Montes Taurus J. Pure Appl. Math. 6 (3), 169–185, 2024

Montes Taurus Journal of Pure and Applied Mathematics
https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

Inverse divisor functions using Dirichlet convolution

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Abstract

The study of convolution sums of restricted divisor functions requires more complex computations because the coefficients in the cusp form or the new form are used. In this article, we introduce the inverse divisor function as a tool to give relatively simple formulae for restricted divisor functions using the Dirichlet convolution and show its values in simple cases. The basic properties of the arithmetic functions needed to obtain the main results are introduced.

Keywords: Dirichlet convolution sum, convolution sum, inverse divisor functions

2020 MSC: 11A05, 11A25

1. Introduction

It is known that the study of the convolution sum of the divisor functions begins with a letter from Besge to Liouville in 1862 [15, p.125]. Ramanujan used only elementary method to find the convolution sums of nine divisor functions [10, 11]. As is well known to us, Glaisher [2, 3, 4] had obtained the results of convolution sums of various restricted divisor functions. Also, Lahri's results [6, 7] are important references.

Focusing on Williams [15], his research group studied the convolution sums of several restricted divisor functions using Liouville identities. We intend to utilize the results of the above studies in this paper. We try to obtain the main results by using Dirichlet convolution among the divisor functions. The notations necessary to write this paper are first introduced below.

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and p be a positive prime integer. We recall some divisor functions and their convolution sums for later use, which appear in number theory area:

The main goal is to find the inverse of a weak function that is often used to find

$$\sigma_k(n) := \sum_{d|n} d^k,$$

and

$$\sigma_k^{\dagger}(n) := \sum_{\substack{d|n \\ \frac{n}{d}\equiv 1 \pmod 2}} d^k,$$

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Received:15 November 2023, Accepted:21 March 2024, Published:6 April 2024
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where $d, n \in \mathbb{N}$ and $k \in \mathbb{N}_0$.

In fact, Srinivasa Ramanujan [11] considered the sums of the type

$$\sigma_r(1)\sigma_s(N-1) + \cdots + \sigma_r(N-1)\sigma_s(1) = \sum_{k=1}^{N} \sigma_r(k)\sigma_s(N-k).$$

Furthermore, the problem of finding the formula for

$$\sum_{\substack{k=1 \\ (k,N-k)=1}}^{N} \sigma_r(k)\sigma_s(N-k)$$

is a bit more complicated than the above problem. The main results of this article are mainly related to the inverses of divisor functions and related results.

Although the divisor function is well known [2], not much is known about σ_k^{-1}. Here, σ_k^{-1} is the inverse function of σ_k obtained by the Dirichlet convolution of σ_k. The purpose of most of this paper is to introduce σ_k^{-1}. Furthermore, rather than discussing the general natural number k in detail, we aim to make it easier for the reader by choosing relatively simple σ_1 and σ_3 and finding their inverse functions. In other words, in this paper, we want to find $\sigma_1^{-1}(n)$, $(\sigma_1^{\dagger})^{-1}(n)$, $(\sigma_3^{\dagger})^{-1}(n)$ and $(\sigma_1^{\dagger}\sigma_3^{\dagger})^{-1}(n)$. Using these results, they can be useful in finding the convolution of divisor functions. The Dirichlet convolution and the inverse of the divisor function are helpful in solving this problem.

To briefly summarise this article, in subsection 1.1 states the basic definitions and properties of arithmetic functions. Subsection 1.2 makes statements about multiplicative functions and completely functions. In section 2, we state in detail the process of finding the inverse of $(\sigma_1\sigma_3)$ and $(\sigma_1^{\dagger}\sigma_3^{\dagger})$, the main goal of this paper, and present their results.

1.1. *Arithmetical functions*

An arithmetical function is generally any function $f(n)$ the positive integers and whose range is a subset of the complex numbers. Let f and g be arithmetical functions and $n \in \mathbb{N}$. The Dirichlet convolution of f and g is defined by

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d). \tag{1.1}$$

Here, d means a positive divisor of n. See [8, 9].

An arithmetic function f^{-1} is called an inverse of f if

$$\left(f * f^{-1}\right)(n) = \left(f^{-1} * f\right)(n) = \delta(n)$$

with

$$\delta(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

An arithmetic function f has an inverse if and only if $f(1) \neq 0$, see [8, Proposition 1.2]. In fact, the arithmetic function f^{-1} satisfy the following:

$$f^{-1}(1) = \frac{1}{f(1)}$$

and

$$f^{-1}(n) = -\frac{1}{f(1)} \sum_{\substack{d|n \\ d>1}} f(d)f^{-1}(n/d), \quad \text{see [8, p. 5].} \tag{1.2}$$

Let

$$\zeta_k(n) := n^k \ \text{ and } \ \zeta(n) := \zeta_0(n) = n^0.$$

Since $\zeta(1) = 1$, ζ has an inverse by (1.2). To find the inverse of the zeta function, we use the well-known Möbius μ-function below.

$$\mu(n) = \begin{cases} 1 & n = 1, \\ (-1)^t & n = p_1 p_2 \dots p_t, \\ 0 & \text{otherwise.} \end{cases}$$

For details on arithmetic functions, refer to [1, p. 152], [8, 9, 12], [13, Chapter 5], [14], etc.

1.2. Multiplicative functions and completely multiplicative functions

An arithmetical function f is called a multiplicative function if $f(n) \neq 0$ for at least one integer n and $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$.

An arithmetical function f is called a completely multiplicative function if $f(n) \neq 0$ for at least one integer n and $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$.

If f and g are arithmetical functions their sum $f + g$ and product fg are defined in the usual way:

$$(f + g)(n) = f(n) + g(n)$$

and

$$(fg)(n) = f(n)g(n)$$

for all positive integer n, see [8, p. 2], [9], [13, Chapter 5]. Furthermore, we define

$$f^k(n) = \underbrace{f(n) \dots f(n)}_{k} \quad \text{with } k \in \mathbb{N}.$$

Proposition 1.1 (*cf.* [8, Proposition 1.8]). *If f is a completely multiplicative function then f^{-1} is $f\mu$.*

Proposition 1.2 (*cf.* [8, p. 6]). *Let f, g and h be arithmetical functions. Then we obtain*

(1) $f * g = g * f$.

(2) $(f * g) * h = f * (g * h)$.

(3) $f * (g + h) = f * g + f * h$ *and* $(f + g) * h = f * h + g * h$.

Proposition 1.3 and 1.4 below are relatively well-known results in [1, p. 152], [8, 15], etc. Proposition 1.4 is necessary for understanding the main results of this article, so we introduce its proof briefly.

