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New Ostrowski type inequalities for (α,β) convex functions

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Abstract

We introduce (α,β) convex functions, which unify and generalize three classes of s-convex functions, and hypergeometric function which two parameters, which generalize classical hypergeometric functions. Some new Ostrowski type inequalities for (α,β) convex functions are established.

Keywords: (α,β) convex function, integral inequality, approximation

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1. Introduction

Definition 1.1 (*cf.* [13])**.** A function $f : [a,b] \to \mathbb{R}$ is called convex, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \tag{1.1}$$

$\forall x, y \in [a,b], \forall t \in [0,1]$.

Let $f : [a,b] \to \mathbb{R}$ be a convex function, then the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.2}$$

is known in the literature as the Hermite-Hadamard inequality (see for instance [13, 15]). In fact, the inequality (1.2) holds if and only if f is a convex function. The Hermite-Hadamard inequality provides approximations for integral mean of a real valued function f. The concept of convex function was extended in many directions and frameworks due to its numerous applications in optimization, variational methods, geometry and artificial intelligence. Hence, the inequality (1.2) has also been extended and generalized for different classes of generalized convex functions (see [13] and the references therein).

The estimate of the difference between left and middle terms in (1.2) is called the Ostrowski type inequality. Such as

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Theorem 1.2 (*cf.* [13]). *Let $f \in AC[a,b]$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2}\left(\frac{b-a}{q+1}\right)^{1/q}\left\|f'\right\|_p,$$

where $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$;

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \left(\frac{b-a}{2}\right)\left\|f'\right\|_1;$$

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \left(\frac{b-a}{4}\right)\left\|f'\right\|_\infty.$$

Theorem 1.3 (*cf.* [13]). *Let $f' \in AC[a,b]$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left(\frac{b-a}{2q+1}\right)^{1/q}\left\|f''\right\|_p,$$

where $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$;

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \left(\frac{b-a}{8}\right)\left\|f''\right\|_1;$$

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{24}\left\|f''\right\|_\infty.$$

Theorem 1.4 (*cf.* [11]). *Let $f \in L[a,b]$, $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$, and $\left|f'\right|^q$ is convex on $[a,b]$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p}\left\{\left(\left|f'(a)\right|^q+3\left|f'(b)\right|^q\right)^{1/q}+\left(3\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)^{1/q}\right\}.$$

If $\frac{a+b}{2}$ is replaced by $x \in [a,b]$, we have

Theorem 1.5 (*cf.* [13]). *Let $f' \in L^\infty[a,b]$, then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \leq \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left\|f'\right\|_\infty. \tag{1.3}$$

The constant $1/4$ is the bet possible.

Theorem 1.6 (*cf.* [1]). *Let $f \in BV[a,b]$, then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \leq \left[\frac{1}{2}+\left|x-\frac{a+b}{2}\right|\right]V_a^b(f). \tag{1.4}$$

The constant $1/2$ is the best possible.

The Ostrowski type inequalities have been developed for other types of functions, and have wide applications in numerical analysis and in the theory of some special estimating error bounds for some means and quadrature rules, etc. (see ([1, 2], [4]-[7], [9]-[17]; and the references therein).

The classical hypergeometric function is defined as follows:

$$F(\alpha,\beta,\gamma,z)=\frac{1}{B(\gamma-\alpha,\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-zt)^{-\beta}dt, \tag{1.5}$$

where $|z| < 1, \gamma > \alpha > 0, \beta > 0$ and

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt, \quad \alpha,\beta > 0 \tag{1.6}$$

is the Beta function.

The paper is categorized as follows:

In Section 2, we introduce (α,β) convex functions, which unify and generalize three classes of s-convex functions, and hypergeometric functions which two parameters, which generalize classical hypergeometric functions. In Section 3, two lemmas. In Section 4, some new Ostrowski type inequalities for (α,β) convex functions are established. These Ostrowski type inequalities provide the estimations of integral mean of a real valued function f.

2. (α,β) convex functions and hypergeometric functions which two parameters

There are three s-convex functions, all of which are generalizations of convex functions (such as, see [8, 13, 16] and the references cite therein. Here, we relax the condition of s-convex set to convex set).

