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Building generating functions for degenerate Simsek-type numbers and polynomials of higher order

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Abstract

The objective of this paper is to build generating functions for new families of special numbers and polynomials, which are called higher order degenerate Peters-type Simsek numbers and polynomials of the second kind. Using generating function methods, we give both some fundamental properties of these functions with some relations among the higher order degenerate Peters-type Simsek numbers and polynomials of the second kind, the Stirling numbers of the first kind, the higher order degenerate Changhee numbers and polynomials, and the higher order Apostol-type Daehee numbers and polynomials. We also give some plots of these numbers and polynomials via Wolfram Cloud. Further, applying a partial derivative operator to these generating functions, we obtain derivative formulas for these new families. Eventually, we present further remarks on our new families including their generating functions.

Keywords: Stirling numbers, Apostol-type Daehee numbers and polynomials, degenerate Changhee numbers and polynomials, degenerate Peters-type Simsek numbers and polynomials, generating functions

2020 MSC: 05A15, 11B73, 35A99

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1. Introduction

Special polynomials and numbers have useful applications in mathematics, engineering, and related fields, including differential equations, functional analysis, mathematical physics, mathematical analysis, etc. It is known that these numbers and polynomials can be found by using a variety of methods, such as differential equation, p-adic analysis, generating function, and probability theory (*cf.* [1]-[28]). Moreover, one of the most important classes for determining the combinatorial identities for special numbers and polynomials are Simsek numbers and polynomials. The Simsek numbers and polynomials play and significant role in connecting the relationships between special numbers. Additionally, it has attracted the notice of many mathematicians to study and investigate degenerate versions of some special numbers and polynomials in recent years. In 1979, Carlitz [2] first studied the degenerate versions of some special numbers and polynomials. Afterwards, many authors gave some interesting results containing the degenerate versions of special numbers and polynomials (*cf.* [5, 9, 17]). For example, the degenerate Simsek numbers was studied by Oussi in [17].

In this paper, we introduce generating functions for the higher order degenerate Peters-type Simsek numbers and polynomials of the second kind. Using these generating functions with their functional equations and differential

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equations, we obtain some novel relations for these new families linked with some special numbers and polynomials. Furthermore, we show some special values and plots of these numbers and polynomials.

We give below some notations and definitions that will be utilized throughout the paper.

Let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of positive integers, real numbers, and complex numbers, respectively, and also $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The falling factorial $(\omega)_p$ is defined by

$$(\omega)_p = \prod_{c=0}^{p-1} (\omega - c)$$

with $(\omega)_0 = 1$ and $p \in \mathbb{N}$. Besides, we assume that $\log z$ denotes the principal branch of the many-valued function with the imaginary part $\mathrm{Im}(\log z)$ constrained by $-\pi < \mathrm{Im}(\log z) \leq \pi$ (*cf.* [1]-[28]).

The Stirling numbers of the first kind are defined by

$$(\log(1+t))^k = k! \sum_{v=0}^{\infty} S_1(v,k) \frac{t^v}{v!} \tag{1.1}$$

and

$$(\omega)_v = \sum_{p=0}^{v} S_1(v,p)\, \omega^p$$

(*cf.* [1]-[28]).

The numbers $S_1(v,k)$ satisfy the following relation:

$$S_1(v+1,k) - S_1(v,k-1) + vS_1(v,k) = 0, \quad (1 \leq k \leq v) \tag{1.2}$$

with

$$\begin{aligned} S_1(v,k) &= 0, \quad (k > v), \\ S_1(v,v) &= 1, \quad (v \in \mathbb{N}_0), \\ S_1(v,0) &= S_1(0,v) = 0, \quad (v \in \mathbb{N}) \end{aligned}$$

(*cf.* [1]-[28]).

Recently, many research on degenerate versions of special numbers and polynomials have been conducted by some authors, see for detail [2, 5, 9, 17]. Some of these are the higher order degenerate Changhee numbers and the higher order degenerate Changhee polynomials, which are defined by means of the following generating functions, respectively:

$$\left(\frac{2\eta}{2\eta + \log(1+\eta t)}\right)^k = \sum_{v=0}^{\infty} Ch_{v,\eta}^{(k)} \frac{t^v}{v!} \tag{1.3}$$

and

$$\left(\frac{2\eta}{2\eta + \log(1+\eta t)}\right)^k \left(1 + \log(1+\eta t)^{\frac{1}{\eta}}\right)^x = \sum_{v=0}^{\infty} Ch_{v,\eta}^{(k)}(x) \frac{t^v}{v!},$$

where $k \in \mathbb{N}$ (*cf.* [5, pp. 300-301, Eqs. (2.16) and (2.21)]).

The numbers $\mathcal{D}_v^{(k)}(\lambda)$ and the polynomials $\mathcal{D}_v^{(k)}(x;\lambda)$ are defined by

$$\left(\frac{\log(1+\lambda t)}{\lambda t}\right)^k = \sum_{v=0}^{\infty} \mathcal{D}_v^{(k)}(\lambda) \frac{t^v}{v!} \tag{1.4}$$

and

$$\left(\frac{\log(1+\lambda t)}{\lambda^{x+1} t}\right)^k (1+\lambda t)^x = \sum_{v=0}^{\infty} \mathcal{D}_v^{(k)}(x;\lambda) \frac{t^v}{v!},$$

where $k \in \mathbb{N}_0$ (*cf.* [19, p. 202, Eqs. (9) and (10a)]).

The numbers $\mathfrak{D}_v^{(k)}(\lambda)$ and the polynomials $\mathfrak{D}_v^{(k)}(x;\lambda)$, called the higher order Apostol-type Daehee numbers and polynomials, are defined by

$$\left(\frac{\log(\lambda)+\log(1+\lambda t)}{\lambda^2 t+\lambda-1}\right)^k=\sum_{v=0}^{\infty}\mathfrak{D}_v^{(k)}(\lambda)\frac{t^v}{v!} \tag{1.5}$$

and

$$\left(\frac{\log(\lambda)+\log(1+\lambda t)}{\lambda^2 t+\lambda-1}\right)^k(1+\lambda t)^x=\sum_{v=0}^{\infty}\mathfrak{D}_v^{(k)}(x;\lambda)\frac{t^v}{v!},$$

where $k \in \mathbb{N}_0$ (*cf.* [19, p. 203, Eq. (15)]).

