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New generating functions and formulas for Bernstein-Stancu basis functions and their applications

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Abstract

One of the main aims of this article is to construct new generating functions for the Bernstein-Stancu basis functions, with the help of hypergeometric series, the descending factorial polynomials, and the ascending factorial polynomials. Another main aims is to reveal novel definitions and formulas of the Bernstein-Stancu basis functions with the help of the Euler gamma function and Beta functions. Furthermore, by blending the higher order Bernoulli polynomials, the Lah numbers and the Stirling numbers represented by the descending factorial polynomials, and the ascending factorial polynomials, we derive many new identities and relations of the Bernstein-Stancu basis functions. Finally, recurrence relations, derivative formulas for the Bernstein-Stancu basis functions are also given. Finally, we give Bèzier-type curves in terms of control points and the generalized Bernstein-Stancu basis functions.

Keywords: Generating functions, gamma and beta functions, Bernoulli numbers and polynomials, special sequences and polynomials, recurrences, curves, factorials

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1. Introduction

In recent years, the Bernstein polynomials have been used most effectively in important areas such as approximation theory, construction of the Bèzier curves and B-spline curves, probability distribution functions, computer geometric design, etc. These polynomials also have applications, especially in mathematical modeling and differential equations. Therefore, many different families of Bernstein-type polynomials have still been studied and investigated. Some of these can be given as the Schurer polynomials, the Kantorovich polynomials, the Stancu polynomials, the q-Bernstein polynomials, the Durrmeyer polynomials, the Favard-Szasz-Mirakjan operators, the Baskakov operators, and the Balazs-Szabados operators, etc. (*cf.* [1]-[35]).

In [35], Simsek and Acikgoz firstly constructed generating functions for the q-Bernstein bases functions. In [1], Araci and Acikgoz gave generating function for the Bernstein bases functions. During the years following these studies, the author concentrated his work on the generating functions for the Bernstein of base functions. The author gave many generalizations and applications of these functions (*cf.* [23]-[32]).

The motivation of this paper is to investigate some fundamental properties of the generalized Bernstein-Stancu basis functions. It is to construct the generating functions of these functions in terms of hypergeometric functions, descending, and ascending factorial polynomials. It is to investigate relations among these functions, the special

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numbers, special polynomials, and special functions. By using generating functions and their functional and derivative equations, many novel formulas involving these bases functions, special numbers and polynomials are derived.

We can use the following definitions and notations in the following parts of this paper:

Let $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, real numbers, and complex numbers, respectively.

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We [32] defined the multidimensional unification of the Bernstein basis functions $\mathcal{G}_n\left(\overrightarrow{x}; \overrightarrow{b}, \overrightarrow{s}, \overrightarrow{k}\right)$ as follows:

$$\begin{aligned} \mathcal{G}_n\left(\overrightarrow{x}; \overrightarrow{b}, \overrightarrow{s}, \overrightarrow{k}\right) &= \binom{n}{b_1s_1 - k_1, b_2s_2 - k_2, \ldots, n - \sum\limits_{j=1}^{m-1}\left(b_js_j - k_j\right)} \\ &\quad \times 2^{\sum\limits_{j=1}^{m} b_j\left(1-s_j\right)} \prod_{v=1}^{m}\left(x_v^{b_vs_v-k_v}\right)\left(1 - \sum_{j=1}^{m} x_j\right)^{n-\sum\limits_{j=1}^{m}\left(b_js_j-k_j\right)}, \end{aligned} \tag{1.1}$$

where $\overrightarrow{x} = (x_1, x_2, \ldots, x_m)$; $x_j \in [0, 1]$, $\overrightarrow{b} = (b_1, b_2, \ldots, b_m)$, $\overrightarrow{s} = (s_1, s_2, \ldots, s_m)$ and $\overrightarrow{k} = (k_1, k_2, \ldots, k_m)$ with $b_j, s_j, k_j \in \mathbb{N}_0$, $j = 1, \ldots, m$ and

$$\sum_{j=1}^{m}\left(b_js_j - k_j\right) \leq n$$

and also

$$\binom{n}{b_1s_1 - k_1, b_2s_2 - k_2, \ldots, b_ms_m - k_m} = \frac{n!}{\prod\limits_{v=1}^{m}(b_vs_v - k_v)!\left(n - \sum\limits_{j=1}^{m}\left(b_js_j - k_j\right)\right)!}.$$