A subset $A \subset \mathbb{R}^n$ is said to be convex, if given any $x, y \in A$ and any $t \in [0, 1]$, we have $tx + (1-t)y \in A$.

Definition 2.1. Let $A \subset \mathbb{R}^n$ be a convex set. A function $f : A \to \mathbb{R}$ is said to be s-convex in the first sense if

$$f(tx + (1-t)y) \le t^s f(x) + (1-t^s)f(y), \tag{2.1}$$

where $x, y \in A$, $t \in [0, 1]$, $s \in (0, 1]$.

Definition 2.2. Let $A \subset \mathbb{R}^n$ be a convex set. A function $f : A \to \mathbb{R}$ is said to be s-convex in the second sense if

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y), \tag{2.2}$$

where $x, y \in A$, $t \in [0, 1]$, $s \in (0, 1]$.

Definition 2.3. Let $A \subset \mathbb{R}^n$ be a convex set. A function $f : A \to \mathbb{R}$ is said to be s-convex in the third sense if

$$f(tx + (1-t)y) \le t^{1/s} f(x) + (1-t^s)^{1/s^2} f(y), \tag{2.3}$$

where $x, y \in A$, $t \in [0, 1]$, $s \in (0, 1]$.

Definition 2.4. Let X be a vector space over the field of real, $A \subset X$. A be a convex set. A function $f : A \to \mathbb{R}$ is said to be (α,β) convex, if

$$f(tx + (1-t)y) \le t^{\alpha\beta} f(x) + (1-t^{\alpha})^{\beta} f(y), \tag{2.4}$$

where $x, y \in A$, $t \in [0, 1]$, $s \in (0, 1]$ and $\alpha,\beta > 0$.

In particular, let $X = \mathbb{R}^n$.

(a) If $\alpha = s, \beta = 1$, then (2.4) reduces to s-convex function in the first sense (2.1).

(b) If $\alpha = 1, \beta = s$, then (2.4) reduces to s-convex function in the second sense (2.2).

(c) If $\alpha = s, \beta = s^{-2}$, then (2.4) reduces to s-convex function in the third sense (2.3).

Hence, the (α,β) convex functions unify and generalize three classes of s-convex function.

Definition 2.5. Let $\gamma > \alpha > 0$, $|z| < 1, \beta > 0$. If there exist two constants $c_1, c_2 > 0$, such that

$$\int_0^1 t^{\alpha-1}(1-t)^{\gamma-1}(1-t^{c_1})^{-c_2}(1-zt)^{-\beta}dt < \infty,$$

then the hypergeometric function which two parameters is defined by

$$H(\alpha,\beta,\gamma,c_1,c_2,z) = \frac{1}{B(\gamma-\alpha,\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-1}(1-t^{c_1})^{-c_2}(1-zt)^{-\beta}dt. \tag{2.5}$$

If $c_1 = 1$, then $H(\alpha,\beta,\gamma,1,c_2,z) = F(\alpha,\beta,\gamma+\alpha-c_2,z)$.

If $c_1 = 1$, $c_2 = \alpha$, then (2.5) reduces to the classical hypergeometric function (1.5).

3. Two lemmas

Lemma 3.1 (*cf.* [3])**.** *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function, $f^{(n-1)} \in AC[a,b]$. Then for all $x \in [a,b]$, we have*

$$\int_a^b f(x)dx = \sum_{k=0}^{n-1} \left\{ \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right\} f^{(k)}(x) + (-1)^n \int_a^b G_n(x,t) f^{(n)}(t)dt, \tag{3.1}$$

where the kernel $G_n : [a,b] \times [a,b] \to \mathbb{R}$ is given by

$$G_n(x,t) = \begin{cases} \frac{(t-a)^n}{n!}, & t \in [a,x], \\ \frac{(t-b)^n}{n!}, & t \in [x,b], \end{cases} \quad x \in [a,b].$$