The Peters-type Simsek numbers of the first kind $Y_v(\lambda)$ and the Peters-type Simsek polynomials of the first kind $Y_v(x;\lambda)$ are defined by

$$\frac{2}{\lambda^2 t+\lambda-1}=\sum_{v=0}^{\infty}Y_v(\lambda)\frac{t^v}{v!} \tag{1.6}$$

and

$$\frac{2(1+\lambda t)^x}{\lambda^2 t+\lambda-1}=\sum_{v=0}^{\infty}Y_v(x;\lambda)\frac{t^v}{v!}$$

(*cf.* [20, p. 568, Eq. (2.13) and p. 572, Eq. (2.19)]).

In [16], Kucukoglu et al. defined the numbers $Y_v^{(k)}(\lambda)$ and the polynomials $Y_v^{(k)}(x;\lambda)$, called the higher order Peters-type Simsek numbers and polynomials of the first kind, respectively,

$$\left(\frac{2}{\lambda^2 t+\lambda-1}\right)^k=\sum_{v=0}^{\infty}Y_v^{(k)}(\lambda)\frac{t^v}{v!} \tag{1.7}$$

and

$$\left(\frac{2}{\lambda^2 t+\lambda-1}\right)^k(1+\lambda t)^x=\sum_{v=0}^{\infty}Y_v^{(k)}(x;\lambda)\frac{t^v}{v!}, \tag{1.8}$$

where $\lambda \in \mathbb{C}$ (or $\mathbb{R}$) and $k \in \mathbb{N}_0$ (*cf.* [16, p. 2340, Eqs. (2.1)-(2.2)]).

Substituting $k = 1$ into (1.7) and (1.8), we get

$$Y_v^{(1)}(\lambda)=Y_v(\lambda)$$

and

$$Y_v^{(1)}(x;\lambda)=Y_v(x;\lambda).$$

The numbers $Y_{v,2}(\lambda)$ and the polynomials $Y_{v,2}(x;\lambda)$, called the Peters-type Simsek numbers and polynomials of the second kind, are defined by, respectively,

$$\frac{2}{\lambda^2 t+2(\lambda-1)}=\sum_{v=0}^{\infty}Y_{v,2}(\lambda)\frac{t^v}{v!} \tag{1.9}$$

and

$$\frac{2}{\lambda^2 t+2(\lambda-1)}(1+\lambda t)^x=\sum_{v=0}^{\infty}Y_{v,2}(x;\lambda)\frac{t^v}{v!} \tag{1.10}$$

(*cf.* [22, p. 630, Eqs. (12) and (14)]).

Note that these numbers and polynomials, given in Eqs. (1.6)-(1.10), are members of the family of the Boole numbers and polynomials, as well as the Peters polynomials and numbers. Thereof, these numbers and polynomials given in Eqs. (1.6)-(1.10) were first called Peters-type Simsek numbers and polynomials by Kucukoglu [13]. We also note that these numbers and polynomials are used in many disciplines, such as combinatorics, probability, and algebra. Moreover, many authors have examined many applications of these numbers and polynomials (see, for details, [3, 4, 6, 7], [10]-[16], [20]-[25], [27, 28]).

The paper is organized as below.

In Section 2, we construct generating functions for the higher order degenerate Peters-type Simsek numbers and polynomials of the second kind. We not only investigate some properties of these numbers and polynomials, but also give many novel relations involving the Stirling numbers of the first kind, the higher order Apostol-type Daehee numbers, and the higher order degenerate Changhee numbers. Moreover, we present some plots with special values of these numbers and polynomials.

In Section 3, applying derivative operator to the generating functions of the higher order degenerate Peters-type Simsek numbers and polynomials of the second kind, we obtain functional equations of these families. With the help of these equations, we give some relations and recurrence formulas involving these numbers and polynomials.

Finally, a conclusion section is provided.

2. Generating functions for higher order degenerate Peters-type Simsek numbers and polynomials of the second kind

In this section, we build new families of higher order degenerate Peters-type Simsek numbers and polynomials of the second kind with generating functions. With the help of these functions, we examine some properties of these numbers and polynomials. Moreover, we derive some new formulas related to some special numbers. Using these formulas with Wolfram Cloud, we present some special selected plots of these numbers and polynomials.

Let $k \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$ (or $\mathbb{R}$) and $\eta \in \mathbb{R} \setminus \{0\}$. We define the higher order degenerate Peters-type Simsek numbers of the second kind $Y_{v,2}^{(k)}(\lambda \mid \eta)$, and the higher order degenerate Peters-type Simsek polynomials of the second kind $Y_{v,2}^{(k)}(x; \lambda \mid \eta)$, by means of the following generating functions, respectively:

$$N(t, k; \lambda, \eta) = \left(\frac{2}{\frac{\lambda^2}{\eta} \log(1 + \eta t) + 2(\lambda - 1)} \right)^k = \sum_{v=0}^{\infty} Y_{v,2}^{(k)}(\lambda \mid \eta) \frac{t^v}{v!} \tag{2.1}$$

and

$$N(t, x, k; \lambda, \eta) = \left(\frac{2}{\frac{\lambda^2}{\eta} \log(1 + \eta t) + 2(\lambda - 1)} \right)^k \left(1 + \frac{\lambda}{\eta} \log(1 + \eta t) \right)^x = \sum_{v=0}^{\infty} Y_{v,2}^{(k)}(x; \lambda \mid \eta) \frac{t^v}{v!}. \tag{2.2}$$

Substituting $x = 0$ into (2.2), we have

$$Y_{v,2}^{(k)}(0; \lambda \mid \eta) = Y_{v,2}^{(k)}(\lambda \mid \eta).$$

Setting $k = 1$ in (2.1) and (2.2), the higher order degenerate Peters-type Simsek numbers and polynomials of the second kind are reduced:

$$Y_{v,2}(\lambda \mid \alpha) := Y_{v,2}^{(1)}(\lambda \mid \alpha)$$

and

$$Y_{v,2}(x; \lambda \mid \alpha) := Y_{v,2}^{(1)}(x; \lambda \mid \alpha),$$

which are called degenerate Peters-type Simsek numbers and polynomials of the second kind from now on.