Proposition 1.3. *Let $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then*

$$(\zeta_k * \zeta)(n) = \sigma_k(n).$$

Proposition 1.4. *Let $k \in \mathbb{N}_0$. $\sigma_k^\dagger$ and σ_k are multiplicative functions.*

Proof. Let $m, n \in \mathbb{N}$ with $\gcd(m, n) = 1$. Then

$$\begin{aligned} \sigma_k^\dagger(mn) &= \sum_{\substack{d \mid mn \\ \frac{mn}{d} \equiv 1 \pmod 2}} d^k = \sum_{\substack{d_1 \mid m, d_2 \mid n, \\ \frac{mn}{d_1 d_2} \equiv 1 \pmod 2,\ \gcd(\frac{m}{d_1}, \frac{n}{d_2}) = 1}} d^k \\ &= \left(\sum_{\substack{d_1 \mid m \\ \frac{m}{d_1} \equiv 1 \pmod 2}} d_1^k \right) \left(\sum_{\substack{d_2 \mid n \\ \frac{n}{d_2} \equiv 1 \pmod 2}} d_2^k \right) \\ &= \sigma_k^\dagger(m) \sigma_k^\dagger(n). \end{aligned}$$

Similarly, σ_k is an also multiplicative function. □

Proposition 1.5 (*cf.* [15, Theorem 3.3]). *Let $n, k \in \mathbb{N}$. Then*

$$\sigma_k^\dagger(n) = \sigma_k(n) - \sigma_k(n/2).$$

2. Formulas for $(\sigma_1\sigma_3)^{-1}$ and $(\sigma_1^\dagger\sigma_3^\dagger)^{-1}$

In this section, we want to find the formulas for $(\sigma_1\sigma_3)^{-1}$ and $(\sigma_1^\dagger\sigma_3^\dagger)^{-1}$. In order to find these formulae, we need the properties of multiplicative and inverse multiplicative functions, which are listed below.

Proposition 2.1. *The followings hold true:*

- ***(1)*** (*cf.* [8, Proposition 1.5])**:** *If f is multiplicative function, then f^{-1} is multiplicative function.*
- ***(2)*** (*cf.* [8, Proposition 1.6])**:** *If f and g are multiplicative functions then $f*g$ and fg are multiplicative functions.*

Lemma 2.2. $(\sigma_1^\dagger)^{-1}$ *is a multiplicative function.*

Proof. By Proposition 1.4, $\sigma_1^\dagger$ is a multiplicative function. Therefore, $(\sigma_1^\dagger)^{-1}$ is multiplicative function by Proposition 2.1 (1). □

Theorem 2.3. *If p is a prime and $\alpha \in \mathbb{N}_0$ then*

$$(\sigma_1^\dagger)^{-1}(p^\alpha) = \begin{cases} 1 & \alpha = 0, \\ -2 & p = 2, \alpha = 1, \\ 0 & p = 2, \alpha \geq 2, \\ -(1+p) & p = odd, \ \alpha = 1, \\ p & p = odd, \ \alpha = 2, \\ 0 & p = odd, \ \alpha \geq 3. \end{cases}$$

Proof. By Proposition 1.5,

$$\sigma_1^\dagger(n) = \begin{cases} \sigma_1(n) & \text{if } n \text{ is odd,} \\ \sigma_1(n) - \sigma_1(\frac{n}{2}) & \text{otherwise.} \end{cases}$$

If $n = 1$ then

$$(\sigma_1^\dagger)^{-1}(1) = \frac{1}{\sigma_1^\dagger(1)} = \frac{1}{\sigma_1(1)} = 1. \tag{2.1}$$

The proof will be divided into the case where p is 2 and the case where p is odd.

First, consider the case where p is 2. By (1.2),

$$\begin{aligned} (\sigma_1^\dagger)^{-1}(2) &= -\frac{1}{\sigma_1^\dagger(1)} \sum_{\substack{d|2 \\ d>1}} \sigma_1^\dagger(d)(\sigma_1^\dagger)^{-1}(2/d) \\ &= -\sigma_1^\dagger(2)(\sigma_1^\dagger)^{-1}(1) \\ &= -\sigma_1^\dagger(2) \\ &= -\sum_{\substack{d|2 \\ 2/d=odd}} d \\ &= -2 \end{aligned} \tag{2.2}$$

and

$$
\begin{aligned}
(\sigma_1^\dagger)^{-1}(2^2) =& -\frac{1}{\sigma_1^\dagger(1)}\sum_{\substack{d|2^2\\ d>1}}\sigma_1^\dagger(d)(\sigma_1^\dagger)^{-1}(2^2/d)\\
=& -(\sigma_1^\dagger(2)(\sigma_1^\dagger)^{-1}(2)+\sigma_1^\dagger(2^2)(\sigma_1^\dagger)^{-1}(1))\\
=& -(-2\sigma_1^\dagger(2)+\sigma_1^\dagger(2^2))\\
=&2\sigma_1^\dagger(2)-\sigma_1^\dagger(2^2)\\
=&0.
\end{aligned} \tag{2.3}
$$

We will use mathematical induction to show $(\sigma_1^\dagger)^{-1}(2^k)=0$ with $k\geq 2$.

We assume that

$$(\sigma_1^\dagger)^{-1}(2^l)=0 \tag{2.4}$$

with $2\leq l\leq k$.

Then, by (2.1), (2.2) and (2.4), we obtain

$$
\begin{aligned}
(\sigma_1^\dagger)^{-1}\left(2^{k+1}\right) =& -\frac{1}{\sigma_1^\dagger(1)}\sum_{\substack{d|2^{k+1}\\ d>1}}\sigma_1^\dagger(d)(\sigma_1^\dagger)^{-1}\left(2^{k+1}/d\right)\\
=& -(\sigma_1^\dagger(2)(\sigma_1^\dagger)^{-1}(2^k)+\sigma_1^\dagger(2^2)(\sigma_1^\dagger)^{-1}(2^{k-1})+\ldots+\sigma_1^\dagger(2^k)(\sigma_1^\dagger)^{-1}(2)+\sigma_1^\dagger(2^{k+1})(\sigma_1^\dagger)^{-1}(1))\\
=& -\left(\sigma_1^\dagger(2^k)(\sigma_1^\dagger)^{-1}(2)+\sigma_1^\dagger(2^{k+1})(\sigma_1^\dagger)^{-1}(1)\right)\\
=& -\left(2^k(-2)+2^{k+1}\right)\\
=&0.
\end{aligned}
$$

This completes the proof of case $n=2^k$ with $k\geq 1$.