Lemma 3.2. *Let $f^{''} \in L[a,b]$, then*

$$\frac{1}{(b-a)^2} \left\{ a^4 f(a) + b^4 f(b) - \frac{1}{b-a} \int_a^b f(u)\omega(u)du \right\} = \int_0^1 t(1-t)(ta + (1-t)b)^4 f^{''}(ta + (1-t)b)dt, \tag{3.2}$$

where

$$w(u) = 30u^4 - 20(a+b)u^3 + 12abu^2. \tag{3.3}$$

Proof. Setting $u = (ta + (1-t)b)$ and by integration by parts, we get

$$\begin{aligned} I &= \int_0^1 t(1-t)(ta + (1-t)b)^4 f^{''}(ta + (1-t)b)dt \\ &= \frac{1}{(b-a)^3} \int_a^b (b-u)(u-a)u^4 f^{''}(u)du \\ &= \frac{1}{(b-a)^3} \int_a^b (6u^5 - 5(a+b)u^4 + 4abu^3) f^{'}(u)du \\ &= \frac{1}{(b-a)^3} \left\{ b^4(b-a)f(b) + a^4(b-a)f(a) - \int_a^b f(u)\omega(u)du \right\} \\ &= \frac{1}{(b-a)^2} \left\{ b^4 f(b) + a^4 f(a) - \frac{1}{b-a} \int_a^b f(u)\omega(u)du \right\}. \end{aligned}$$

The Lemma is proved. □

4. Main results

Theorem 4.1. *Let $f : [a,b] \to \mathbb{R}$ be n times differentiable function such that $f^{(n)} \in L[a,b]$ and $f^{(n)}(x) \neq 0$ for all $x \in [a,b]$. If $\left|f^{(n)}\right|$ is (α,β) convex, then for all $x \in [a,b]$, we have*

$$\begin{aligned} &\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \left\{ \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right\} f^{(k)}(x) \right| \\ \leq\ & \frac{(x-a)^{n+1}}{n!} \left\{ \frac{1}{\alpha\beta + n + 1} \left|f^{(n)}(x)\right| + \frac{1}{\alpha} B\left(\frac{n+1}{\alpha}, \beta + 1\right) \left|f^{(n)}(a)\right| \right\} \\ &+ \frac{(b-x)^{n+1}}{n!} \left\{ B(\alpha + \beta + 1, n+1) \left|f^{(n)}(b)\right| + \frac{1}{\alpha} \left[\sum_{k=0}^{n} (-1)^k \binom{n}{k} B\left(\frac{k+1}{\alpha}, \beta + 1\right) \right] \left|f^{(k)}(x)\right| \right\}. \end{aligned} \tag{4.1}$$

Proof. By Lemma 3.1, we have

$$\begin{aligned}\triangle(f) &= \left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\left\{\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right\}f^{(k)}(x)\right| \\ &\leq \int_a^x \frac{(u-a)^n}{n!}\left|f^{(n)}(u)\right|du + \int_x^b \frac{(b-u)^n}{n!}\left|f^{(n)}(u)\right|du \\ &= I_1 + I_2. \end{aligned} \tag{4.2}$$

Let $u = (1-t)a + tx$ and by $\left|f^{(n)}\right|$ is (α,β) convex, we have

$$\begin{aligned} I_1 &= \frac{(x-a)^{n+1}}{n!}\int_0^1 t^n\left|f^{(n)}((1-t)a+tx)\right|dt \\ &\leq \frac{(x-a)^{n+1}}{n!}\int_0^1 t^n\left\{t^{\alpha\beta}\left|f^{(n)}(x)\right| + (1-t^\alpha)^\beta\left|f^{(n)}(a)\right|\right\}dt \\ &= \frac{(x-a)^{n+1}}{n!}\left\{|f^n(x)|\int_0^1 t^{n+\alpha\beta}dt + \left|f^{(n)}(a)\right|\int_0^1 t^n(1-t^\alpha)^\beta dt\right\} \\ &= \frac{(x-a)^{n+1}}{n!}\left\{\frac{1}{\alpha\beta+n+1}\left|f^{(n)}(x)\right| + \frac{1}{\alpha}B\left(\frac{n+1}{\alpha},\beta+1\right)\left|f^{(n)}(a)\right|\right\}. \end{aligned} \tag{4.3}$$