For $k = 0$ in (2.1) and (2.2), we obtain

$$Y_{0,2}^{(0)}(\lambda \mid \eta) = 1; \qquad Y_{v,2}^{(0)}(\lambda \mid \eta) = 0 \quad (v \in \mathbb{N})$$

and

$$Y_{v,2}^{(0)}(x; \lambda \mid \eta) = \sum_{m=0}^{v} (x)_m S_1(v, m) \lambda^m \eta^{v-m}.$$

It follows that, for $\eta \to 0$, the Eqs. (2.1) and (2.2) reduce to the following relations:

$$\lim_{\eta\to 0} Y_{v,2}^{(k)}(\lambda \mid \eta) = Y_{v,2}^{(k)}(\lambda)$$

and

$$\lim_{\eta\to 0} Y_{v,2}^{(k)}(x;\lambda \mid \eta) = Y_{v,2}^{(k)}(x;\lambda),$$

where the numbers $Y_{v,2}^{(k)}(\lambda)$ and the polynomials $Y_{v,2}^{(k)}(x;\lambda)$ denote the higher order Peters-type Simsek numbers of the second kind and the higher order Peters-type Simsek polynomials of the second kind defined by the following generating functions, respectively:

$$\left(\frac{2}{\lambda^2 t + 2(\lambda-1)}\right)^k = \sum_{v=0}^{\infty} Y_{v,2}^{(k)}(\lambda)\frac{t^v}{v!} \tag{2.3}$$

and

$$\left(\frac{2}{\lambda^2 t + 2(\lambda-1)}\right)^k (1+\lambda t)^x = \sum_{v=0}^{\infty} Y_{v,2}^{(k)}(x;\lambda)\frac{t^v}{v!}, \tag{2.4}$$

which are special cases of a unification and generalization of the Peters-type Simsek numbers and polynomials, denoted by $Y_{n,m}^{(k,a,b)}(\lambda)$ and $Y_{n,m}^{(k,a,b)}(x;\lambda)$, when $a = b = 1$ and $m = 2$ (see, for details, [14, p. 80, Eqs. (3.1)-(3.2)]).

When $k = 1$ in (2.3) and (2.4), we have

$$Y_{v,2}^{(1)}(\lambda) = Y_{v,2}(\lambda)$$

and

$$Y_{v,2}^{(1)}(x;\lambda) = Y_{v,2}(x;\lambda).$$

Using (1.7) and (2.3), it follows that

$$2^k \sum_{v=0}^{\infty} Y_{v,2}^{(k)}(\lambda)\frac{t^v}{v!} = \sum_{v=0}^{\infty} \frac{Y_v^{(k)}(\lambda)}{2^v}\frac{t^v}{v!}.$$

When the coefficients of $\frac{t^v}{v!}$ on the two sides of the previous equation are compared, we get the following result:

Corollary 2.1. *Let $v, k \in \mathbb{N}_0$. Then we have*

$$Y_{v,2}^{(k)}(\lambda) = \frac{Y_v^{(k)}(\lambda)}{2^{v+k}}. \tag{2.5}$$

Setting $k = 1$ in (2.5), we get

$$Y_{v,2}(\lambda) = \frac{Y_v(\lambda)}{2^{v+1}}$$

(*cf.* [22]).

Theorem 2.2. *Let $\lambda \in \mathbb{C}$ (or $\mathbb{R}$) with $\lambda \neq 1$. For $v \in \mathbb{N}$ and $k \in \mathbb{N}_0$, we have*

$$\sum_{r=0}^{v}\binom{v}{r}\sum_{p=0}^{k}\binom{k}{p} p!\,(\lambda-1)^{k-p}\lambda^{2p}2^{-p}\eta^{r-p}S_1(r,p)\,Y_{v-r,2}^{(k)}(\lambda \mid \eta) = 0,$$

where $Y_{0,2}^{(k)}(\lambda|\eta) = \frac{1}{(\lambda-1)^k}$.

Proof. From (2.1), we have

$$2^k = \sum_{p=0}^{k}\binom{k}{p}\left(\frac{\lambda^2}{\eta}\log(1+\eta t)\right)^p (2\lambda-2)^{k-p}\sum_{v=0}^{\infty} Y_{v,2}^{(k)}(\lambda \mid \eta)\frac{t^v}{v!}. \tag{2.6}$$

Using (1.1) and (2.6), we obtain

$$1=\sum_{p=0}^{k}\binom{k}{p}\frac{p!\lambda^{2p}(\lambda-1)^{k-p}}{(2\eta)^{p}}\sum_{v=0}^{\infty}S_{1}(v,p)\eta^{v}\frac{t^{v}}{v!}\sum_{v=0}^{\infty}Y_{v,2}^{(k)}(\lambda\mid\eta)\frac{t^{v}}{v!}.$$

Thus,

$$1=\sum_{v=0}^{\infty}\sum_{r=0}^{v}\binom{v}{r}\sum_{p=0}^{k}\binom{k}{p}\frac{p!\lambda^{2p}(\lambda-1)^{k-p}}{(2\eta)^{p}}S_{1}(r,p)\eta^{r}Y_{v-r,2}^{(k)}(\lambda\mid\eta)\frac{t^{v}}{v!}.$$

When the coefficients of $\frac{t^v}{v!}$ on the two sides of the previous equation are compared, we get the desired result. □

Theorem 2.3. *Let $\lambda\in\mathbb{C}$ (or $\mathbb{R}$) with $\lambda\neq 1$ and $v,k\in\mathbb{N}_0$. We have*

$$Y_{v,2}^{(k)}(\lambda\mid\eta)=\frac{1}{(\lambda-1)^{k}}\sum_{m=0}^{v}(-1)^{m}\binom{m+k-1}{m}\left(\frac{\lambda^{2}}{2\eta(\lambda-1)}\right)^{m}m!S_{1}(v,m)\eta^{v}. \tag{2.7}$$