Generating functions for the $\mathcal{G}_n\left(\overrightarrow{x}; \overrightarrow{b}, \overrightarrow{s}, \overrightarrow{k}\right)$ are given by

$$\begin{aligned} F_{\mathcal{B}}\left(t, \overrightarrow{x}; \overrightarrow{b}, \overrightarrow{s}, \overrightarrow{k}\right) &= \sum_{n=0}^{\infty} \mathcal{G}_n\left(\overrightarrow{x}; \overrightarrow{b}, \overrightarrow{s}, \overrightarrow{k}\right)\frac{t^n}{n!} \\ &= \frac{2^{\sum\limits_{j=1}^{m}\left(b_j-k_j\right)} \prod\limits_{v=1}^{m} x_v^{b_vs_v-k_v}\left(\frac{t}{2}\right)^{\sum\limits_{j=1}^{m} b_js_j-k_j}}{\prod\limits_{j=1}^{m}\left(b_js_j - k_j\right)!} e^{t\left(1-\sum\limits_{j=1}^{m} x_j\right)}, \end{aligned} \tag{1.2}$$

(*cf.* [32]).

Note that there is one generating function for each value of $\overrightarrow{b}$, $\overrightarrow{s}$ and $\overrightarrow{k}$ (*cf.* [32]).

Remark 1.1. Substituting $\overrightarrow{s} = \overrightarrow{1} = (1, 1, \ldots, 1)$ and $\overrightarrow{k} = \overrightarrow{0} = (0, 0, \ldots, 0)$ into (1.1), it is easy to see that

$$\begin{aligned} \mathcal{G}_n\left(\overrightarrow{x}; \overrightarrow{b}, \overrightarrow{1}, \overrightarrow{0}\right) &= \binom{n}{b_1, b_2, \ldots, b_m}\prod_{v=1}^{m} x_v^{b_v}\left(1 - \sum_{j=1}^{m} x_j\right)^{n-\sum\limits_{j=1}^{m} b_j} \\ &= P^{n}_{b_1,b_2,\ldots,b_m}(x_1, x_2, \ldots, x_m), \end{aligned}$$

where

$$\binom{n}{b_1, b_2, \ldots, b_m} = \frac{n!}{b_1!b_2!\ldots b_m!\left(n - \sum\limits_{j=1}^{m} b_j\right)!}$$

(*cf.* [18, p. 51, Eq. (13)], [2, 17]).

Remark 1.2. By using generating functions for the multidimensional of Bernstein basis functions (1.1), the Hausdorff distance and machine learning algorithm, Kucukoglu et al. [17] presented very interesting open problems as well as giving an application for a real-world problem involving human facial expression recognition with the help of curvature of Bèzier curves supported by statistical evaluations on machine learning feature vectors.

Remark 1.3. Substituting $s_1 = 1$ and $m = 1$ into (1.1), it is also easy to get the Bernstein basis functions:

$$\mathcal{G}_n\left(x; b_1, \overrightarrow{1}, \overrightarrow{0}\right) = \binom{n}{b_1} x_1^{b_1} (1 - x_1)^{n-b_1} =: B_{b_1}^n(x_1),$$

which are given by the following generating functions:

$$F_{\mathcal{B}}(t, w; b_1) = \frac{(tw)^{b_1}}{b_1!} e^{-t(w-1)} = \sum_{n=0}^{\infty} \frac{B_{b_1}^n(w)}{n!} t^n \tag{1.3}$$

(*cf.* [1, 11, 16], [32]-[35]). Note that there is one generating function for each value of w and b_1.

