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A note on characteristic ideal bundles and structure of Lie algebra bundles

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Abstract

We study, properties of completely semi-simple Lie algebra bundles and characteristic semi-simple (css) Lie algebra bundles over arbitrary characteristic. Further, the decomposition theorem is proved over general topological space.

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1. Introduction

Groups that are simultaneously smooth manifolds are called Lie groups. Sophus Lie (1842–1899), who established the theory of continuous transformation groups, is honored by the name Lie groups. Lie originally introduced Lie groups to represent the continuous symmetries of differential equations, a concept that is analogous to the use of finite groups in Galois theory to model the discrete symmetries of algebraic equations.

There is a close relationship between Lie groups and Lie algebras. Every Lie group generates a Lie algebra, or its tangent space at the identity. Conversely, a corresponding linked Lie group exists for any finite-dimensional Lie algebra over real or complex numbers. Because of this correlation, it is possible to examine how Lie groups are categorized and structured in relation to Lie algebras. It is possible to expand the theory of Lie algebra bundles and Lie group bundles from the theory of Lie algebras and Lie groups (*cf.* [6, 7, 8]).

In particular, in quantum field theory, the structure theory of Lie algebra bundles is essential to both mathematics and physics. It has been thoroughly studied for many years by a variety of mathematicians, including theoretical physicists and algebraists who are experts in representation theory worldwide.

Characteristic ideal characteristics of Lie algebras were examined by George Seligman [10]. Additionally, a large number of authors investigated the structure theory of Lie algebras through characteristic ideals (*cf.* [1]-[3], [8, 9]). We

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study several features of completely semi-simple Lie algebra bundles and characteristic semi-simple (css) Lie algebra bundles in this paper. Further, we employ methods akin to those demonstrated by Jean Dieudonne [4] to demonstrate the decomposition theorem for bundles of Lie algebras.

1.1. Preliminaries

In this section, we introduce the notions of characteristic ideals and characteristic semi-simple Lie algebra bundles (*cf.* [6, 8]).

Definition 1.1. Let ξ be a Lie algebra bundle. A derivation D of ξ is a linear map $D : \xi \to \xi$ with $D[u, v] = [D(u), v] + [u, D(v)]$ for all $u, v \in \xi_x$, for all $x \in X$.

Definition 1.2. Let ξ be a Lie algebra bundle. The derivation $l \to [l_0, l]$, where l_0 is any fixed element of ξ_x, is called an inner derivation of ξ. We denote this inner derivation by ad_{l_0}.

Definition 1.3. A subalgebra bundle ζ of ξ is called an ideal bundle if $[\zeta, \xi] \subseteq \xi$.

Definition 1.4. An ideal ζ in ξ is called a characteristic ideal bundle of ξ if $D(\zeta) \subseteq \zeta$ for every derivation D of ξ.

Definition 1.5. We define a derived series of ξ by

$$\xi^{(0)} = \xi,\ \xi^{(1)} = [\xi, \xi],\ \xi^{(2)} = \left[\xi^{(1)}, \xi^{(1)}\right], \ldots,\ \xi^{(i+1)} = \left[\xi^{(i)}, \xi^{(i)}\right].$$

ξ is called solvable ideal bundle if $\xi^{(n)} = 0$ for some $n \geq 1$.

Definition 1.6. The radical (c-radical) bundle of ξ is the maximal solvable (characteristic) ideal bundle of ξ.

Definition 1.7. If the c-radical bundle of a Lie algebra bundle ξ is (0), then ξ is called characteristic semi-simple (css) Lie algebra bundle.

Definition 1.8. Let ξ be a Lie algebra bundle. Then ξ is called characteristic-simple (c-simple) bundle if its only characteristic ideal bundles are ξ, (0), and if $[\xi, \xi] = xi$.

Definition 1.9. A Lie algebra bundle ξ is called completely semi-simple if ξ can be written as a direct sum of ideal bundles in ξ which are characteristic-simple algebra bundles.

Definition 1.10. We define a sequence of ideal bundles of ξ by

$$\xi^{0} = \xi,\ \xi^{1} = [\xi, \xi],\ \xi^{2} = \left[\xi, \xi^{1}\right], \ldots,\ \xi^{i+1} = \left[\xi, \xi^{i}\right].$$

ξ is called nilpotent ideal bundle if $\xi^{n} = 0$ for some $n \geq 1$.

Definition 1.11. The nil-radical bundle of ξ is the largest nilpotent ideal bundle of ξ.