Proof. By using (2.1), we have

$$\sum_{v=0}^{\infty}Y_{v,2}^{(k)}(\lambda\mid\eta)\frac{t^{v}}{v!}=\frac{1}{(\lambda-1)^{k}\left(\frac{\lambda^{2}}{2\eta(\lambda-1)}\log(1+\eta t)+1\right)^{k}}.$$

Assume that $\left|\frac{\lambda^2}{2\eta(\lambda-1)}\log(1+\eta t)\right|<1$, and using binomial series expansion in the above equation, we obtain

$$\sum_{v=0}^{\infty}Y_{v,2}^{(k)}(\lambda\mid\eta)\frac{t^{v}}{v!}=(\lambda-1)^{-k}\sum_{m=0}^{\infty}\binom{m+k-1}{m}\left(\frac{\lambda^{2}}{2\eta(1-\lambda)}\right)^{m}(\log(1+\eta t))^{m}.$$

Joining the above equation with (1.1) and using some calculations, we get

$$\sum_{v=0}^{\infty}Y_{v,2}^{(k)}(\lambda\mid\eta)\frac{t^{v}}{v!}=\frac{1}{(\lambda-1)^{k}}\sum_{v=0}^{\infty}\sum_{m=0}^{v}\binom{m+k-1}{m}\left(\frac{\lambda^{2}}{2\eta(1-\lambda)}\right)^{m}m!\eta^{v}S_{1}(v,m)\frac{t^{v}}{v!}.$$

When the coefficients of $\frac{t^v}{v!}$ on the two sides of the previous equation are compared, we obtain the asserted in Eq. (2.7). □

Remark 2.4. For $\eta\to 0$ and $k=1$, Eq. (2.7) is reduced to the following result which is given by Simsek [22, p. 632, Eq. (21)]:

$$Y_{v,2}(\lambda)=(-1)^{v}v!\lambda^{2v}2^{-v}(\lambda-1)^{-v-1}.$$

Using (2.7), the first few values of the numbers $Y_{v,2}^{(k)}(\lambda\mid\eta)$ for v are given as follows:

$$\begin{aligned}
Y_{0,2}^{(k)}(\lambda\mid\eta)&=\frac{1}{(\lambda-1)^{k}},\\
Y_{1,2}^{(k)}(\lambda\mid\eta)&=-\frac{\lambda^{2}k}{2(\lambda-1)^{k+1}},\\
Y_{2,2}^{(k)}(\lambda\mid\eta)&=\frac{1}{(\lambda-1)^{k}}\left(\frac{k\eta\lambda^{2}}{2(\lambda-1)}+\frac{k(k+1)\lambda^{4}}{4(\lambda-1)^{2}}\right),\\
Y_{3,2}^{(k)}(\lambda\mid\eta)&=\frac{1}{(\lambda-1)^{k}}\left(-\frac{k\eta^{2}\lambda^{2}}{\lambda-1}-\frac{3k(k+1)\eta\lambda^{4}}{4(\lambda-1)^{2}}-\frac{k(k+1)(k+2)\lambda^{6}}{8(\lambda-1)^{3}}\right),
\end{aligned}$$

$$
\begin{aligned}
Y_{4,2}^{(k)}(\lambda \mid \eta) &= \frac{1}{(\lambda-1)^k}\left(\frac{3k\eta^3\lambda^2}{\lambda-1}+\frac{11k(k+1)\eta^2\lambda^4}{4(\lambda-1)^2}+\frac{3k(k+1)(k+2)\eta\lambda^6}{4(\lambda-1)^3}\right)\\
&\quad+\frac{1}{(\lambda-1)^k}\left(\frac{k(k+1)(k+2)(k+3)\lambda^8}{16(\lambda-1)^4}\right),\\
Y_{5,2}^{(k)}(\lambda \mid \eta) &= \frac{1}{(\lambda-1)^k}\left(-\frac{12k\eta^4\lambda^2}{\lambda-1}-\frac{25k(k+1)\eta^3\lambda^4}{2(\lambda-1)^2}-\frac{35k(k+1)(k+2)\eta^2\lambda^6}{8(\lambda-1)^3}\right)\\
&\quad+\frac{1}{(\lambda-1)^k}\left(-\frac{5k(k+1)(k+2)(k+3)\eta\lambda^8}{8(\lambda-1)^4}-\frac{k(k+1)(k+2)(k+3)(k+4)\lambda^{10}}{32(\lambda-1)^5}\right).
\end{aligned}
$$

From (2.7) with Wolfram Cloud [29], some plots of the numbers $Y_{v,2}^{(k)}(\lambda \mid \eta)$ for the special cases are presented in Figure 1 and Figure 2.

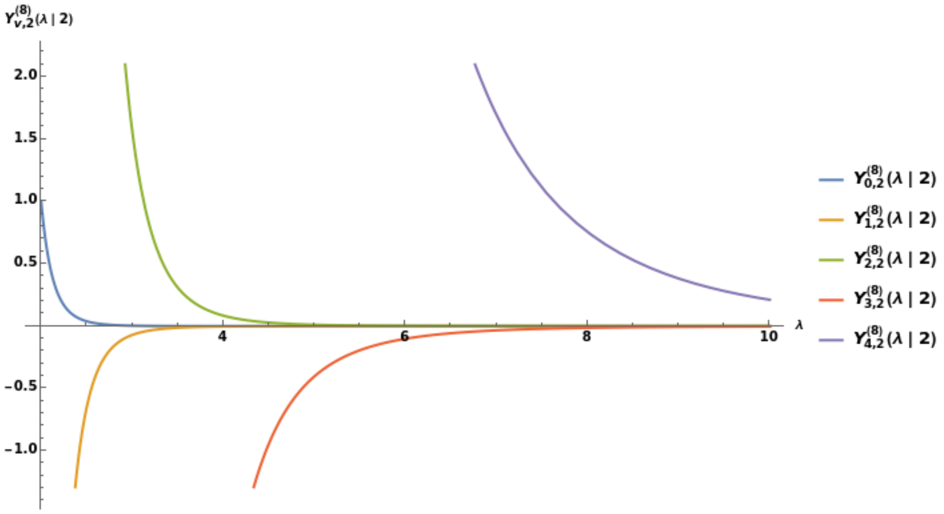

Figure 1. Plots of the numbers $Y_{v,2}^{(8)}(\lambda \mid 2)$ when $v \in \{0,1,2,3,4\}$ and $\lambda \in [2,10]$