Let p be an odd prime. Then, by (1.2) and (2.1),

$$
\begin{aligned}
(\sigma_1^\dagger)^{-1}(p) =& -\frac{1}{\sigma_1^\dagger(1)}\sum_{\substack{d|p\\ d>1}}\sigma_1^\dagger(d)(\sigma_1^\dagger)^{-1}(p/d)\\
=& -\left(\sigma_1^\dagger(p)(\sigma_1^\dagger)^{-1}(1)\right)\\
=& -\sigma_1(p)\\
=& -(1+p),
\end{aligned} \tag{2.5}
$$

$$
\begin{aligned}
(\sigma_1^\dagger)^{-1}(p^2) =& -\frac{1}{\sigma_1^\dagger(1)}\sum_{\substack{d|p^2\\ d>1}}\sigma_1^\dagger(d)(\sigma_1^\dagger)^{-1}(p^2/d)\\
=& -(\sigma_1^\dagger(p)(\sigma_1^\dagger)^{-1}(p)+(\sigma_1^\dagger(p^2)(\sigma_1^\dagger)^{-1}(1))\\
=& -\left(-(1+p)^2+\left(1+p+p^2\right)\right)\\
=&p
\end{aligned} \tag{2.6}
$$

and

$$
\begin{aligned}
(\sigma_1^\dagger)^{-1}(p^3) =& -\frac{1}{\sigma_1^\dagger(1)}\sum_{\substack{d|p^3\\ d>1}}\sigma_1^\dagger(d)(\sigma_1^\dagger)^{-1}(p^3/d)\\
=& -\left((1+p)p+\left(1+p+p^2\right)(-(1+p))+\left(1+p+p^2+p^3\right)\right)\\
=&0.
\end{aligned}
$$

We will use mathematical induction to show

$$(\sigma_1^{\dagger})^{-1}(p^{\alpha}) = 0 \tag{2.7}$$

with $\alpha \geq 3$.

Suppose that $(\sigma_1^{\dagger})^{-1}(p^{\alpha}) = 0$ with $3 \leq \alpha \leq k$.

Then, by (2.1), (2.5), (2.6) and (2.7), we obtain

$$\begin{aligned}
(\sigma_1^{\dagger})^{-1}(p^{k+1}) &= -\frac{1}{\sigma_1^{\dagger}(1)} \sum_{\substack{d|p^{k+1}\\ d>1}} \sigma_1^{\dagger}(d)(\sigma_1^{\dagger})^{-1}(p^{k+1}/d) \\
&= -\left(\sigma_1^{\dagger}(p)(\sigma_1^{\dagger})^{-1}(p^k) + \sigma_1^{\dagger}(p^2)(\sigma_1^{\dagger})^{-1}(p^{k-1}) + \ldots + \sigma_1^{\dagger}(p^{k+1})(\sigma_1^{\dagger})^{-1}(1)\right) \\
&= -\left(\sigma_1^{\dagger}(p^{k-1})(\sigma_1^{\dagger})^{-1}(p^2) + \sigma_1^{\dagger}(p^k)(\sigma_1^{\dagger})^{-1}(p) + \sigma_1^{\dagger}(p^{k+1})(\sigma_1^{\dagger})^{-1}(1)\right) \\
&= -\left(\left(1 + p + \ldots + p^{k-1}\right)p + \left(1 + p + \ldots + p^k\right)(-(1+p)) \right.\\
&\quad \left. + \left(1 + p + p^2 + \ldots + p^{k+1}\right)\right) \\
&= 0.
\end{aligned}$$

These complete the proof of Theorem 2.3. □

Theorem 2.4. *Let $n = p_1^{\alpha_1} \ldots p_t^{\alpha_t}$ with distinct odd primes p_i $(1 \leq i \leq t)$ and $\alpha_i \in \mathbb{N}$. Then*

$$(\sigma_1^{\dagger})^{-1}(n) = \begin{cases}
1 & n = 1, \\
-2 & n = 2, \\
(-1)^t \prod_{i=1}^t (p_i + 1) & n = p_1 \ldots p_t, \\
\prod_{i=1}^t p_i & n = p_1^2 \ldots p_t^2, \\
(-1)^{t-r} \left(\prod_{i=1}^r p_i\right)\left(\prod_{i=r+1}^t (p_i + 1)\right) & n = p_1^2 \ldots p_r^2 p_{r+1} \ldots p_t, \\
-2(-1)^t \prod_{i=1}^t (p_i + 1) & n = 2p_1 \ldots p_t, \\
-2 \prod_{i=1}^t p_i & n = 2p_1^2 \ldots p_t^2, \\
-2(-1)^{t-r} \left(\prod_{i=1}^r p_i\right)\left(\prod_{i=r+1}^t (p_i + 1)\right) & n = 2p_1^2 \ldots p_r^2 p_{r+1} \ldots p_t, \\
0 & \textit{otherwise}.
\end{cases}$$

Proof. Since $(\sigma_1^{\dagger})^{-1}$ is a multiplicative function,

$$\begin{aligned}
(\sigma_1^{\dagger})^{-1}(p_1 \ldots p_t) &= (\sigma_1^{\dagger})^{-1}(p_1) \ldots (\sigma_1^{\dagger})^{-1}(p_t) \\
&= (-(1 + p_1)) \ldots (-(1 + p_t)) \\
&= (-1)^t \prod_{i=1}^t (1 + p_i)
\end{aligned}$$

and

$$\begin{aligned}
(\sigma_1^{\dagger})^{-1}(p_1^2 \ldots p_t^2) &= (\sigma_1^{\dagger})^{-1}(p_1^2) \ldots (\sigma_1^{\dagger})^{-1}(p_t^2) \\
&= \prod_{i=1}^t p_i
\end{aligned}$$

by Theorem 2.3.

Similarly, by Theorem 2.3, we obtain

$$\begin{aligned}(\sigma_1^{\dagger})^{-1}(p_1^2\dots p_r^2 p_{r+1}\dots p_t) =&(\sigma_1^{\dagger})^{-1}(p_1^2)\dots(\sigma_1^{\dagger})^{-1}(p_r^2)(\sigma_1^{\dagger})^{-1}(p_{r+1})(\sigma_1^{\dagger})^{-1}(p_t)\\ =&(-1)^{t-r}\Big(\prod_{i=1}^{r}p_i\Big)\Big(\prod_{i=r+1}^{t}(p_i+1)\Big),\end{aligned}$$

$$\begin{aligned}(\sigma_1^{\dagger})^{-1}(2p_1\dots p_t) =&(\sigma_1^{\dagger})^{-1}(2)(\sigma_1^{\dagger})^{-1}(p_1)\dots(\sigma_1^{\dagger})^{-1}(p_t)\\ =&-2(-1)^t\prod_{i=1}^{t}(p_i+1),\end{aligned}$$

$$\begin{aligned}(\sigma_1^{\dagger})^{-1}(2p_1^2\dots p_t^2) =&(\sigma_1^{\dagger})^{-1}(2)(\sigma_1^{\dagger})^{-1}(p_1^2)\dots(\sigma_1^{\dagger})^{-1}(p_t^2)\\ =&-2\prod_{i=1}^{t}p_i\end{aligned}$$

and

$$\begin{aligned}(\sigma_1^{\dagger})^{-1}(2p_1^2\dots p_r^2 p_{r+1}\dots p_t) &= (\sigma_1^{\dagger})^{-1}(2)(\sigma_1^{\dagger})^{-1}(p_1^2)\cdots(\sigma_1^{\dagger})^{-1}(p_r^2)\\ &\quad\times(\sigma_1^{\dagger})^{-1}(p_{r+1})\cdots(\sigma_1^{\dagger})^{-1}(p_t)\\ &= (-2)(-1)^{t-r}\left(\prod_{i=1}^{r}p_i\right)\left(\prod_{i=r+1}^{t}p_i+1\right).\end{aligned}$$