The well-known generalized falling factorial (or descending factorial polynomials) $w^{(m,\theta)}$, which is homogeneous polynomial in w and θ of degree m, with increment θ defined by

$$w^{(m,\theta)} = \prod_{v=0}^{m-1} (w - v\theta)$$

for $m \in \mathbb{N}$, with the convention $w^{(0,\theta)} = 1$ (*cf.* [3, 14, 15], [19, Eq. (2.11)], [38]). If $\theta \neq 0$, we have

$$w^{(m,\theta)} = \theta^m \left(\theta^{-1} w\right)^{(m)}$$

with

$$w^{(m,0)} = w^m.$$

The ascending factorial polynomials are defined by

$$w_{(m,\theta)} = \prod_{v=0}^{m-1} (w + v\theta) \tag{1.4}$$

for $m \in \mathbb{N}$, with the convention $w_{(0,\theta)} = 1$ and $w_{(m,0)} = w^m$ (*cf.* [19, Eq. (2.11)]). It is also well-known that $w_{(m)}$ denote the Pochhammer symbol, (also known as the rising factorial polynomial), which is also given in terms of the Euler gamma function:

$$\begin{aligned} w_{(m,1)} =: w_{(m)} &= \prod_{v=0}^{m-1} (w + v) \\ &= \frac{\Gamma(w + m)}{\Gamma(w)}, \end{aligned} \tag{1.5}$$

where

$$\Gamma(w) = \int_0^{\infty} t^{w-1} e^{-t} dt$$

with $\mathrm{Re}(w) > 1$ (*cf.* [30, 36]).

From (1.4) and (1.5), it is clear that the ascending factorial polynomials are also given in terms of the Euler gamma function as follows:

$$w_{(m,b)} = \frac{b^m \Gamma\left(\frac{w}{b} + m\right)}{\Gamma\left(\frac{w}{b}\right)}. \tag{1.6}$$

The Bernstein-Stancu polynomials are defined by

$$S_v^{<b>}[g](w)=\sum_{m=0}^{v}P_{v,m}^{<b>}(w)\,g\left(\frac{m}{v}\right),$$

where $w\in\mathbb{C}$ and $b\geq 0$ and $P_{v,m}^{<b>}(w)$ denote the Bernstein-Stancu basis functions, which are defined by

$$P_{v,m}^{<b>}(w)=\frac{\binom{v}{m}}{\prod\limits_{c=0}^{v-1}(1+cb)}\prod_{c=0}^{m-1}(w+cb)\prod_{c=0}^{v-m-1}(1-w+cb) \tag{1.7}$$

(*cf.* [8, p. 68]).

The subsequent sections of this article are succinctly outlined in the following manner:

In Section 2, we construct a new generating function for the Bernstein-Stancu polynomials with the aid of the hypergeometric series. By using descending and ascending factorial polynomials properties, we derive some formulas for the Bernstein-Stancu polynomials.

In Section 3, we construct generating functions for the Bernstein-Stancu basis functions. We also give some properties of these functions.

In Section 4, we give derivative formulas for the Bernstein-Stancu basis functions.

In Section 5, we give Bèzier-type curves in terms of control points and the generalized Bernstein-Stancu basis functions with their open problems.

In Section 6, we give observation on the Bernstein-Stancu basis functions in terms of approach to the techniques of topological and functional analysis.

In Section 7, we give conclusion.

2. Formulas and generating functions for the Bernstein-Stancu polynomials

In this section, we construct a new generating function for the Bernstein-Stancu polynomials with the aid of the hypergeometric series. By using descending and ascending factorial polynomials properties, we derive some formulas for the Bernstein-Stancu polynomials.