Let $v = (1-t)x + tb$ and by $\left|f^{(n)}\right|$ is $(\alpha\beta)$ convex, we have

$$\begin{aligned} I_2 &= \frac{(b-x)^{n+1}}{n!}\int_0^1 (1-t)^n\left|f^{(n)}((1-t)x+tb)\right|dt \\ &\leq \frac{(b-x)^{n+1}}{n!}\int_0^1 (1-t)^n\left\{t^{\alpha\beta}\left|f^{(n)}(b)\right| + (1-t^\alpha)^\beta\left|f^{(n)}(x)\right|\right\}dt \\ &= \frac{(b-x)^{n+1}}{n!}\left\{\left|f^{(n)}(b)\right|\int_0^1 (1-t)^n t^{\alpha\beta}dt + \left|f^{(n)}(x)\right|\int_0^1 (1-t)^n(1-t^\alpha)^\beta dt\right\} \\ &= \frac{(b-x)^{n+1}}{n!}\left\{B(\alpha\beta+1,n+1)\left|f^{(n)}(b)\right| + \left[\sum_{k=0}^{n}(-1)^k\binom{n}{k}\int_0^1 t^k(1-t^\alpha)^\beta dt\right]\left|f^{(n)}(x)\right|\right\} \\ &= \frac{(b-x)^n}{n!}\left\{B(\alpha\beta+1,n+1)\left|f^{(n)}(b)\right| + \frac{1}{\alpha}\left[\sum_{k=0}^{n}(-1)^k\binom{n}{k}B\left(\frac{k+1}{\alpha},\beta+1\right)\right]\left|f^{(n)}(x)\right|\right\}. \end{aligned} \tag{4.4}$$

A combination of (4.2)-(4.4) gives the required result (4.1). □

In case $\alpha = s, \beta = 1$ in Theorem 4.1, we have

Corollary 4.2. *Let $f : [a,b] \to \mathbb{R}$ be n times differentiable function such that $f^{(n)} \in L[a,b]$ and $f^{(n)} \neq 0$ for all $x \in [a,b]$. If $\left|f^{(n)}\right|$ be an s-convex function in the first sense, then for all $x \in [a,b]$, we have*

$$\begin{aligned}\left|\int_0^1 f(t)dt - \sum_{k=0}^{n-1}\left\{\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right\}f^{(k)}(x)\right| &\leq \frac{(x-a)^{n+1}}{n!}\left\{\frac{1}{s+n+1}\left|f^{(n)}(x)\right| + \frac{1}{s}B\left(\frac{n+1}{s},2\right)\left|f^{(n)}(a)\right|\right\} \\ &+ \frac{(b-x)^{n+1}}{n!}\left\{B(s+2,n+1)\left|f^{(n)}(b)\right| + \frac{1}{s}\sum_{k=0}^{n}(-1)^k\binom{n}{k}B\left(\frac{k+1}{s},2\right)\left|f^{(n)}(x)\right|\right\}. \end{aligned} \tag{4.5}$$

In case $\alpha = 1, \beta = s$ in Theorem 4.1, we have

Corollary 4.3. *Let $f : [a,b] \to \mathbb{R}$ be n times differentiable function such that $f^{(n)} \in L[a,b]$, and $f^{(n)}(x) \neq 0$ for all $x \in [a,b]$. If $\left|f^{(n)}\right|$ be an s-convex function in the second sense, then for all $x \in [a,b]$, we have*

$$\left|\int_0^1 f(t)dt - \sum_{k=0}^{n-1}\left\{\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right\} f^{(k)}(x)\right| \leq \frac{(x-a)^{n+1}}{n!}\left\{\frac{1}{s+n+1}\left|f^{(n)}(x)\right| + B(n+1,s+1)\left|f^{(n)}(a)\right|\right\}$$
$$+\frac{(b-x)^{n+1}}{n!}\left\{B(s+1,n+1)\left|f^{(n)}(b)\right| + \sum_{k=0}^{n}(-1)^k\binom{n}{k}B(k+1,s+1)\left|f^{(n)}(x)\right|\right\}. \tag{4.6}$$