Other cases are trivial. □

Lemma 2.5. *Let p be a prime and $\alpha \in \mathbb{N}_0$. Then*

$$(\sigma_3^{\dagger})^{-1}(p^{\alpha}) = \begin{cases}1 & \alpha = 0,\\ -2^3 & p=2, \alpha=1,\\ 0 & p=2, \alpha\geq 2,\\ -p^3-1 & p=odd,\ \alpha=1,\\ p^3 & p=odd,\ \alpha=2,\\ 0 & p=odd,\ \alpha\geq 3.\end{cases}$$

Proof. By Proposition 1.5,

$$\sigma_3^{\dagger}(n) = \begin{cases}\sigma_3(n) & \text{if } n \text{ is odd},\\ \sigma_3(n)-\sigma_3(\frac{n}{2}) & \text{otherwise.}\end{cases}$$

From (1.2), we obtain

$$(\sigma_3^{\dagger})^{-1}(1) = \frac{1}{\sigma_3^{\dagger}(1)} = \frac{1}{\left(\sum\limits_{\substack{d|1\\ 1/d:odd}} d^3\right)} = 1. \tag{2.8}$$

We prove it by dividing it into a case where p is 2 and an odd number of cases.

First, consider the case where p is 2 By (1.2),

$$\begin{aligned}(\sigma_3^\dagger)^{-1}(2) &= -\frac{1}{\sigma_3^\dagger(1)}\sum_{\substack{d|2\\ d>1}}\sigma_3^\dagger(d)(\sigma_3^\dagger)^{-1}(2/d)\\ &= -\frac{1}{\sigma_3^\dagger(1)}\sigma_3^\dagger(2)(\sigma_3^\dagger)^{-1}(1)\\ &= -\sigma_3^\dagger(2)\\ &= -\left(\sum_{\substack{d|2\\ 2/d:odd}} d^3\right)\\ &= -2^3 \end{aligned} \tag{2.9}$$

and

$$\begin{aligned}(\sigma_3^\dagger)^{-1}(2^2) &= -\frac{1}{\sigma_3^\dagger(1)}\sum_{\substack{d|2^2\\ d>1}}\sigma_3^\dagger(d)(\sigma_3^\dagger)^{-1}(2^2/d)\\ &= -(\sigma_3^\dagger(2)(\sigma_3^\dagger)^{-1}(2)+\sigma_3^\dagger(2^2)(\sigma_3^\dagger)^{-1}(1))\\ &= -\left(\sum_{\substack{d|2\\ 2/d=odd}} d^3(-2^3)+\sum_{\substack{d|2^2\\ 2^2/d=odd}} d^3\right)\\ &=0. \end{aligned} \tag{2.10}$$

We will use mathematical induction to show $(\sigma_3^\dagger)^{-1}(2^k)=0$ with $k\geq 2$.
We assume that

$$(\sigma_3^\dagger)^{-1}(2^l)=0 \tag{2.11}$$

with $2\leq l\leq k$.

Then, by (2.8), (2.9) and (2.11), we obtain

$$\begin{aligned}(\sigma_3^\dagger)^{-1}(2^{k+1}) &= -\frac{1}{\sigma_3^\dagger(1)}\sum_{\substack{d|2^{k+1}\\ d>1}}\sigma_3^\dagger(d)(\sigma_3^\dagger)^{-1}(2^{k+1}/d)\\ &= -(\sigma_3^\dagger(2)(\sigma_3^\dagger)^{-1}(2^k)+\ldots+\sigma_3^\dagger(2^k)(\sigma_3^\dagger)^{-1}(2)+\sigma_3^\dagger(2^{k+1})(\sigma_3^\dagger)^{-1}(1))\\ &= -(\sigma_3^\dagger(2^k)(\sigma_3^\dagger)^{-1}(2)+\sigma_3^\dagger(2^{k+1}))\\ &= -\left(\sum_{\substack{d|2^k\\ 2^k/d=odd}} d^3(-2^3)+\sum_{\substack{d|2^{k+1}\\ 2^{k+1}/d=odd}} d^3\right)\\ &= -(-2^3\cdot 2^{3k}+(2^{k+1})^3)\\ &=0. \end{aligned}$$

This completes the proof of case $n=2^k$ with $k\geq 1$.

Let p be an odd prime and $t\in\mathbb{N}_0$. Then, by Proposition 1.5,

$$\sigma_3^\dagger(p^t)=\sigma_3(p^t)-\sigma_3\left(\frac{p^t}{2}\right)=\sigma_3(p^t).$$

Therefore, we obtain

$$(\sigma_3^\dagger)^{-1}(p^t) = (\sigma_3)^{-1}(p^t). \tag{2.12}$$

Also, by Proposition 1.3,

$$\sigma_3(n) = \sum_{d|n} d^3 = (\zeta_3 * \zeta)(n).$$

Since ζ_k is a completely multiplicative function,

$$\zeta_3^{-1} = \zeta_3\mu$$

and

$$\zeta^{-1} = \zeta\mu$$

by Proposition 1.1. Hence,

$$\zeta_3^{-1} * \sigma_3 * \zeta^{-1} = \zeta_3^{-1} * (\zeta_3 * \zeta) * \zeta^{-1} = \delta$$

and

$$(\zeta_3\mu) * \sigma_3 * (\zeta_0\mu) = \delta. \tag{2.13}$$

Using (2.13) and the fact Dirichlet convolution $*$ is commutative, we obtain

$$\sigma_3^{-1} = (\zeta_3\mu) * (\zeta_0\mu) = (\mu\zeta_3) * \mu. \tag{2.14}$$

Now let's examine the value of $\sigma_3^{-1}(p^\alpha)$.