The function $P_{v,m}^{<b>}(w)$ is given in terms of the ascending factorial polynomials as follows:

$$P_{v,m}^{<b>}(w)=\binom{v}{m}\frac{w_{(m,b)}(1-w)_{(v-m,b)}}{1_{(v,b)}}, \tag{2.1}$$

where $m,v\in\mathbb{N}_0$ with $v\geq m$ and $w\in\mathbb{C}$.

From (1.4), it is easy to see the following known formula:

$$w_{(m,b)}=(-1)^m\binom{-\frac{w}{b}}{m}b^m m!, \tag{2.2}$$

where $b>0$, and

$$\binom{x}{m}=\frac{x^{(m,1)}}{m!}=\frac{x(x-1)\cdots(x-m+1)}{m!}. \tag{2.3}$$

With the aid of (2.2), we modify definition of the Bernstein-Stancu basis functions as follows:

Theorem 2.1. *Let $m,v\in\mathbb{N}_0$ with $v\geq m$ and $w\in\mathbb{C}$. For $b>0$, we have*

$$P_{v,m}^{<b>}(w)=\frac{\binom{-\frac{w}{b}}{m}\binom{\frac{w-1}{b}}{v-m}}{\binom{-\frac{1}{b}}{v}}. \tag{2.4}$$

By using the descending factorial polynomials with (2.3), we can rewrite the equation (1.7) as follows:

$$P^{<b>}_{v,m}(w) = \binom{v}{m} \frac{\left(-\frac{w}{b}\right)^{(m)} \left(\frac{w-1}{b}\right)^{(v-m)}}{\left(-\frac{1}{b}\right)^{(v)}}. \tag{2.5}$$

By combining (2.5) with the following formula, given by Milne-Thomson [19, p. 133], for developing the ascending factorial polynomial $y^{(v)}$ by the MacLaurin's theorem, for $v > 0$:

$$y^{(v)} = \sum_{j=0}^{v} \binom{v}{j} \frac{j}{v} y^{j} B^{(v)}_{v-j}, \tag{2.6}$$

where $B^{(v)}_n$ denote the Bernoulli numbers of order v, defined by

$$\left(\frac{t}{e^t - 1}\right)^{v} = \sum_{n=0}^{\infty} B^{(v)}_n \frac{t^n}{n!}$$

(*cf.* [19, 36]), and using (2.6) with (2.4), we give the function $P^{<b>}_{v,m}(w)$ in terms of the higher order Bernoulli numbers by the following theorem:

Theorem 2.2. *Let $m, v \in \mathbb{N}$ with $v \geq m$ and $w \in \mathbb{C}$. For $b > 0$, we have*

$$P^{<b>}_{v,m}(w) = \sum_{s=1}^{m} \sum_{d=1}^{v-m} (-1)^s \binom{m-1}{s-1} \binom{v-m-1}{d-1} \binom{v}{m} \frac{w^s (w-1)^d B^{(m)}_{m-s} B^{(v-m)}_{v-m-d}}{b^{s+d} \left(-\frac{1}{b}\right)^{(v)}}. \tag{2.7}$$

By applying the well-known Chu-Vandermonde identity (*cf.* [30, p. 4. Eq. (1.9)]) to the sum of the equation (2.4), we get

$$\sum_{m=0}^{v} P^{<b>}_{v,m}(w) = \frac{1}{\binom{-\frac{1}{b}}{v}} \sum_{m=0}^{v} \binom{-\frac{w}{b}}{m} \binom{-\frac{1-w}{b}}{v-m}.$$

After some elementary calculations, we arrive at the following theorem, which gives us sum of the Bernstein-Stancu basis function:

Theorem 2.3 (Sum of the Bernstein-Stancu basis function)**.**

$$\sum_{m=0}^{v} P^{<b>}_{v,m}(w) = 1.$$

Remark 2.4. In this study, Theorem 2.3, proved by the Chu-Vandermonde formula, may also be proven by methods other than that of given here. In other words, the proof of the Theorem 2.3 can also be given with the help of finite binomial expansion. The literature can be consulted for different methods similar to that of this theorem for Bernstein-type basis functions. For example, in [24], the author used the generating unctions techniques to prove the properties Bernstein-type basis functions.