2. Main results

In this section, we discuss some properties of css and completely semi-simple Lie algebra bundles over arbitrary field. Further, we apply Dieudonne's [4] methods to prove the Lie algebra bundle decomposition theorem.

Theorem 2.1. *Let ξ be a Lie algebra bundle over arbitrary field F, with the centre* $\{0\}$. *Then ξ is css if and only if $D(\xi)$ is semi-simple.*

Proof. Let $\mathscr{R}$ represent ξ's characteristic radical. Since $ad(Dr) \in ad\mathscr{R}$, $ad\mathscr{R}$ is the solvable ideal bundle of $D(\xi)$ for any $r \in \mathscr{R}_x$ and $D \in D(\xi_x)$, $[adr, D] = adD(r)$. Given that $D(\xi)$ is semi-simple, $ad\mathscr{R} = 0$ and $\mathscr{R}$ are in the center of ξ in this case. Therefore, $\mathscr{R} = 0$ and ξ are css.

On the other hand, let ξ be css. ξ is isomorphic to $ad(\xi)$, and $ad(\xi)$ is an ideal bundle of $D(\xi)$ since ξ has zero center. Assume that $\mathfrak{R}$ represents the radical of $D(\xi)$. As a result, the ideal bundle of $D(\xi)$ and $ad(\xi)$ is solvable and

$\mathfrak{R} \cap ad(\xi)$. Then, $\mathfrak{R} \cap ad(\xi) = ad(\xi_1)$, where the solvable characteristic ideal bundle of ξ is denoted by ξ_1. However, since ξ is css, $\xi_1 = \{0\}$, and as a result, $\mathfrak{R} \cap ad(\xi) = ad(\xi_1) = 0$, Consequently, we have

$$[\mathfrak{R}, ad(\xi)] = 0. \tag{2.1}$$

For all $D \in \mathfrak{R}$, and for all $u,\ v \in \xi_x$, we have

$$[u, Dv] = [D, ad_v]\,(u).$$

Therefore, for all $u \in \xi_x$, $[u, Dv] = [D, ad_v](u) = 0$. For every $v \in \xi$, Dv is therefore at the center of ξ; as a result, $D = 0$ and $\mathfrak{R} = 0$. $D(\xi)$ is hence semi-simple. □

Lemma 2.2. *In a characteristic-simple Lie algebra bundle ξ, each proper ideal bundle μ is nilpotent.*

Proof. $D(\mu^n) \subseteq \mu^{n-1}$ for all $n \geq 1$ for all $D \in D(\xi)$. Since the sequence μ^n is a chain of descending ideal bundles, $\mu^n = \mu^{n+1}$ for some n must exist. Since $D(\mu^n) = D(\mu^{n+1}) \subseteq \mu^n$ for any $D \in D(\xi)$, μ^n is then a characteristic ideal bundle of ξ. Given that ξ is characteristically simple, $\mu^n = 0$. □

Theorem 2.3. *A semi-simple Lie algebra bundle is the direct sum of simple ideal bundles if and only if it is completely semi-simple Lie algebra bundle.*

Proof. The necessary part is explicit. The way we demonstrate the sufficient component is as follows: The decomposition of $\xi = \xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n$, where each ξ_i is a characteristic-simple ideal bundle of ξ, exists if ξ is completely semi-simple. Should ξ_i exist and it not be simple, then ξ_i has a valid ideal bundle $\mu \neq 0$. In light of the aforementioned Lemma 2.2, μ is a nilpotent ideal bundle of ξ_i and, hence, a nilpotent ideal bundle of ξ, which runs counter to ξ's semi-simplicity, since ξ_i is a simple ideal bundle of ξ, all of it is. □

Theorem 2.4. *Every non-zero ideal bundle of ξ is semi-simple if the semi-simple Lie algebra bundle ξ is completely semi-simple.*

Proof. If ξ is completely semi-simple then from Theorem 2.3, ξ has the decomposition

$$\xi = \xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n,$$

where ξ_i for each $i = 1, 2, 3, \ldots, n$ is simple ideals in ξ. Let $J \neq 0$ be any ideal of ξ. Since $J \cap \xi_i \neq 0$, and so $\xi_i \subseteq J$. Suppose that $\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_k}$ is contained in J and $\xi_{i_{k+1}}, \xi_{i_{k+2}}, \ldots, \xi_{i_s}$ is not contained in J. Then

$$\xi_{i_1} \oplus \xi_{i_2} \oplus \cdots \oplus \xi_{i_k} \subseteq J. \tag{2.2}$$

