\,dt$$
$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\xi)_{\lambda r}(\gamma)_r\,x^r}{\Gamma(\sigma r+\mu)\,(\nu)_{\phi r}\,r!}\,\frac{u^s}{s!}(a)_s\,B^m_{p,q}(\rho_1+\sigma r+\eta s,\rho_2-\rho_1+\delta s). \tag{2.9}$$

Proof. Taking LHS of (2.9), expanding the exponential function, applying the definition of generalized Mittag-Leffler function (1.9), and then changing the order of integration and summation, we have

$$\begin{aligned}
&\int_0^1 (t)^{\rho_1-1}(1-t)^{\rho_2-\rho_1-1}(1-ut^{\eta}(1-t)^{\delta})^{-a} e^{\left[\frac{-p}{(t)^m}-\frac{q}{(1-t)^m}\right]} E^{\xi,\lambda,\gamma}_{\sigma,\mu,\nu,\phi}(xt^{\sigma})\, dt \\
&= \sum_{r=0}^{\infty} \frac{(\xi)_{\lambda r}(\gamma)_r\, x^r}{\Gamma(\sigma r+\mu)\,(\nu)_{\phi r}\, r!} \int_0^1 (t)^{\rho_1+\sigma r-1}(1-t)^{\rho_2-\rho_1-1}(1-ut^{\eta}(1-t)^{\delta})^{-a}\, e^{\left[\frac{-p}{(t)^m}-\frac{q}{(1-t)^m}\right]}\, dt \\
&= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\xi)_{\lambda r}(\gamma)_r\, x^r}{\Gamma(\sigma r+\mu)\,(\nu)_{\phi r}\, r!}\, \frac{u^s}{s!}(a)_s \int_0^1 (t)^{\rho_1+\sigma r+s\eta-1}(1-t)^{\rho_2-\rho_1+\delta s-1}\, e^{\left[\frac{-p}{(t)^m}-\frac{q}{(1-t)^m}\right]}\, dt. \qquad (2.10)
\end{aligned}$$

Using the definition of generalized beta function (1.20) in the above equation (2.10), we get the required result (2.9).

Corollary 2.7. *On setting $p = q = 0$ in* (2.9)*, we obtain the underlying result:*

$$\begin{aligned}
&\int_0^1 (t)^{\rho_1-1}(1-t)^{\rho_2-\rho_1-1}(1-ut^{\eta}(1-t)^{\delta})^{-a}\, E^{\xi,\lambda,\gamma}_{\sigma,\mu,\nu,\phi}(xt^{\sigma})\, dt \\
&= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\xi)_{\lambda r}(\gamma)_r\, x^r}{\Gamma(\sigma r+\mu)\,(\nu)_{\phi r}\, r!}\, \frac{u^s}{s!}(a)_s\, B(\rho_1+\sigma r+s\eta, \rho_2-\rho_1+\delta s). \qquad (2.11)
\end{aligned}$$

3. Partly unilateral and partly bilateral generating function

In this section we derived a generating function involving the product of three generalized Mittag-Leffler function. An interesting result on generating function was given by Exton [5]. The modified form of his result due to Pathan and Yasmeen [11] is as follows:

$$\exp\left(u+v-\frac{wv}{u}\right) = \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} u^m v^n F_n^m(w), \qquad (3.1)$$

where

$$F_n^m(w) = {}_1F_1\left[-n; m+1; w\right]/m!n! = L_n^{(m)}(w)/(m+n)!\,. \qquad (3.2)$$

Here, $L_n^{(m)}(w)$ denotes the classical Laguerre polynomial (see [4, 15]) and

$$m^* = \max(0, -m),$$

so that all factorials of negative integers have meaning ($m = 0, 1, 2, \dots$).