Combining (1.6) with (2.1), we get the following theorem:

Theorem 2.5. *Let $m, v \in \mathbb{N}_0$ with $v \geq m$ and $w \in \mathbb{C}$. For $b > 0$, we have*

$$P^{<b>}_{v,m}(w) = \binom{v}{m} \frac{\Gamma\left(\frac{1}{b}\right) \Gamma\left(\frac{w}{b} + m\right) \Gamma\left(\frac{1-w}{b} + v - m\right)}{\Gamma\left(\frac{w}{b}\right) \Gamma\left(\frac{1-w}{b}\right) \Gamma\left(\frac{1}{b} + v\right)}, \tag{2.8}$$

where $\operatorname{Re}\left\{\frac{w}{b} + m\right\} > 0$ *and* $\operatorname{Re}\left\{\frac{1-w}{b} + v - m\right\} > 0$.

Substituting the following identity:

$$\Gamma(x)\Gamma(y) = \Gamma(x+y)B(x,y),$$

(where

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

denote the Beta function with $x > 0$ and $y > 0$ (*cf.* [36])) into (2.8), we get the following theorem:

Theorem 2.6. *Let $m, v \in \mathbb{N}_0$ with $v \geq m$ and $w \in \mathbb{C}$. For $b > 0$, we have*

$$P_{v,m}^{<b>}(w) = \binom{v}{m} \frac{B\left(\frac{w}{b} + m, \frac{1-w}{b} + v - m\right)}{B\left(\frac{w}{b}, \frac{1-w}{b}\right)}, \tag{2.9}$$

where

$$\mathrm{Re}\left\{\frac{w}{b} + m\right\} > 0,\ \mathrm{Re}\left\{\frac{1-w}{b} + v - m\right\} > 0,\ \mathrm{Re}\left\{\frac{w}{b}\right\} > 0,\ \mathrm{Re}\left\{\frac{1-w}{b}\right\} > 0.$$

By using (2.9), we also get the following corollary which gives us an integral representation for the functions $P_{v,m}^{<b>}(w)$:

Corollary 2.7. *Let $m, v \in \mathbb{N}_0$ with $v \geq m$ and $w \in \mathbb{C}$. For $b > 0$, we have*

$$P_{v,m}^{<b>}(w) = \binom{v}{m} \frac{\int_0^1 t^{\frac{w}{b}+m-1}(1-t)^{\frac{1-w}{b}+v-m-1}dt}{\int_0^1 t^{\frac{w}{b}-1}(1-t)^{\frac{1-w}{b}-1}dt},$$

where

$$\mathrm{Re}\left\{\frac{w}{b} + m\right\} > 0,\ \mathrm{Re}\left\{\frac{1-w}{b} + v - m\right\} > 0,\ \mathrm{Re}\left\{\frac{w}{b}\right\} > 0,\ \mathrm{Re}\left\{\frac{1-w}{b}\right\} > 0.$$

By using the following identity:

$$\Gamma\left(\frac{1}{2} + v\right) = \frac{(2v)!\sqrt{\pi}}{4^v v!},$$

(*cf.* [36]), we give some applications of the equations (2.8) and (2.9) as follows:

When $b = 2$, we have

$$P_{v,m}^{\langle 2\rangle}(w) = \binom{v}{m} \frac{4^v v! \Gamma\left(\frac{w}{2} + m\right)\Gamma\left(\frac{1-w}{2} + v - m\right)}{(2v)!\Gamma\left(\frac{w}{2}\right)\Gamma\left(\frac{1-w}{2}\right)}.$$