It is easily obtained that

$$\sigma_3^{-1}(1) = (\mu\zeta_3 * \mu)(1) = 1,$$

$$\sigma_3^{-1}(p) = (\mu\zeta_3 * \mu)(p) = \sum_{d|p} \mu\zeta_3(d)\mu(p/d) = -1 - p^3$$

and

$$\sigma_3^{-1}\left(p^2\right) = (\mu\zeta_3 * \mu)\left(p^2\right) = \sum_{d|p^2} \mu\zeta_3(d)\mu\left(p^2/d\right) = p^3.$$

If $\alpha \geq 3$ then

$$\begin{aligned}
\sigma_3^{-1}(p^\alpha) =& (\mu\zeta_3 * \mu)(p^\alpha) \\
=& \sum_{d|p^\alpha} \mu(d)\zeta_3(d)\mu(p^\alpha/d) \\
=& \mu(1)\zeta_3(1)\mu(p^\alpha) + \ldots + \mu(p^{\alpha-1})\zeta_3(p^{\alpha-1})\mu(p) \\
=& 0.
\end{aligned}$$

Therefore, Lemma 2.5 is proved. □

By Proposition 1.4 and 2.1, $(\sigma_3^\dagger)^{-1}$ is a multiplicative function.

So, by Lemma 2.5, we obtain Theorem 2.6.

Theorem 2.6. *Let $n = p_1^{\alpha_1} \ldots p_t^{\alpha_t}$ be a positive integer with distinct odd primes p_i $(1 \leq i \leq t)$ and $\alpha_i \in \mathbb{N}$. Then*

$$(\sigma_3^\dagger)^{-1}(n) = \begin{cases}
1 & n = 1, \\
-2^3 & n = 2, \\
(-1)^t \prod_{i=1}^t (p_i^3 + 1) & n = p_1 \ldots p_t, \\
\prod_{i=1}^t p_i^3 & n = p_1^2 \ldots p_t^2, \\
(-1)^{t-r} \prod_{i=1}^r p_i^3 \prod_{i=r+1}^t (p_i^3 + 1) & n = p_1^2 \ldots p_r^2 p_{r+1} \ldots p_t, \\
(-2)^3(-1)^t \prod_{i=1}^t (p_i^3 + 1) & n = 2p_1 \ldots p_t, \\
(-2)^3 \prod_{i=1}^t p_i^3 & n = 2p_1^2 \ldots p_t^2, \\
(-2)^3(-1)^{t-r} \prod_{i=1}^r p_i^3 \prod_{i=r+1}^t (p_i^3 + 1) & n = 2p_1^2 \ldots p_r^2 p_{r+1} \ldots p_t, \\
0 & n = otherwise.
\end{cases}$$

It is possible to obtain the $(\sigma_1\sigma_3)^{-1}$ function directly, but it is a little easier to obtain the $(\sigma_1\sigma_3)^{-1}$ function using the proposition below. Therefore, we state the following proposition below.

Proposition 2.7 (*cf.* [12, p. 621])**.** *Let f_i and g_i be completely multiplicative functions for $i = 1, 2$. Then*

$$f_1 f_2 * f_1 g_2 * f_2 g_1 * g_1 g_2 = (f_1 * g_1)(f_2 * g_2) * u,$$

where

$$u(n) = \begin{cases} f_1 f_2 g_1 g_2 \left(n^{1/2}\right) & \text{if } n \text{ is square,} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.8. *Let p be a prime number, $t \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}$. Then*

$$(\sigma_1\sigma_3)^{-1}(p^t) = \begin{cases} 1 & t = 0, \\ -\sigma_1(p)\sigma_3(p) & t = 1, \\ p\left(\sigma_2(p^3) + 3p^3\right) & t = 2, \\ -2p^{4\alpha}\sigma_1(p)\sigma_3(p) & t = 2\alpha + 1,\ \alpha \geq 1, \\ p^{4\alpha+1}\left(\sigma_2(p^3) + 4p^3\right) & t = 2\alpha + 2,\ \alpha \geq 1. \end{cases}$$

Proof. If $t = 0$ then

$$(\sigma_1\sigma_3)^{-1}(1) = \frac{1}{\sigma_1(1)\sigma_3(1)} = 1 \tag{2.15}$$

by (1.2).

It is well known that $\zeta_k(n) = n^k$ is a completely multiplicative function with $k \in \mathbb{N}_0$. Thus, we derive from (2.7) that

$$(\sigma_1\sigma_3) * v = (\zeta_1 * \zeta_0)(\zeta_3 * \zeta_0) * v = \zeta_4 * \zeta_1 * \zeta_3 * \zeta_0.$$

Here,

$$v(n) = \begin{cases} \zeta_4(n^{1/2}) & \text{if } n \text{ is square,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, by Proposition 1.1,

$$(\sigma_1\sigma_3)^{-1} = \zeta_4\mu * \zeta_1\mu * \zeta_3\mu * \zeta_0\mu * v = \zeta_4\mu * \zeta_1\mu * \zeta_3\mu * \mu * v.$$

It is easily checked that

$$(\zeta_1\mu * \zeta_3\mu)\,(p^\alpha) = \begin{cases} -\left(p + p^3\right) & if\ \alpha = 1, \\ p^4 & if\ \alpha = 2, \\ 0 & if\ \alpha \geq 3 \end{cases}$$

and

$$(\mu * v)(p^\alpha) = \begin{cases} p^{2\alpha} & if\ \alpha\ is\ even, \\ -p^{2(\alpha-1)} & if\ \alpha\ is\ odd. \end{cases}$$

Using these, we obtain

$$\begin{aligned} ((\zeta_1\mu * \zeta_3\mu) * (\mu * v))(p) &= \sum_{d|p} (\zeta_1\mu * \zeta_3\mu)(d)(\mu * v)(p/d) \\ &= (\zeta_1\mu * \zeta_3\mu)(1)(\mu * v)(p) + (\zeta_1\mu * \zeta_3\mu)(p)(\mu * v)(1) \\ &= -1 - \left(p + p^3\right) \\ &= -1 - p - p^3 \\ &= -\left(1 + p + p^3\right), \end{aligned}$$

$$
\begin{aligned}
((\zeta_1\mu * \zeta_3\mu) * (\mu * \nu))(p^{2\alpha}) = &\sum_{d|p^{2\alpha}} (\zeta_1\mu * \zeta_3\mu)(d)(\mu * \nu)(p^{2\alpha}/d) \\
=&(\zeta_1\mu * \zeta_3\mu)(1)(\mu * \nu)\left(p^{2\alpha}\right) + (\zeta_1\mu * \zeta_3\mu)(p)(\mu * \nu)\left(p^{2\alpha-1}\right) \\
&+ (\zeta_1\mu * \zeta_3\mu)\left(p^2\right)(\mu * \nu)\left(p^{2\alpha-2}\right) \\
=&p^{4\alpha} + \left(p + p^3\right)p^{4(\alpha-1)} + p^4 p^{4(\alpha-1)} \\
=&2p^{4\alpha} + p^{4\alpha-3} + p^{4\alpha-1}
\end{aligned}
$$

and

$$
\begin{aligned}
((\zeta_1\mu * \zeta_3\mu) * (\mu * \nu))(p^{2\alpha+1}) = &\sum_{d|p^{2\alpha+1}} (\zeta_1\mu * \zeta_3\mu)(d)(\mu * \nu)(p^{2\alpha+1}/d) \\
=&(\zeta_1\mu * \zeta_3\mu)(1)(\mu * \nu)\left(p^{2\alpha+1}\right) + (\zeta_1\mu * \zeta_3\mu)(p)(\mu * \nu)\left(p^{2\alpha}\right) \\
&+ (\zeta_1\mu * \zeta_3\mu)\left(p^2\right)(\mu * \nu)\left(p^{2\alpha-1}\right) \\
=&-p^{4\alpha} - \left(p + p^3\right)p^{4\alpha} - p^4 p^{4(\alpha-1)} \\
=&-2p^{4\alpha} - p^{4\alpha+1} - p^{4\alpha+3}
\end{aligned}
$$

with $\alpha \geq 1$.