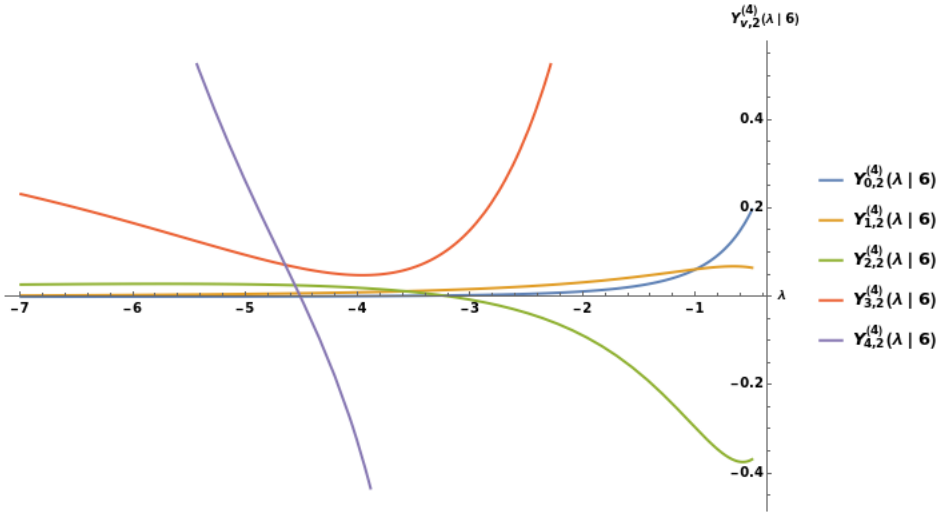

Figure 2. Plots of the numbers $Y_{v,2}^{(4)}(\lambda \mid 6)$ when $v \in \{0,1,2,3,4\}$ and $\lambda \in [-7,-0.5]$

Theorem 2.5. *For $v, k \in \mathbb{N}_0$, we have*

$$Y_{v,2}^{(k)}(x;\lambda \mid \eta) = \sum_{r=0}^{v} \binom{v}{r} \sum_{m=0}^{v-r} (x)_m \, Y_{r,2}^{(k)}(\lambda \mid \eta)\, S_1(v-r,m)\, \lambda^m \eta^{v-r-m}. \tag{2.8}$$

Proof. In view of (2.1) and (2.2), we find that

$$\sum_{v=0}^{\infty} Y_{v,2}^{(k)}(x;\lambda \mid \eta) \frac{t^v}{v!} = \sum_{v=0}^{\infty} Y_{v,2}^{(k)}(\lambda \mid \eta) \frac{t^v}{v!} \sum_{m=0}^{\infty} (x)_m \left(\frac{\lambda}{\eta}\right)^m \frac{(\log(1+\eta t))^m}{m!}.$$

From the above equation and (1.1), we obtain

$$\sum_{v=0}^{\infty} Y_{v,2}^{(k)}(x;\lambda \mid \eta) \frac{t^v}{v!} = \sum_{v=0}^{\infty} Y_{v,2}^{(k)}(\lambda \mid \eta) \frac{t^v}{v!} \sum_{v=0}^{\infty} \sum_{m=0}^{v} (x)_m \left(\frac{\lambda}{\eta}\right)^m S_1(v,m)\, \eta^v \frac{t^v}{v!}.$$

Therefore, we have

$$\sum_{v=0}^{\infty} Y_{v,2}^{(k)}(x;\lambda \mid \eta) \frac{t^v}{v!} = \sum_{v=0}^{\infty} \sum_{r=0}^{v} \binom{v}{r} \sum_{m=0}^{v-r} Y_{r,2}^{(k)}(\lambda \mid \eta)\, S_1(v-r,m)\, (x)_m\, \lambda^m \eta^{v-r-m} \frac{t^v}{v!}.$$

When the coefficients of $\frac{t^v}{v!}$ on the two sides of the previous equation are compared, we get the desired result. $\square$

Remark 2.6. For $\eta \to 0$ and $k = 1$, Eq. (2.8) reduced to the following result:

$$Y_{v,2}(x;\lambda) = \sum_{r=0}^{v} \binom{v}{r} Y_{r,2}(\lambda)\, (x)_{v-r}\, \lambda^{v-r}$$

(*cf.* [22, p. 630, Eq. (15)]).

From (2.8), the first few values of the polynomials $Y_{v,2}^{(k)}(x;\lambda \mid \eta)$ for v are given as follows:

$$\begin{aligned}
Y_{0,2}^{(k)}(x;\lambda \mid \eta) &= \frac{1}{(\lambda-1)^k},\\
Y_{1,2}^{(k)}(x;\lambda \mid \eta) &= \frac{x\lambda}{(\lambda-1)^k} - \frac{\lambda^2 k}{2(\lambda-1)^{k+1}},\\
Y_{2,2}^{(k)}(x;\lambda \mid \eta) &= \frac{1}{(\lambda-1)^k}\left(-x\lambda\eta + x(x-1)\lambda^2 - \frac{kx\lambda^3}{\lambda-1} + \frac{k\eta\lambda^2}{2\lambda-2} + \frac{k(k+1)\lambda^4}{4(\lambda-1)^2}\right),\\
Y_{3,2}^{(k)}(x;\lambda \mid \eta) &= \frac{1}{(\lambda-1)^k}\left(2x\eta^2\lambda - 3(x-1)x\eta\lambda^2 + \frac{3kx\eta\lambda^3}{2\lambda-2} + x(x-1)(x-2)\lambda^3 - \frac{3kx(x-1)\lambda^4}{2\lambda-2}\right)\\
&\quad + \frac{1}{(\lambda-1)^k}\left(+3x\lambda\left(\frac{k\eta\lambda^2}{2\lambda-2} + \frac{k(k+1)\lambda^4}{4(\lambda-1)^2}\right) - \frac{k\eta^2\lambda^2}{\lambda-1} - \frac{3k(k+1)\eta\lambda^4}{4(\lambda-1)^2} - \frac{k(k+1)(k+2)\lambda^6}{8(\lambda-1)^3}\right).
\end{aligned}$$

From (2.8) with Wolfram Cloud [29], some plots of the polynomials $Y_{v,2}^{(k)}(x;\lambda \mid \eta)$ for special cases are presented in Figure 3 and Figure 4.