Setting $w = \frac{1}{2}$ in the equation (2.9), we have

$$P_{v,m}^{<b>}\left(\frac{1}{2}\right) = \binom{v}{m} \frac{B\left(\frac{1}{2b} + m, \frac{1}{2b} + v - m\right)}{B\left(\frac{1}{2b}, \frac{1}{2b}\right)}.$$

Since

$$\frac{\pi}{\sin(\pi w)} = \Gamma(1-w)\Gamma(w)$$

(*cf.* [36]), substituting $b = 1$ into (2.8) gives

$$P_{v,m}^{<1>}(w) = \binom{v}{m} \frac{\sin(\pi w)}{\pi} B(1 - w + v - m, w + m). \tag{2.10}$$

Substituting $w = k \in \mathbb{N}$ into the equation (2.10), we get

$$P_{v,m}^{<1>}(k) = 0.$$

Substituting $w = k + \frac{1}{2}$; $(k \in \mathbb{N})$ into the equation (2.10), for $v \geq m$, we also get

$$P_{v,m}^{<1>}\left(k + \frac{1}{2}\right) = (-1)^k \binom{v}{m} \frac{B\left(v - m - \frac{k}{2}, 1 + m + \frac{k}{2}\right)}{\pi}.$$

Furthermore, for $k = 2$, the previous equation reduces to the following interesting formula:

$$P_{v,m}^{<1>}\left(\frac{5}{2}\right) = \frac{m + 1}{\pi(v - m)(v - m - 1)}.$$

3. Generating functions for the function $P_{v,m}^{<b>}(w)$

In this section, we construct generating functions for the function $P_{v,m}^{<b>}(w)$. We also give some properties of this function.

Recall that the generalized hypergeometric function ${}_kF_m$ is given by

$${}_kF_m\left(\begin{array}{c} x_1, \ldots, x_k \\ w_1, \ldots, w_m \end{array}; u\right) = \sum_{n=0}^{\infty} \left(\frac{\prod_{j=1}^{k} \left(x_j\right)_{(n)}}{\prod_{j=1}^{m} \left(w_j\right)_{(n)}}\right) \frac{u^n}{n!},$$

or, equivalently, by

$${}_kF_m\left(x_1, \ldots, x_k; w_1, \ldots, w_m; u\right) = \sum_{n=0}^{\infty} \left(\frac{\prod_{j=1}^{k} \left(x_j\right)_{(n)}}{\prod_{j=1}^{m} \left(w_j\right)_{(n)}}\right) \frac{u^n}{n!},$$

where the above series converges for all u if $k < m + 1$, and for $|u| < 1$ if $k = m + 1$. Assuming that all parameters have general values, real or complex, except for the w_j, $j = 1, 2, \ldots, m$ none of which are equal to zero or a negative integer (*cf.* [3, 20, 31, 36]).

In this section, we set the following generating function for $P_{v,m}^{<b>}(w)$:

$$F_s(t, w; m, b) = \sum_{v=0}^{\infty} P_{v,m}^{<b>}(w) \frac{t^v}{v!}. \tag{3.1}$$

Formulas for the functions $F_s(t, w; m, b)$, in terms of the generalized hypergeometric functions, are given in the next theorem:

Theorem 3.1. *Let $m \in \mathbb{N}_0$ and $w \in \mathbb{C}$. For $b > 0$, we have*

$$F_s(t, w; m, b) = \frac{\left(\frac{w}{b}\right)_m}{m! \left(\frac{1}{b}\right)_m} t^m \, {}_1F_1\left(\begin{array}{c} \frac{1-w}{b} \\ \frac{1}{b} + m \end{array}; t\right) \tag{3.2}$$

or

$$F_s(t, w; m, b) = \frac{\left(\frac{w}{b}\right)_m}{m! \left(\frac{1}{b}\right)_m} t^m \, {}_1F_1\left(\frac{1 - w}{b}; \frac{1}{b} + m; t\right). \tag{3.3}$$

Note that there is one generating function for each value of m and b.