Result 3.1. With Exton's work serving as our inspiration, we are able to arrive at the following generating relation for the generalized Mittag-Leffler function (1.9) given as

$$E^{\xi_1,\lambda_1,\gamma_1}_{\sigma_1,\mu_1,\nu_1,\phi_1}(u)\, E^{\xi_2,\lambda_2,\gamma_2}_{\sigma_2,\mu_2,\nu_2,\phi_2}(v)\, E^{\xi_3,\lambda_3,\gamma_3}_{\sigma_3,\mu_3,\nu_3,\phi_3}\left(\frac{-wv}{u}\right) = \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} \frac{u^m v^n}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)} \left\{{}^{\xi_i,\lambda_i,\gamma_i}_{\sigma_i,\mu_i,\nu_i,\phi_i}F^m_n(w)\right\}, \qquad (3.3)$$

where

$$\begin{aligned}
\left\{{}^{\xi_i,\lambda_i,\gamma_i}_{\sigma_i,\mu_i,\nu_i,\phi_i}F^m_n(w)\right\} &= \sum_{l=0}^{\infty} \frac{(\xi_1)_{\lambda_1(m+l)}(\gamma_1)_{(m+l)}(\xi_2)_{\lambda_2(n-l)}(\gamma_2)_{(n-l)}(\xi_3)_{\lambda_3 l}(\gamma_3)_l(-w)^l}{(\mu_1)_{\sigma_1(m+l)}(\nu_1)_{\phi_1(m+l)}(\mu_2)_{\sigma_2(n-l)}(\nu_2)_{\phi_2(n-l)}(\mu_3)_{\sigma_3 l}(\nu_3)_{\phi_3 l}} \\
&\quad \times \frac{1}{\Gamma(l+m+1)\Gamma(n-l+1)\Gamma(l+1)}. \qquad (3.4)
\end{aligned}$$

Proof. To obtain our result, consider the product of generalized Mittag-Leffler function

$$P(w,u,v) = E^{\xi_1,\lambda_1,\gamma_1}_{\sigma_1,\mu_1,\nu_1,\phi_1}(u)\, E^{\xi_2,\lambda_2,\gamma_2}_{\sigma_2,\mu_2,\nu_2,\phi_2}(v)\, E^{\xi_3,\lambda_3,\gamma_3}_{\sigma_3,\mu_3,\nu_3,\phi_3}\left(\frac{-wv}{u}\right).$$

Now, expanding the function in series form, we get

$$\begin{aligned} P(w,u,v) &= \sum_{r=0}^{\infty} \frac{(\xi_1)_{\lambda_1 r}(\gamma_1)_r\, u^r}{\Gamma(\sigma_1 r+\mu_1)(\nu_1)_{\phi_1 r}\, r!} \sum_{k=0}^{\infty} \frac{(\xi_2)_{\lambda_2 k}(\gamma_2)_k\, v^k}{\Gamma(\sigma_2 k+\mu_2)(\nu_2)_{\phi_2 k}\, k!} \sum_{l=0}^{\infty} \frac{(\xi_3)_{\lambda_3 l}(\gamma_3)_l\, (-w)^l\, v^l}{\Gamma(\sigma_3 l+\mu_3)(\nu_3)_{\phi_3 l}\, u^l\, l!} \\ &= \sum_{r=0}^{\infty} \frac{(\xi_1)_{\lambda_1 r}(\gamma_1)_r\, u^{r-l}}{\Gamma(\sigma_1 r+\mu_1)(\nu_1)_{\phi_1 r}\, r!} \sum_{k=0}^{\infty} \frac{(\xi_2)_{\lambda_2 k}(\gamma_2)_k\, v^{k+l}}{\Gamma(\sigma_2 k+\mu_2)(\nu_2)_{\phi_2 k}\, k!} \sum_{l=0}^{\infty} \frac{(\xi_3)_{\lambda_3 l}(\gamma_3)_l\, (-w)^l}{\Gamma(\sigma_3 l+\mu_3)(\nu_3)_{\phi_3 l}\, l!}. \end{aligned} \tag{3.5}$$