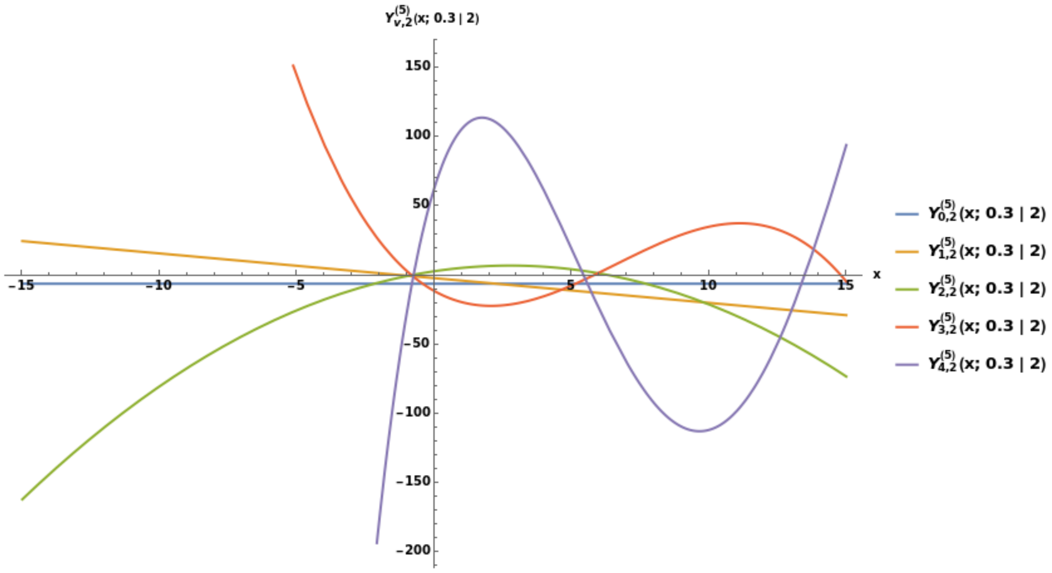

Figure 3. Plots of the polynomials $Y_{v,2}^{(5)}(x; 0.3 \mid 2)$ when $v \in \{0, 1, 2, 3, 4\}$ and $x \in [-15, 15]$

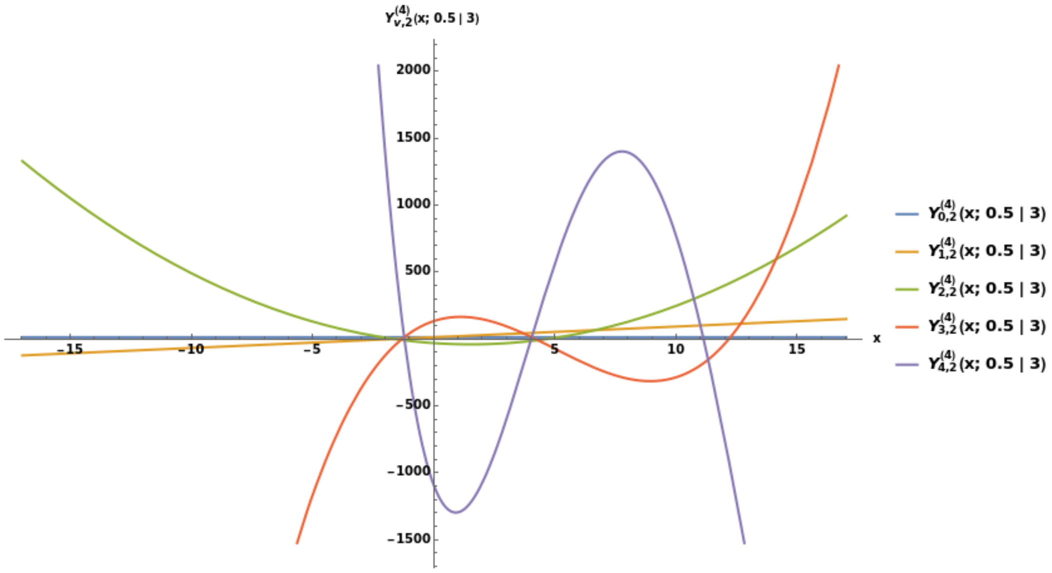

Figure 4. Plots of the polynomials $Y_{v,2}^{(4)}(x; 0.5 \mid 3)$ when $v \in \{0, 1, 2, 3, 4\}$ and $x \in [-17, 17]$

Theorem 2.7. *For $v \in \mathbb{N}$ and $k \in \mathbb{N}_0$, we have*

$$\sum_{p=0}^{k}\binom{k}{p}\left(\frac{\lambda^2}{2}\right)^p(\lambda-1)^{k-p}(v)_p\sum_{r=0}^{v-p}\binom{v-p}{r}\mathcal{D}_r^{(k)}(\eta)\,Y_{v-p-r,2}^{(k)}(\lambda \mid \eta)=0.$$

Proof. From (2.1), we can write

$$2^k=\sum_{p=0}^{k}\binom{k}{p}\frac{\lambda^{2p}}{\eta^p}(\log(1+\eta t))^p(2\lambda-2)^{k-p}\sum_{v=0}^{\infty}Y_{v,2}^{(k)}(\lambda \mid \eta)\frac{t^v}{v!}.$$