Let $A := \zeta_1\mu * \zeta_3\mu * \mu * \nu$.

Then

$$
A(p^t) = \begin{cases} -(1 + p + p^3) & t = 1, \\ 2p^{4\alpha} + p^{4\alpha-3} + p^{4\alpha-1} & t = 2\alpha\ (\alpha \geq 1), \\ -2p^{4\alpha} - p^{4\alpha+1} - p^{4\alpha+3} & t = 2\alpha + 1\ (\alpha \geq 1). \end{cases}
$$

Therefore, for $\alpha \geq 1$

$$
\begin{aligned}
\zeta_4\mu * A(p) = &\sum_{d|p} \zeta_4\mu(d)A(p/d) \\
=&\zeta_4\mu(1)A(p) + \zeta_4\mu(p)A(1) \\
=&-\left(1 + p + p^3\right) - p^4 \\
=&-\sigma_1(p)\sigma_3(p),
\end{aligned}
$$

$$
\begin{aligned}
\zeta_4\mu * A(p^2) = &\sum_{d|p^2} \zeta_4\mu(d)A(p^2/d) \\
=&\zeta_4\mu(1)A\left(p^2\right) + \zeta_4\mu(p)A(p) \\
=&A\left(p^2\right) - p^4 A(p) \\
=&2p^4 + p + p^3 + p^4\left(1 + p + p^3\right) \\
=&p\left(3p^3 + 1 + p^2 + p^4 + p^6\right) \\
=&p\left(\sigma_2(p^3) + 3p^3\right),
\end{aligned}
$$

$$\begin{aligned}
\zeta_4\mu * A\left(p^{2\alpha+1}\right) &= \sum_{d|p^{2\alpha+1}} \zeta_4\mu(d)A\left(p^{2\alpha+1}/d\right) \\
&= \zeta_4\mu(1)A\left(p^{2\alpha+1}\right) + \zeta_4\mu(p)A\left(p^{2\alpha}\right) \\
&= A\left(p^{2\alpha+1}\right) - p^4 A\left(p^{2\alpha}\right) \\
&= -2p^{4\alpha} - p^{4\alpha+1} - p^{4\alpha+3} - p^4\left(2p^{4\alpha} + p^{4\alpha-3} + p^{4\alpha-1}\right) \\
&= -2p^{4\alpha} - p^{4\alpha+1} - p^{4\alpha+3} - 2p^{4\alpha+4} - p^{4\alpha+1} - p^{4\alpha+3} \\
&= -2p^{4\alpha}(p^4 + p^3 + p + 1) \\
&= -2p^4\sigma_1(p)\sigma_3(p)
\end{aligned}$$

and

$$\begin{aligned}
\zeta_4\mu * A(p^{2\alpha+2}) &= \sum_{d|p^{2\alpha+2}} \zeta_4\mu(d)A(p^{2\alpha+2}/d) \\
&= \zeta_4\mu(1)A\left(p^{2\alpha+2}\right) + \zeta_4\mu(p)A\left(p^{2\alpha+1}\right) \\
&= A\left(p^{2\alpha+2}\right) - p^4 A\left(p^{2\alpha+1}\right) \\
&= 2p^{4(\alpha+1)} + p^{4(\alpha+1)-3} + p^{4(\alpha+1)-1} + p^4\left(2p^{4\alpha} + p^{4\alpha+1} + p^{4\alpha+3}\right) \\
&= 2p^{4(\alpha+1)} + p^{4\alpha+1} + p^{4\alpha+3} + 2p^{4\alpha+4} + p^{4\alpha+5} + p^{4\alpha+7} \\
&= p^{4\alpha+1}\left(4p^3 + 1 + p^2 + p^4 + p^6\right) \\
&= p^{4\alpha+1}\left(4p^3 + \sigma_2(p^3)\right).
\end{aligned}$$

These complete the proof of Lemma 2.8. □

Proposition 2.9. *$(\sigma_1\sigma_3)$ and $(\sigma_1\sigma_3)^{-1}$ are multiplicative functions.*

Proof. We know that σ_1 and σ_3 are multiplicative functions by Proposition 1.4. Then $\sigma_1\sigma_3$ is a multiplicative function. Therefore, $(\sigma_1\sigma_3)^{-1}$ is a multiplicative function by Proposition 2.1. □

Theorem 2.10. *The followings hold true:*

- ***(1)*** $(\sigma_1\sigma_3)^{-1}(n) = 1.$
- ***(2)*** *If p_i $(1 \le i \le m)$ is distinct prime integers and k_i $(t+1 \le i \le m)$ are positive integers then*

$$\begin{aligned}
&(\sigma_1\sigma_3)^{-1}\left(p_1 \dots p_r p_{r+1}^2 \dots p_t^2 p_{t+1}^{2k_{t+1}+1} \dots p_\delta^{2k_\delta+1} p_{\delta+1}^{2k_{\delta+1}+2} \dots p_m^{2k_m+2}\right) \\
&= (-1)^r(-2)^{\delta-t} \prod_{i=1}^{r}\left(1 + p_i + p_i^3 + p_i^4\right) \prod_{i=r+1}^{t} p_i\left(1 + p_i^2 + 3p_i^3 + p_i^4 + p_i^6\right) \\
&\quad \times \prod_{i=t+1}^{\delta} p_i^{4k_i}\left(1 + p_i + p_i^3 + p_i^4\right) \prod_{i=\delta+1}^{m} p_i^{4k_i+1}\left(1 + p_i^2 + 4p_i^3 + p_i^4 + p_i^6\right).
\end{aligned}$$

Proof. By (1.2),

$$(\sigma_1\sigma_3)^{-1}(1) = \frac{1}{(\sigma_1\sigma_3)(1)} = 1.$$

Recall that $(\sigma_1\sigma_3)^{-1}$ is a multiplicative function by Proposition 2.9.