Alternative forms for $F_s(t,w;m,b)$ are given by

$$F_s(t,w;m,b)=\frac{\Gamma\left(\frac{1}{b}+m\right)}{\Gamma\left(\frac{1-w}{b}\right)\Gamma\left(m+\frac{w}{b}\right)}\int_0^1 t^{\frac{1-w}{b}-1}(1-t)^{m+\frac{w}{b}-1}\,dt,$$

where

$${}_1F_1\left(\begin{matrix} b \\ c \end{matrix};t\right)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\,dt,$$

which is known as the Kummer confluent hypergeometric function, and

$$F_s(t,w;m,b)=\frac{\Gamma\left(\frac{1}{b}\right)\Gamma\left(\frac{w}{b}+m\right)}{\Gamma(m+1)\Gamma\left(\frac{w}{b}\right)\Gamma\left(\frac{1}{b}+m\right)}\,{}_1F_1\left(\begin{matrix} \frac{1-w}{b} \\ \frac{1}{b}+m \end{matrix};t\right)t^m.$$

Proof of Theorem 3.1. Combining (3.1) with (2.1), we get

$$F_s(t,w;m,b)=\frac{t^m}{m!}\left(\frac{w}{b}\right)_{(m)}\sum_{v=m}^{\infty}\frac{\left(\frac{1-w}{b}\right)_{(v-m)}}{\left(\frac{1}{b}\right)_{(v)}}\frac{t^{v-m}}{(v-m)!}.$$

Substituting the following identity

$$\left(\frac{1}{b}\right)_{(v)}=\left(\frac{1}{b}\right)_{(m)}\left(\frac{1}{b}+m\right)_{(v-m)}$$

into the above equation yields

$$F_s(t,w;m,b)=\frac{t^m}{m!}\frac{\left(\frac{w}{b}\right)_{(m)}}{\left(\frac{1}{b}\right)_{(m)}}\sum_{v=m}^{\infty}\frac{\left(\frac{1-w}{b}\right)_{(v-m)}}{\left(\frac{1}{b}+m\right)_{(v-m)}}\frac{t^{v-m}}{(v-m)!}.$$

This gives us the desired result with the help of

$${}_1F_1\left(\begin{matrix} x \\ y \end{matrix};t\right)=\sum_{m=0}^{\infty}\frac{x_{(m)}}{y_{(m)}}\frac{t^m}{m!},$$

where $y>0$. □

4. Derivative formulas for the function $P_{v,m}^{<b>}(w)$

In this section we give derivative formulas for the function $P_{v,m}^{<b>}(w)$.

Differentiating (2.4) j times, with the aid of the Leibniz formula, we obtain

$$\begin{aligned}\frac{d^j}{dw^j}\left\{P_{v,m}^{<b>}(w)\right\} &= \binom{-\frac{1}{b}}{v}^{-1}\frac{d^j}{dw^j}\left\{\binom{-\frac{w}{b}}{m}\binom{-\frac{1-w}{b}}{v-m}\right\}\\ &= \frac{1}{\binom{-\frac{1}{b}}{v}}\sum_{k=0}^{j}\binom{j}{k}\frac{1}{m!(v-m)!}\frac{d^j}{dw^j}\left\{\left(-\frac{w}{b}\right)^{(m)}\right\}\frac{d^{j-k}}{dw^{j-k}}\left\{\left(\frac{w-1}{b}\right)^{(v-m)}\right\}.\end{aligned}$$

Combining the above equation with the following formula (*cf.* [19, p. 133]):

$$\frac{d^j}{dw^j}\left\{x^{(m)}\right\}=m^{(j)}B_{m-j}^{(m+1)}(x+1), \tag{4.1}$$

where $B_n^{(v)}(x)$ denote the Bernoulli polynomials of order v, defined by

$$\left(\frac{t}{e^t-1}\right)^v e^{tx} = \sum_{n=0}^{\infty} B_n^{(v)}(x)\frac{t^n}{n!}$$