One may get to the answer after some rearrangement, which is justified by the fact that the series given above converges absolutely. To get there, one must first replace $r-l$ and $k+l$ with m and n, respectively. This brings one closer to the solution

$$\begin{aligned} P(w,u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} & \frac{u^m v^n}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)} \\ = \sum_{l=0}^{\infty} & \frac{(\xi_1)_{\lambda_1(m+l)}(\gamma_1)_{(m+l)}(\xi_2)_{\lambda_2(n-l)}(\gamma_2)_{(n-l)}(\xi_3)_{\lambda_3 l}(\gamma_3)_l(-w)^l}{(\mu_1)_{\sigma_1(m+l)}(\nu_1)_{\phi_1(m+l)}(\mu_2)_{\sigma_2(n-l)}(\nu_2)_{\phi_2(n-l)}(\mu_3)_{\sigma_3 l}(\nu_3)_{\phi_3 l}} \\ \times & \frac{1}{\Gamma(l+m+1)\Gamma(n-l+1)\Gamma(l+1)}, \end{aligned} \tag{3.6}$$

which is the required result. □

4. Special cases

In this section, we settle the following useful integral operators involving generalized Wright hypergeometric functions as special cases of our main results:

(i) On setting $\xi = \gamma = 1$ in (2.1), we have

$$\int_0^1 t^{\zeta-1}(1-t)^{\eta-1} e^{\left[\frac{-p}{t^m} - \frac{q}{(1-t)^m}\right]}\, {}_2\Psi_2\left[\begin{matrix} (1,\lambda), & (1,1); \\ & & xt^{\sigma} \\ (\mu,\sigma), & (\nu,\phi); \end{matrix}\right] dt = \frac{1}{\Gamma(\nu)} \sum_{r=0}^{\infty} \frac{(1)_{\lambda r}\, x^r}{\Gamma(\sigma r+\mu)\,(\nu)_{\phi r}}\, B^m_{p,q}(\sigma r+\zeta,\eta). \tag{4.1}$$

(ii) On setting $\sigma = \mu = \xi = \gamma = 1$ in (2.1), we have

$$\int_0^1 t^{\zeta-1}(1-t)^{\eta-1} e^{\left[\frac{-p}{t^m} - \frac{q}{(1-t)^m}\right]}\, {}_1\Psi_1\left[\begin{matrix} (1,\lambda)\ ; & \\ & xt \\ (\nu,\phi); & \end{matrix}\right] dt = \frac{1}{\Gamma(\nu)} \sum_{r=0}^{\infty} \frac{(1)_{\lambda r}\ \ x^r}{(\nu)_{\phi r}\ r!}\, B^m_{p,q}(r+\zeta,\eta). \tag{4.2}$$

(iii) On setting $\lambda = \phi = 1$ in (2.1), we have

$$\int_0^1 t^{\zeta-1}(1-t)^{\eta-1} e^{\left[\frac{-p}{t^m} - \frac{q}{(1-t)^m}\right]}\, {}_2\Psi_2\left[\begin{matrix} (\xi,1), & (\gamma,1); \\ & & xt^{\sigma} \\ (\mu,\sigma), & (\nu,1); \end{matrix}\right] dt = \frac{\Gamma(\xi)\ \Gamma(\gamma)}{\Gamma(\nu)} \sum_{r=0}^{\infty} \frac{(\xi)_r\ (\gamma)_r\ x^r}{\Gamma(\sigma r+\mu)\ (\nu)_r\ r!}\, B^m_{p,q}(\sigma r+\zeta,\eta). \tag{4.3}$$

(iv) On setting $\xi = \gamma = 1$ in (2.5), we have

$$\int_{\zeta_1}^{\zeta_2} (t-\zeta_1)^{\rho_1-1}(\zeta_2-t)^{\rho_2-1}(\zeta_1\tau_1+\tau_2)^{\delta} e^{\left[\frac{-p}{(t-\zeta_1)^m}-\frac{q}{(\zeta_2-t)^m}\right]} {}_2\Psi_2\left[\begin{matrix}(1,\lambda), & (1,1); \\ & & x(\zeta_2-t)^f \\ (\mu,\sigma), & (\nu,\phi);\end{matrix}\right] dt$$
$$= \frac{(\zeta_1\tau_1+\tau_2)^{\delta}}{\Gamma(\nu)} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-p)^s}{s!}\frac{(-q)^k}{k!}\frac{(1)_{\lambda r}\ x^r}{\Gamma(\sigma r+\mu)\ (\nu)_{\phi r}} B(\rho_1-ms,\rho_2+fr-mk) \tag{4.4}$$
$$\times\left[(\zeta_2-\zeta_1)^{\rho_1+\rho_2+fr-ms-mk-1}\right] {}_2F_1\left[\rho_1-ms,-\delta;\ \rho_1+\rho_2+fr-ms-mk;\ \frac{-(\zeta_2-\zeta_1)\tau_1}{\zeta_1\tau_1+\tau_2}\right].$$