So, we obtain

$$\begin{aligned}(\sigma_1\sigma_3)^{-1}(p_1\dots p_r) =&(\sigma_1\sigma_3)^{-1}(p_1)\dots(\sigma_1\sigma_3)^{-1}(p_r)\\ =&(-\sigma_1(p_1)\sigma_3(p_1))\dots(-\sigma_1(p_r)\sigma_3(p_r))\\ =&(-1)^r\prod_{i=1}^{r}\sigma_1(p_i)\sigma_3(p_i)\\ =&(-1)^r\prod_{i=1}^{r}\left(1+p_i+p_i^3+p_i^4\right),\end{aligned}$$

$$\begin{aligned}(\sigma_1\sigma_3)^{-1}\left(p_{r+1}^2\dots p_t^2\right) =&(\sigma_1\sigma_3)^{-1}(p_{r+1}^2)\dots(\sigma_1\sigma_3)^{-1}(p_{r+1}^2)\\ =&\prod_{i=r+1}^{t}p_i\left(\sigma_2(p_i^3)+3p_i^3\right)\\ =&\prod_{i=r+1}^{t}p_i\left(1+p_i^2+3p_i^3+p_i^4+p_i^6\right),\end{aligned}$$

$$\begin{aligned}(\sigma_1\sigma_3)^{-1}\left(p_{t+1}^{2k_{t+1}+1}\dots p_\delta^{2k_\delta+1}\right) =&(\sigma_1\sigma_3)^{-1}\left(p_{t+1}^{2k_{t+1}+1}\right)\dots(\sigma_1\sigma_3)^{-1}\left(p_\delta^{2k_\delta+1}\right)\\ =&(-2)^{\delta-t}\prod_{i=t+1}^{\delta}p_i^{4k_i}\sigma_1(p_i)\sigma_3(p_i)\\ =&(-2)^{\delta-t}\prod_{i=t+1}^{\delta}p_i^{4k_i}\left(1+p_i+p_i^3+p_i^4\right)\end{aligned}$$

and

$$\begin{aligned}(\sigma_1\sigma_3)^{-1}\left(p_{\delta+1}^{2k_{\delta+1}+2}\dots p_m^{2k_m+2}\right) =&(\sigma_1\sigma_3)^{-1}\left(p_{\delta+1}^{2k_{\delta+1}+2}\right)\dots(\sigma_1\sigma_3)^{-1}\left(p_m^{2k_m+2}\right)\\ =&\prod_{i=\delta+1}^{m}p_i^{4k_i+1}\left(\sigma_2(p_i^3)+4p_i^3\right)\\ =&\prod_{i=\delta+1}^{m}p_i^{4k_i+1}\left(1+p_i^2+4p_i^3+p_i^4+p_i^6\right).\end{aligned}$$

Then

$$\begin{aligned}&(\sigma_1\sigma_3)^{-1}\left(p_1\dots p_r p_{r+1}^2\dots p_t^2 p_{t+1}^{2k_{t+1}+1}\dots p_\delta^{2k_\delta+1}p_{\delta+1}^{2k_{\delta+1}+2}\dots p_m^{2k_m+2}\right)\\ =\;&(\sigma_1\sigma_3)^{-1}(p_1\dots p_r)(\sigma_1\sigma_3)^{-1}\left(p_{r+1}^2\dots p_t^2\right)(\sigma_1\sigma_3)^{-1}\left(p_{t+1}^{2k_{t+1}+1}\dots p_\delta^{2k_\delta+1}\right)(\sigma_1\sigma_3)^{-1}\left(p_{\delta+1}^{2k_{\delta+1}+2}\dots p_m^{2k_m+2}\right)\\ =\;&(-1)^r(-2)^{\delta-t}\prod_{i=1}^{r}\left(1+p_i+p_i^3+p_i^4\right)\prod_{i=r+1}^{t}p_i\left(1+p_i^2+3p_i^3+p_i^4+p_i^6\right)\\ &\times\prod_{i=t+1}^{\delta}p_i^{4k_i}\left(1+p_i+p_i^3+p_i^4\right)\prod_{i=\delta+1}^{m}p_i^{4k_i+1}\left(1+p_i^2+4p_i^3+p_i^4+p_i^6\right).\end{aligned}$$

These complete the proof of Theorem 2.10. □

Lemma 2.11. *Let p be a prime number, $t \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}$. Then*

$$(\sigma_1^\dagger\sigma_3^\dagger)^{-1}(p^t) = \begin{cases} 1 & t = 0, \\ -2^4 & p = 2 \text{ and } t = 1, \\ 0 & p = 2 \text{ and } t \geq 2, \\ -\sigma_1(p)\sigma_3(p) & p = \text{odd prime and } t = 1, \\ p(\sigma_2(p^3) + 3p^3) & p = \text{odd prime and } t = 2, \\ -2p^{4\alpha}\sigma_1(p)\sigma_3(p) & p = \text{odd prime and } t = 2\alpha + 1,\ \alpha \geq 1, \\ p^{4\alpha+1}(\sigma_2(p^3) + 4p^3) & p = \text{odd prime and } t = 2\alpha + 2,\ \alpha \geq 1. \end{cases}$$

Proof. If $t = 0$, then

$$(\sigma_1^\dagger\sigma_3^\dagger)^{-1}(1) = \frac{1}{(\sigma_1^\dagger\sigma_3^\dagger)(1)} = \frac{1}{\left(\sum\limits_{\substack{d|1 \\ 1/d=odd}} d\right)\left(\sum\limits_{\substack{d|1 \\ 1/d=odd}} d^3\right)} = 1 \tag{2.16}$$

by (1.2).
Also, if $p = 2$ then

$$\begin{aligned}(\sigma_1^\dagger\sigma_3^\dagger)^{-1}(2) &= \frac{1}{(\sigma_1^\dagger\sigma_3^\dagger)(1)} \sum_{\substack{d|2 \\ d>1}} (\sigma_1^\dagger\sigma_3^\dagger)(d)(\sigma_1^\dagger\sigma_3^\dagger)^{-1}(2/d) \\ &= -(\sigma_1^\dagger\sigma_3^\dagger)(2)(\sigma_1^\dagger\sigma_3^\dagger)^{-1}(1) \\ &= -2^4 \end{aligned} \tag{2.17}$$

and

$$\begin{aligned}(\sigma_1^\dagger\sigma_3^\dagger)^{-1}(2^2) &= \frac{1}{\sigma_1^\dagger\sigma_3^\dagger(1)} \sum_{\substack{d|2^2 \\ d>1}} \sigma_1^\dagger\sigma_3^\dagger(d)(\sigma_1^\dagger\sigma_3^\dagger)^{-1}(2^2/d) \\ &= -\left(\sigma_1^\dagger\sigma_3^\dagger(2)(\sigma_1^\dagger\sigma_3^\dagger)^{-1}(2) + \sigma_1^\dagger\sigma_3^\dagger(2^2)\right) \\ &= 0 \end{aligned}$$

by (1.2).