Since $\xi_{i_{k+j}} \cap J = 0$ for $j = 1, 2, \ldots$, we have $[\xi_{i_{k+j}}, J] \subseteq \xi_{i_{k+j}} \cap J$ and therefore

$$[J, \xi_{i_{k+1}} \oplus \xi_{i_{k+2}} \oplus \cdots \oplus \xi_{i_s}] = 0. \tag{2.3}$$

For $x \in J \cap \xi_{i_{k+1}} \oplus \xi_{i_{k+2}} \oplus \cdots \oplus \xi_{i_s}$, using (2.2) and (2.3) gives $x = 0$. This show that $J \oplus \xi_{i_{k+1}} \oplus \xi_{i_{k+2}} \oplus \cdots \oplus \xi_{i_s}$ is the direct sum of linear spaces. Also

$$\begin{aligned} \xi &= (\xi_{i_1} \oplus \xi_{i_2} \oplus \cdots \oplus \xi_{i_k}) \oplus (\xi_{i_{k+1}} \oplus \xi_{i_{k+2}} \oplus \cdots \oplus \xi_{i_s}) \\ &\subseteq J \oplus (\xi_{i_{k+1}} \oplus \xi_{i_{k+2}} \oplus \cdots \oplus \xi_{i_s}) \\ &\subseteq \xi. \end{aligned}$$

We obtain $J = (\xi_{i_1} \oplus \xi_{i_2} \oplus \cdots \oplus \xi_{i_k})$. Let η be any solvable ideal of J, then

$$[\eta, \xi] = [\eta, J \oplus (\xi_{i_{k+1}} \oplus \xi_{i_{k+2}} \oplus \cdots \oplus \xi_{i_s})] = [\eta, J] = \eta.$$

Thus, in ξ, η is solvable ideal. However, since ξ is semi-simple, $\eta = 0$ and J is semi-simple.

□

Theorem 2.5. *If there is a symmetric invariant nondegenerate bilinear form κ on ξ and ξ does not contain any abelian ideal bundles, then a Lie algebra bundle ξ is completely semi-simple.*

Proof. Given that η contains $[\eta, \eta]$, an ideal bundle of ξ. Assume that η is a minimum ideal bundle in ξ. Then, $[\eta, \eta]$ is either (0) or η; the first case is ruled out because η would then be abelian, and so, $[\eta, \eta] = \eta$. The subbundle of ξ orthogonal to η is η^1. Since κ is invariant, η^1 is an ideal bundle in ξ. Moreover, the relations $X \in \eta$, $Y \in \eta^1$, $Z \in \xi$ imply $\kappa(X, [Z, Y]) = \kappa([X, Z], Y) = 0$ since $[X, Z] \in \eta$. Since η is minimal, the intersection $\eta \cap \eta^1$ can only be (0) or η. Let's demonstrate that the second case, or that the relation $\eta \subset \eta^1$, cannot hold. Subsequently, $\kappa(X, Y) = 0$ would hold for all elements $X, Y \in \eta$. Nevertheless, if $A \in \eta$, then $A = \sum_i [B_i, C_i]$, where B_i and C_i are in η, can be expressed. Next, for each $X \in \xi$,

$$\kappa(A, X) = \sum_i \kappa([B_i, C_i], X) = \sum_i \kappa(B_i, [C_i, X]) = 0.$$

However, this defies the presumption that κ is nondegenerate since $[C_i, X] \in \eta$. Since κ is nondegenerate, ξ is the direct sum of the two ideal bundles, η and η^1. Consequently, $\eta \cap \eta^1 = (0)$. However, η^1 cannot include any non-zero abelian ideal bundle because such an ideal bundle would also be an ideal bundle in ξ. Instead, the restriction to $\eta^1 \times \eta^1$ of the form κ is then a symmetric invariant nondegenerate bilinear form. Completing the proof by induction. □

3. Conclusion

Jacobson provided an example to demonstrate that there exists a semi-simple Lie algebra L, which is not the direct sum of simple ideals, in the situation of characteristic $p \neq 0$ (*cf.* [5]). In this paper, we have proved that the derivation algebra bundle $D(\xi)$ is semi-simple, but its ideal bundle $ad(\xi) \equiv \xi$ is not, if ξ is css Lie algebra bundle over field of characteristic $p \neq 0$, but not semi-simple. We conclude that the semi-simple Lie algebra bundle structure theorem, obtained in characteristic 0, cannot be extended to an arbitrary field in general.

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