(v) On setting $\sigma = \mu = \xi = \gamma = 1$ in (2.5), we have

$$\int_{\zeta_1}^{\zeta_2} (t-\zeta_1)^{\rho_1-1}(\zeta_2-t)^{\rho_2-1}(\zeta_1\tau_1+\tau_2)^{\delta} e^{\left[\frac{-p}{(t-\zeta_1)^m}-\frac{q}{(\zeta_2-t)^m}\right]} {}_1\Psi_1\left[\begin{matrix}(1,\lambda)\ ; \\ & x(\zeta_2-t)^f \\ (\nu,\phi);\end{matrix}\right] dt$$
$$= \frac{(\zeta_1\tau_1+\tau_2)^{\delta}}{\Gamma(\nu)} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-p)^s}{s!}\frac{(-q)^k}{k!}\frac{(1)_{\lambda r}\ x^r}{(\nu)_{\phi r}\ r!} B(\rho_1-ms,\rho_2+fr-mk) \tag{4.5}$$
$$\times\left[(\zeta_2-\zeta_1)^{\rho_1+\rho_2+fr-ms-mk-1}\right] {}_2F_1\left[\rho_1-ms,-\delta;\ \rho_1+\rho_2+fr-ms-mk;\ \frac{-(\zeta_2-\zeta_1)\tau_1}{\zeta_1\tau_1+\tau_2}\right].$$

(vi) On setting $\lambda = \phi = 1$ in (2.5), we have

$$\int_{\zeta_1}^{\zeta_2} (t-\zeta_1)^{\rho_1-1}(\zeta_2-t)^{\rho_2-1}(\zeta_1\tau_1+\tau_2)^{\delta} e^{\left[\frac{-p}{(t-\zeta_1)^m}-\frac{q}{(\zeta_2-t)^m}\right]} {}_2\Psi_2\left[\begin{matrix}(\xi,1), & (\gamma,1); \\ & & x(\zeta_2-t)^f \\ (\mu,\sigma), & (\nu,1);\end{matrix}\right] dt$$
$$= \frac{\Gamma(\xi)\Gamma(\gamma)(\zeta_1\tau_1+\tau_2)^{\delta}}{\Gamma(\nu)} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-p)^s}{s!}\frac{(-q)^k}{k!}\frac{(\xi)_r\ (\gamma)_r\ x^r}{\Gamma(\sigma r+\mu)\ (\nu)_r\ r!} B(\rho_1-ms,\rho_2+fr-mk) \tag{4.6}$$
$$\times\left[(\zeta_2-\zeta_1)^{\rho_1+\rho_2+fr-ms-mk-1}\right] {}_2F_1\left[\rho_1-s,-\delta;\ \rho_1+\rho_2+fr-s-k;\ \frac{-(\zeta_2-\zeta_1)\tau_1}{\zeta_1\tau_1+\tau_2}\right].$$

(vii) On setting $\xi = \gamma = 1$ in (2.9), we have

$$\int_0^1 (t)^{\rho_1-1}(1-t)^{\rho_2-\rho_1-1}(1-ut^{\eta}(1-t)^{\delta})^{-a} e^{\left[\frac{-p}{(t)^m}-\frac{q}{(1-t)^m}\right]} {}_2\Psi_2\left[\begin{matrix}(1,\lambda), & (1,1); \\ & & xt^{\sigma} \\ (\mu,\sigma), & (\nu,\phi);\end{matrix}\right] dt$$
$$= \frac{1}{\Gamma(\nu)} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(1)_{\lambda r}\ x^r}{\Gamma(\sigma r+\mu)\ (\nu)_{\phi r}}\ \frac{u^s}{s!}(a)_s\ B_{p,q}^{m}(\rho_1+\sigma r+s\eta,\ \rho_2-\rho_1+\delta s). \tag{4.7}$$