We will use mathematical induction to show

$$(\sigma_1^\dagger\sigma_3^\dagger)^{-1}(2^t) = 0$$

with $t \geq 2$.

We assume that

$$(\sigma_1^\dagger\sigma_3^\dagger)^{-1}(2^l) = 0 \tag{2.18}$$

with $2 \leq l \leq t$.

Then, by (2.16), (2.17) and (2.18), we obtain

$$
\begin{aligned}
(\sigma_1^\dagger\sigma_3^\dagger)^{-1}(2^{t+1}) &= -\frac{1}{(\sigma_1^\dagger\sigma_3^\dagger)(1)}\sum_{\substack{d|2^{t+1}\\ d>1}}(\sigma_1^\dagger\sigma_3^\dagger)(d)(\sigma_1^\dagger\sigma_3^\dagger)^{-1}(2^{t+1}/d)\\
&= -\Big((\sigma_1^\dagger\sigma_3^\dagger)(2^t)(\sigma_1^\dagger\sigma_3^\dagger)^{-1}(2) + (\sigma_1^\dagger\sigma_3^\dagger)(2^{t+1})(\sigma_1^\dagger\sigma_3^\dagger)^{-1}(1)\Big)\\
&= -\Big((\sigma_1^\dagger\sigma_3^\dagger)(2^t)\cdot(-2^4) + (\sigma_1^\dagger\sigma_3^\dagger)(2^{t+1})\Big)\\
&= -\Big(\Big(\sum_{\substack{d|2^t\\ 2^t/d=odd}} d \sum_{\substack{d|2^t\\ 2^t/d=odd}} d^3\Big)(-2^4) + \Big(\sum_{\substack{d|2^{t+1}\\ 2^{t+1}/d=odd}} d \sum_{\substack{d|2^{t+1}\\ 2^{t+1}/d=odd}} d^3\Big)\Big)\\
&=0.
\end{aligned}\tag{2.19}
$$

Therefore, $(\sigma_1^\dagger\sigma_3^\dagger)^{-1}(2^t) = 0$ with $t \geq 2$. This completes the proof of case $p = 2$.

Now consider the odd prime number p. Then

$$
\begin{aligned}
(\sigma_1^\dagger\sigma_3^\dagger)(p^t) &= \Big(\sum_{\substack{d|p^t\\ p^t/d=odd}} d\Big)\Big(\sum_{\substack{d|p^t\\ p^t/d=odd}} d^3\Big)\\
&= \Big(\sum_{d|p^t} d\Big)\Big(\sum_{d|p^t} d^3\Big)\\
&= (\sigma_1\sigma_3)(p^t).
\end{aligned}\tag{2.20}
$$

So, for prime numbers when p is odd, the result can be obtained using Lemma 2.8. Therefore, the proof of Lemma 2.11 is completed. □

By Lemma 2.11, we obtain the following theorem below.

Theorem 2.12. *Let p_i $(1 \leq i \leq r)$ be distinct odd primes. Then we have*

(1) $(\sigma_1^\dagger\sigma_3^\dagger)^{-1}(1) = 1$.

(2) $(\sigma_1^\dagger\sigma_3^\dagger)^{-1}(2) = -2^4$.

(3) *If $n_1 = p_1 \cdots p_r$ is a positive integer then* $(\sigma_1^\dagger\sigma_3^\dagger)^{-1}(n_1) = (-1)^r(\sigma_1(n_1)\sigma_3(n_1)$.

(4) *If $n_2 = n_1^2 = (p_1 \cdots p_r)^2$ is a positive integer then*

$$(\sigma_1^\dagger\sigma_3^\dagger)^{-1}(n_2) = n_1\prod_{i=1}^{r}(1 + p_i^2 + 3p_i^3 + p_i^4 + p_i^6).$$

(5) *If $n_3 = (p_1^{2\alpha_1+2} \cdots p_r)^2$ is a positive integer then*

$$(\sigma_1^\dagger\sigma_3^\dagger)^{-1}(n_2) = n_1\prod_{i=1}^{r}(1 + p_i^2 + 3p_i^3 + p_i^4 + p_i^6).$$

$$
(\sigma_1^{\dagger}\sigma_3^{\dagger})^{-1}(n) = \begin{cases} 1 & if\ n = 1, \\ -2^4 & if\ n = 2, \\ (-1)^r(-2)^{\delta-t}\prod_{i=1}^{r}(1+p_i+p_i^3+p_i^4) & \\ \prod_{i=r+1}^{t} p_i(1+p_i^2+3p_i^3+p_i^4+p_i^6) & if\ n = p_1\dots p_r p_{r+1}^2\dots p_t^2 \\ \prod_{i=t+1}^{\delta} p_i^{4k_i}(1+p_i+p_i^3+p_i^4) & p_{t+1}^{2k_{t+1}+1}\dots p_\delta^{2k_\delta+1} p_{\delta+1}^{2k_{\delta+1}+2}\dots p_m^{2k_m+2} \\ \prod_{i=\delta+1}^{m} p_i^{4k_i+1}(1+p_i^2+4p_i^3+p_i^4+p_i^6) & \\ (-1)^r(-2)^{4+\delta-t}\prod_{i=1}^{r}(1+p_i+p_i^3+p_i^4) & \\ \prod_{i=r+1}^{t} p_i(1+p_i^2+3p_i^3+p_i^4+p_i^6) & if\ n = 2p_1\dots p_r p_{r+1}^2\dots p_t^2 \\ \prod_{i=t+1}^{\delta} p_i^{4k_i}(1+p_i+p_i^3+p_i^4) & p_{t+1}^{2k_{t+1}+1}\dots p_\delta^{2k_\delta+1} p_{\delta+1}^{2k_{\delta+1}+2}\dots p_m^{2k_m+2} \\ \prod_{i=\delta+1}^{m} p_i^{4k_i+1}(1+p_i^2+4p_i^3+p_i^4+p_i^6) & \\ 0 & if\ n = otherwise. \end{cases}
$$

3. Conclusion

There are many well-known results about the convolution sums of restricted functions. However, it is well known that it is somewhat difficult to obtain a formula for the convolution sums of divisor functions under the condition of relatively prime. The Dirichlet convolution is used to obtain these formulas. What we need is the result of the inverse of the restricted divisor functions. This article gives these formulae, which are meaningful and effective for finding the convolution sums of various restricted divisor functions. It is expected that various results will be derived in the future using this article.

Acknowledgments

This paper is dedicated to Professor Yilmaz Simsek on the occasion of his 60th Anniversary.
The authors are thankful to the referees and the editors for their careful corrections, suggestions and valuable comments for the improvement of the manuscript.

Author Contributions: All authors are equally contributed.

Conflict of Interest: There are no conflict of interest.

Funding (Financial Disclosure): This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (No. 2021R1F1A1051093).

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