Joining the above equation with (1.4), we get

$$1=\sum_{p=0}^{k}\binom{k}{p}\frac{\lambda^{2p}(\lambda-1)^{k-p}}{2^{p}}\sum_{v=0}^{\infty}\mathcal{D}_{v}^{(k)}(\eta)\frac{t^{v+p}}{v!}\sum_{v=0}^{\infty}Y_{v,2}^{(k)}(\lambda\mid\eta)\frac{t^{v}}{v!}.$$

Hence

$$1=\sum_{v=0}^{\infty}\sum_{p=0}^{k}\binom{k}{p}2^{-p}(\lambda-1)^{k-p}\lambda^{2p}(v)_{p}\sum_{r=0}^{v-p}\binom{v-p}{r}\mathcal{D}_{r}^{(k)}(\eta)Y_{v-p-r,2}^{(k)}(\lambda\mid\eta)\frac{t^{v}}{v!}.$$

When the coefficients of $\frac{t^v}{v!}$ on the two sides of the previous equation are compared, we complete the proof. ☐

Theorem 2.8. *For $v\in\mathbb{N}_0$ and $k\in\mathbb{N}$, we have*

$$Y_{v,2}^{(k)}\left(\frac{1\mp i\sqrt{3}}{2}\mid\eta\right)=\left(\frac{2}{-1\mp i\sqrt{3}}\right)^{k}Ch_{v,\eta}^{(k)}$$

where $i^2=-1$.

Proof. Substituting

$$\lambda=\frac{1\mp i\sqrt{3}}{2},$$

where $i^2=-1$, into (2.1), we get

$$\sum_{v=0}^{\infty}Y_{v,2}^{(k)}\left(\frac{1\mp i\sqrt{3}}{2}\mid\eta\right)\frac{t^{v}}{v!}=\frac{1}{\left(\frac{-1\mp i\sqrt{3}}{2}\right)^{k}}\left(\frac{2}{2+\frac{1}{\eta}\log(1+\eta t)}\right)^{k}. \tag{2.9}$$

Combining (2.9) with (1.3), we obtain

$$\sum_{v=0}^{\infty}Y_{v,2}^{(k)}\left(\frac{1\mp i\sqrt{3}}{2}\mid\eta\right)\frac{t^{v}}{v!}=\frac{1}{\left(\frac{-1\mp i\sqrt{3}}{2}\right)^{k}}\sum_{v=0}^{\infty}Ch_{v,\eta}^{(k)}\frac{t^{v}}{v!}.$$

When the coefficients of $\frac{t^v}{v!}$ on the two sides of the previous equation are compared, we obtain the desired result. ☐

Theorem 2.9. *For $v,k\in\mathbb{N}_0$, we have*

$$\mathfrak{D}_{v}^{(k)}(\lambda)=\sum_{r=0}^{v}\sum_{p=0}^{k}\binom{v}{r}(k)_{p}(\log\lambda)^{k-p}2^{v-r}\lambda^{r}S_{1}(r,p)Y_{v-r,2}^{(k)}(\lambda).$$

Proof. From (1.5) and (2.3), we can write

$$\sum_{v=0}^{\infty}\frac{\mathfrak{D}_{v}^{(k)}(\lambda)}{2^{v}}\frac{t^{v}}{v!}=\left(\log\lambda+\log\left(1+\frac{\lambda t}{2}\right)\right)^{k}\sum_{v=0}^{\infty}Y_{v,2}^{(k)}(\lambda)\frac{t^{v}}{v!}.$$

Utilizing the above equation and (1.1), we obtain

$$\sum_{v=0}^{\infty}\frac{\mathfrak{D}_{v}^{(k)}(\lambda)}{2^{v}}\frac{t^{n}}{n!}=\sum_{p=0}^{k}(k)_{p}(\log\lambda)^{k-p}\sum_{v=0}^{\infty}S_{1}(v,p)\left(\frac{\lambda}{2}\right)^{v}\frac{t^{v}}{v!}\sum_{v=0}^{\infty}Y_{v,2}^{(k)}(\lambda)\frac{t^{v}}{v!}.$$

Thus

$$\sum_{v=0}^{\infty}\frac{\mathfrak{D}_{v}^{(k)}(\lambda)}{2^{v}}\frac{t^{v}}{v!}=\sum_{p=0}^{k}(k)_{p}(\log\lambda)^{k-p}\sum_{v=0}^{\infty}\sum_{r=0}^{v}\binom{v}{r}\left(\frac{\lambda}{2}\right)^{r}S_{1}(r,p)Y_{v-r,2}^{(k)}(\lambda)\frac{t^{v}}{v!}.$$

When the coefficients of $\frac{t^v}{v!}$ on the two sides of the previous equation are compared, we get the desired result. ☐

3. Partial derivative equations of higher order degenerate Peters-type Simsek numbers and polynomials of the second kind

In this section, we give some derivative formulas for the higher order degenerate Peters-type Simsek numbers and polynomials of the second kind by applying the partial derivative operator to the generating functions for these numbers.

Theorem 3.1. *For $v, k \in \mathbb{N}_0$, we have*

$$\frac{\partial}{\partial\lambda}\left\{Y_{v,2}^{(k)}(\lambda \mid \eta)\right\} = -kY_{v,2}^{(k+1)}(1;\lambda \mid \eta).$$

Proof. By applying the derivative operator $\frac{\partial}{\partial\lambda}$ to Eq. (2.1), we get

$$\frac{\partial}{\partial\lambda}\{N(t,k;\lambda,\eta)\} = -k\left(\frac{2}{\frac{\lambda^2}{\eta}\log(1+\eta t)+2(\lambda-1)}\right)^{k+1}\left(\frac{\lambda}{\eta}\log(1+\eta t)+1\right). \tag{3.1}$$

Hence

$$\sum_{v=0}^{\infty}\frac{\partial}{\partial\lambda}\left\{Y_{v,2}^{(k)}(\lambda \mid \eta)\right\}\frac{t^v}{v!} = -k\sum_{v=0}^{\infty}Y_{v,2}^{(k+1)}(1;\lambda \mid \eta)\frac{t^v}{v!}.$$