In case $\alpha = s, \beta = s^{-2}$ in Theorem 4.1, we have

Corollary 4.4. *Let $f : [a,b] \to \mathbb{R}$ be n times differentiable function such that $f^{(n)} \in L[a,b]$ and $f^{(n)} \neq 0$ for all $x \in [a,b]$. If $\left|f^{(n)}\right|$ be an s-convex function in the third in the third sense, then for all $x \in [a,b]$, we have*

$$\begin{aligned}
&\left|\int_0^1 f(t)dt - \sum_{k=0}^{n-1}\left\{\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right\} f^{(k)}(x)\right| \\
\leq\ & \frac{(x-a)^{n+1}}{n!}\left\{\frac{1}{s^{-1}+n+1}\left|f^{(n)}(x)\right| + \frac{1}{s}B\left(\frac{n+1}{s}, s^{-2}+1\right)\left|f^{(n)}(a)\right|\right\} \\
&+\frac{(b-x)^{n+1}}{n!}\left\{B(s+s^{-2}+1,n+1)|f^{(n)}(b)| + \frac{1}{s}\sum_{k=0}^{n}(-1)^k\binom{n}{k}B\left(\frac{k+1}{s}, s^{-2}+1\right)\left|f^{(n)}(x)\right|\right\}.
\end{aligned} \tag{4.7}$$

Theorem 4.5. *Let $[a,b] \subset (0,\infty)$, $f'' \in L[a,b]$ and $|f''|^p$ is (α,β) convex on $[a,b]$, $1 \leq p < \infty$, $\frac{1}{p}+\frac{1}{q} = 1$ and for $p = 1$, define $q = \infty$, $\frac{1}{\infty} = 0$, then*

$$\begin{aligned}
&\frac{1}{(b-a)^2}\left|a^4 f(a) + b^4 f(b) - \frac{1}{b-a}\int_a^b f(u)\omega(u)du\right| \\
\leq\ & \frac{C_p b^4}{6}\left\{F\left(2,-4,4,1-\frac{a}{b}\right)\right\}^{1/q} \\
&\times\left\{\left|f''(a)\right|\left[F\left(\alpha\beta+2,-4,\alpha\beta+4,1-\frac{a}{b}\right)\right]^{1/p} + \left|f''(b)\right|\left[H\left(2,-4,2,\alpha,-\beta,1-\frac{a}{b}\right)\right]^{1/p}\right\},
\end{aligned} \tag{4.8}$$

where $\omega(u)$ is defined by (3.3) *and*

$$C_p = \begin{cases} 1, & p \geq 1, \\ 2^{(1/p)-1}, & 0 < p < 1. \end{cases}$$

Proof. By Lemma 3.2 and using the Hölder inequality, we have

$$\begin{aligned}
\frac{1}{(b-a)^2}\left|a^4 f(a) + b^4 f(b) - \frac{1}{b-a}\int_a^b f(u)\omega(u)du\right| &\leq \int_0^1 t(1-t)(ta+(1-t)b)^4\left|f''(ta+(1-t)b)\right|dt \\
&\leq I_1 \times I_2,
\end{aligned} \tag{4.9}$$

where

$$I_1 = \left\{\int_0^1 t(1-t)(ta+(1-t)b)^4 dt\right\}^{1/q}$$

and

$$I_2 = \left\{\int_0^1 t(1-t)(ta+(1-t)b)^4\left|f''(ta+(1-t)b)\right|^p dt\right\}^{1/p}.$$

By (1.5), we get

$$I_1 = \left\{b^4\int_0^1 t(1-t)\left[1-\left(1-\frac{a}{b}\right)t\right]^4 dt\right\}^{1/q} = \left\{\frac{b^4}{6}F\left(2,-4,4,1-\frac{a}{b}\right)\right\}^{1/q}. \tag{4.10}$$

By using the (α,β) convexity of $\left|f''\right|^p$ on $[a,b]$ and C_p inequality, we obtain

$$\begin{aligned}
I_2 &= \left\{\int_0^1 t(1-t)(ta+(1-t)b)^4 \left|f''(ta+(1-t)b)\right|^p dt\right\}^{1/p}\\
&\le \left\{\frac{b^4}{6}\int_0^1 t(1-t)\left[1-\left(1-\frac{a}{b}\right)t\right]^4 \left[t^{\alpha\beta}\left|f''(a)\right|^p + (1-t^\alpha)^\beta \left|f''(b)\right|^p\right] dt\right\}^{1/p}\\
&= \left(\frac{b^4}{6}\right)^{1/p}\left\{\left|f''(a)\right|^p \int_0^1 t^{\alpha\beta}(1-t)\left[1-\left(1-\frac{a}{b}\right)t\right]^4 dt + \left|f''(b)\right|^p \int_0^1 t(1-t)(1-t^\alpha)^\beta\left[1-\left(1-\frac{a}{b}\right)t\right]^4 dt\right\}^{1/p}\\
&= C_p\left(\frac{b^4}{6}\right)^{1/p}\left\{\left|f''(a)\right|\left[F\left(\alpha\beta+2,-4,\alpha\beta+4,1-\frac{a}{b}\right)\right]^{1/p} + \left|f''(b)\right|\left[H\left(2,-4,2,\alpha,-\beta,1-\frac{a}{b}\right)\right]^{1/p}\right\}. \qquad (4.11)
\end{aligned}$$