(viii) On setting $\sigma = \mu = \xi = \gamma = 1$ in (2.9), we have

$$\int_0^1 (t)^{\rho_1-1}(1-t)^{\rho_2-\rho_1-1}(1-ut^{\eta}(1-t)^{\delta})^{-a} e^{\left[\frac{-p}{(t)^m}-\frac{q}{(1-t)^m}\right]} {}_1\Psi_1\left[\begin{matrix}(1,\lambda)\ ; \\ & xt \\ (\nu,\phi);\end{matrix}\right] dt$$
$$= \frac{1}{\Gamma(\nu)} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(1)_{\lambda r}\ x^r}{(\nu)_{\phi r}\ r!}\ \frac{u^s}{s!}(a)_s\ B_{p,q}^{m}(\rho_1+r+s\eta,\ \rho_2-\rho_1+\delta s). \tag{4.8}$$

(ix) On setting $\lambda = \phi = 1$ in (2.9), we have

$$\int_0^1 (t)^{\rho_1-1}(1-t)^{\rho_2-\rho_1-1}(1-ut^{\eta}(1-t)^{\delta})^{-a} e^{\left[\frac{-p}{(t)^m}-\frac{q}{(1-t)^m}\right]} {}_2\Psi_2\left[\begin{array}{cc} (\xi,1), & (\gamma,1); \\ & & xt^{\sigma} \\ (\mu,\sigma), & (\nu,1); \end{array}\right] dt$$

$$= \frac{\Gamma(\xi)\Gamma(\gamma)}{\Gamma(\nu)} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\xi)_r(\gamma)_r\, x^r}{\Gamma(\sigma r+\mu)\,(\nu)_r\, r!}\, \frac{u^s}{s!}(a)_s\, B_{p,q}^{m}(\rho_1+\sigma r+s\eta,\ \rho_2-\rho_1+\delta s). \tag{4.9}$$

5. Conclusion

In our present investigation we have studied number of generating functions of generalized Mittag-Leffler function. The computed results yield a unification and an extension of established findings previously provided by different researchers. From the results (2.1), (2.5) and (2.9), we can easily drive numerous and new integral formulas with several special functions due to the close relationship of the generalized Wright function. The result (3.4), which is partly unilateral and partly bilateral mixed generating function of the general nature in the literature, can be specialized in providing several exciting and potentially useful formulas. For instances, if we write $\phi_i = 1$ and $\gamma_i = \nu_i$ (where $i = 1, 2, 3, \ldots$), (3.3) reduces to the following known result for the Mittag-Leffler function defined by Shukla and Prajapati [18]:

$$\begin{aligned} E_{\sigma_1,\mu_1,\gamma_1,1}^{\xi_1,\lambda_1,\gamma_1}(s)\, E_{\sigma_2,\mu_2,\gamma_2,1}^{\xi_2,\lambda_2,\gamma_2}(t)\, E_{\sigma_3,\mu_3,\gamma_3,1}^{\xi_3,\lambda_3,\gamma_3}\left(\frac{-xt}{s}\right) &= E_{\sigma_1,\,\mu_1}^{\xi_1,\,\lambda_1}(s)\, E_{\sigma_2,\,\mu_2}^{\xi_2,\,\lambda_2}(t)\, E_{\sigma_3,\,\mu_3}^{\xi_3,\,\lambda_3}\left(\frac{-xt}{s}\right) \\ &= \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} \frac{s^m t^n}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)} \left\{{}_{\sigma_i,\mu_i,\gamma_i,1}^{\xi_i,\lambda_i,\gamma_i} F_n^m(x)\right\} \end{aligned} \tag{5.1}$$

where

$$\left\{{}_{\sigma_i,\mu_i,\gamma_i,1}^{\xi_i,\lambda_i,\gamma_i} F_n^m(x)\right\} = \sum_{l=0}^{\infty} \frac{(\xi_1)_{\lambda_1(m+l)}(\xi_2)_{\lambda_2(n-l)}(\xi_3)_{\lambda_3 l}(-x)^l}{(\mu_1)_{\sigma_1(m+l)}(\mu_2)_{\sigma_2(n-l)}(\mu_3)_{\sigma_3 l}} \frac{1}{\Gamma(l+m+1)\Gamma(n-l+1)\Gamma(l+1)}.$$

In a similar way from (3.3), we obtained various new and interesting results with different arguments after some suitable parametric replacements.

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