When the coefficients of $\frac{t^v}{v!}$ on the two sides of the previous equation are compared, we get the desired result. □

Theorem 3.2. *For $v, k \in \mathbb{N}$, we have*

$$Y_{v,2}^{(k+1)}(\lambda \mid \eta) = -\frac{2}{k\lambda^2}\left(Y_{v+1,2}^{(k)}(\lambda \mid \eta) + \eta v Y_{v,2}^{(k)}(\lambda \mid \eta)\right).$$

Proof. Taking partial derivative of Eq. (2.1) with respect to t, we get

$$\frac{\partial}{\partial t}\{N(t,k;\lambda,\eta)\} = -\frac{k\lambda^2}{2(1+\eta t)}\left(\frac{2}{\frac{\lambda^2}{\eta}\log(1+\eta t)+2(\lambda-1)}\right)^{k+1}. \tag{3.2}$$

Thus,

$$-\frac{k\lambda^2}{2}\sum_{v=0}^{\infty}Y_{v,2}^{(k+1)}(\lambda \mid \eta)\frac{t^v}{v!} = \sum_{v=0}^{\infty}Y_{v+1,2}^{(k)}(\lambda \mid \eta)\frac{t^v}{v!} + \eta\sum_{v=0}^{\infty}vY_{v,2}^{(k)}(\lambda \mid \eta)\frac{t^v}{v!}.$$

When the coefficients of $\frac{t^v}{v!}$ on the two sides of the previous equation are compared, we arrive the desired result. □

Theorem 3.3. *For $v \in \mathbb{N}$ and $k \in \mathbb{N}_0$, we have*

$$\begin{aligned}-\lambda\sum_{r=0}^{v-1}(-1)^r\binom{v}{r+1}r!\eta^r\left(Y_{v-r,2}^{(k)}(\lambda \mid \eta) + \eta(v-1-r)Y_{v-1-r,2}^{(k)}(\lambda \mid \eta)\right)\\ = Y_{v+1,2}^{(k)}(\lambda \mid \eta) + \eta v Y_{v,2}^{(k)}(\lambda \mid \eta) + \frac{\lambda^2 k}{2}Y_{v,2}^{(k+1)}(1;\lambda \mid \eta).\end{aligned}$$

Proof. Combining (3.2) with (3.1), we have

$$\frac{\partial}{\partial t}\{N(t,k;\lambda,\eta)\} = \frac{\lambda^2}{2(1+\eta t)\left(\frac{\lambda}{\eta}\log(1+\eta t)+1\right)}\frac{\partial}{\partial\lambda}\{N(t,k;\lambda,\eta)\}.$$

Therefore,

$$\left(\frac{\lambda}{\eta}\log(1+\eta t)+1\right)\left(\sum_{v=0}^{\infty}Y_{v+1,2}^{(k)}(\lambda \mid \eta)\frac{t^v}{v!} + \eta\sum_{v=0}^{\infty}vY_{v,2}^{(k)}(\lambda \mid \eta)\frac{t^v}{v!}\right) = -\frac{\lambda^2 k}{2}\sum_{v=0}^{\infty}Y_{v,2}^{(k+1)}(1;\lambda \mid \eta)\frac{t^v}{v!}.$$

Joining the above equation with

$$\frac{\log(1+t)}{t}=\sum_{v=0}^{\infty}\frac{(-1)^{v}t^{v}}{v+1},$$

we get

$$\left(\lambda t\sum_{v=0}^{\infty}\frac{(-1)^{v}\eta^{v}t^{v}}{v+1}+1\right)\left(\sum_{v=0}^{\infty}Y_{v+1,2}^{(k)}(\lambda\mid\eta)\frac{t^{v}}{v!}+\eta\sum_{v=0}^{\infty}vY_{v,2}^{(k)}(\lambda\mid\eta)\frac{t^{v}}{v!}\right)=-\frac{\lambda^{2}k}{2}\sum_{v=0}^{\infty}Y_{v,2}^{(k+1)}(1;\lambda\mid\eta)\frac{t^{v}}{v!}.$$

After some algebraic operations, we obtain

$$\begin{aligned}\lambda\sum_{v=1}^{\infty}\sum_{r=0}^{v-1}\frac{(-1)^{r}\eta^{r}Y_{v-r,2}^{(k)}(\lambda\mid\eta)}{(r+1)(v-1-r)!}t^{v} &+ \eta\lambda\sum_{v=1}^{\infty}\sum_{r=0}^{v-1}\frac{(-1)^{r}\eta^{r}(v-1-r)Y_{v-1-r,2}^{(k)}(\lambda\mid\eta)}{(r+1)(v-1-r)!}t^{v}\\ &+ \sum_{v=1}^{\infty}Y_{v+1,2}^{(k)}(\lambda\mid\eta)\frac{t^{v}}{v!}+\eta\sum_{v=1}^{\infty}vY_{v,2}^{(k)}(\lambda\mid\eta)\frac{t^{v}}{v!}\\ &= -\frac{\lambda^{2}k}{2}\sum_{v=1}^{\infty}Y_{v,2}^{(k+1)}(1;\lambda\mid\eta)\frac{t^{v}}{v!}.\end{aligned}$$

When the coefficients of t^{v} on the two sides of the previous equation are compared, we get the assertion result. $\square$

4. Conclusion

In this paper, the higher order degenerate Peters-type Simsek numbers and polynomials of the second kind and their generating functions were constructed. Using these functions, we investigated some fundamental properties of these new special numbers and polynomials. We gave some identities and formulas related to these numbers and polynomials, the higher order degenerate Changhee numbers, the higher order Apostol-type Daehee numbers, and the Stirling numbers of the first kind. We also gave some calculation formulas for the higher order degenerate Peters-type Simsek numbers and polynomials of the second kind. Utilizing these calculation formulas, we both calculated these numbers and polynomials and presented their plots via Wolfram Cloud. Further, applying a partial derivative operator to these new generating functions, some formulas for these numbers and polynomials were found. Consequently, since generating functions have a variety of applications, the results of this paper have the potential to shed light on the increasing attempts to use mathematical modeling tools in different areas of mathematics, engineering, and many other related areas.

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