A combination of (4.9)-(4.11) gives the required result (4.8). □

In case $\alpha = s, \beta = 1$ in Theorem 4.5, we have

Corollary 4.6. *Let $[a,b] \subset (0,\infty)$, $f'' \in L[a,b]$ and $\left|f''\right|^p$ be an s-convex function in the first sense on $[a,b]$, $1 \le p < \infty$, $\frac{1}{p}+\frac{1}{q}=1$ and for $p=1$, define $q=\infty$, $\frac{1}{\infty}=0$, then*

$$\begin{aligned}
&\frac{1}{(b-a)^2}\left|a^4 f(a) + b^4 f(b) - \frac{1}{b-a}\int_0^1 f(u)\omega(u)du\right| \qquad (4.12)\\
\le\ &\frac{C_p b^4}{6}\left\{F\left(2,-4,4,1-\frac{a}{b}\right)\right\}^{1/q}\left\{\left|f''(a)\right|\left[F\left(s+2,-4,s+4,1-\frac{a}{b}\right)\right]^{1/p} + \left|f''(b)\right|\left[H\left(2,-4,2,s,-1,1-\frac{a}{b}\right)\right]^{1/p}\right\}.
\end{aligned}$$

In case $\alpha = 1, \beta = s$ in Theorem 4.5, we have

Corollary 4.7. *Let $[a,b] \subset (0,\infty)$, $f'' \in L[a,b]$ and $\left|f''\right|^p$ be an s-convex function in the second sense on $[a,b]$, $1 \le p < \infty$, $\frac{1}{p}+\frac{1}{q}=1$ and for $p=1$, define $q=\infty$, $\frac{1}{\infty}=0$, then*

$$\begin{aligned}
&\frac{1}{(b-a)^2}\left|a^4 f(a) + b^4 f(b) - \frac{1}{b-a}\int_a^b f(u)\omega(u)du\right| \qquad (4.13)\\
\le\ &\frac{C_p b^4}{6}\left\{F\left(2,-4,4,1-\frac{a}{b}\right)\right\}^{1/q}\left\{\left|f''(a)\right|\left[F\left(s+2,-4,s+4,1-\frac{a}{b}\right)\right]^{1/p} + \left|f''(b)\right|\left[F\left(2,-4,4-s,1-\frac{a}{b}\right)\right]^{1/p}\right\}.
\end{aligned}$$

In case $\alpha = s, \beta = s^{-2}$ in Theorem 4.5, we have

Corollary 4.8. *Let $[a,b] \subset (0,\infty)$, $f'' \in L[a,b]$ and $\left|f''\right|^p$ be an s-convex function in the third sense on $[a,b]$, $1 \le p < \infty$, $\frac{1}{p}+\frac{1}{q}=1$ and for $p=1$, define $q=\infty$, $\frac{1}{\infty}=0$, then*

$$\begin{aligned}
&\frac{1}{(b-a)^2}\left|a^4 f(a) + b^4 f(b) - \frac{1}{b-a}\int_a^b f(u)\omega(u)du\right|\\
\le\ &\frac{C_p b^4}{6}\left\{F\left(2,-4,4,1-\frac{a}{b}\right)\right\}^{1/q}\\
&\times\left\{\left|f''(a)\right|\left[F\left(s^{-1}+2,-4,s^{-1}+4,1-\frac{a}{b}\right)\right]^{1/p} + \left|f''(b)\right|\left[H\left(2,-4,2,s,-s^{-2},1-\frac{a}{b}\right)\right]^{1/p}\right\}.
\end{aligned}$$

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