(*cf.* [19, 36]), we get the following theorem:

Theorem 4.1. *Let $j \in \mathbb{N}_0$. Let $m, v \in \mathbb{N}_0$ with $v \geq m$ and $w \in \mathbb{C}$. For $b > 0$, we have*

$$\frac{d^j}{dw^j}\left\{P_{v,m}^{<b>}(w)\right\} = \frac{\binom{v}{m}\left(-\frac{1}{b}\right)^j}{\left(-\frac{1}{b}\right)^{(v)}} \sum_{k=0}^{j} (-1)^j \binom{j}{k} m^{(j)} m^{(j-k)} B_{m-j}^{(m+1)}\left(-\frac{w}{b}+1\right) B_{v-m+k-j}^{(v-m+1)}\left(\frac{w-1}{b}+1\right). \tag{4.2}$$

Differentiating (2.7) j times, with the aid of the Leibniz formula, we also obtain a higher order derivative formula for $P_{v,m}^{<b>}(w)$ given in the following theorem:

Theorem 4.2. *Let $j \in \mathbb{N}_0$. Let $m, v \in \mathbb{N}_0$ with $v \geq m$ and $w \in \mathbb{C}$. For $b > 0$, we have*

$$\frac{d^j}{dw^j}\left\{P_{v,m}^{<b>}(w)\right\} = \sum_{s=1}^{m}\sum_{d=1}^{v-m}(-1)^s \binom{m-1}{s-1}\binom{v-m-1}{d-1}\binom{v}{m} \frac{B_{m-s}^{(m)} B_{v-m-d}^{(v-m)}}{b^{s+d}\left(-\frac{1}{b}\right)^{(v)}} \sum_{k=0}^{j}\binom{j}{k} s^{(k)} d^{(j-k)} w^{s-k}(w-1)^{d-j+k}. \tag{4.3}$$

Applying the partial derivative operator $\frac{\partial}{\partial w}$ to (2.1) with respect to w, we find the following partial differential formula for the polynomials $P_{v,m}^{<b>}(w)$, we get

$$\frac{\partial}{\partial w}\left\{P_{v,m}^{<b>}(w)\right\} = \frac{1}{b} P_{v,m}^{<b>}(w) \left(\sum_{j=0}^{m-1} \frac{1}{\frac{w}{b}+j} - \sum_{k=0}^{v-m-1} \frac{1}{\frac{1-w}{b}+k}\right).$$

The above formula can be expressed in terms of the generalized harmonic functions (*cf.* [4, 33, 36]), which are given by

$$H_m(w) = \sum_{j=1}^{m} \frac{1}{w+j}$$

with $H_0(w) = 0$, w is an indeterminate and $m \in \mathbb{N}$, as follows:

Theorem 4.3. *Let $m, v \in \mathbb{N}$ with $v \geq m$ and $w \in \mathbb{C}$. For $b > 0$, we have*

$$\frac{\partial}{\partial w}\left\{P_{v,m}^{<b>}(w)\right\} = \frac{1}{b}\left(H_{m-1}\left(\frac{w}{b}\right) - H_{v-m-1}\left(\frac{1-w}{b}\right)\right) P_{v,m}^{<b>}(w).$$

5. Further remarks on Bèzier-type curves in terms of the functions $P_{v,m}^{<b>}(w)$

The classical Bèzier curves are constructed by the Bernstein polynomials and control points. By using same method, the Bèzier-type curves are given in terms of control points and the generalized Bernstein-Stancu basis functions as follows:

$$B(w; v, m; b) = \sum_{m=0}^{v} Q_m P_{v,m}^{<b>}(w),$$

where Q_m, $m \in \{0, 1, 2, \ldots, v\}$, denote the control points.

For detailed properties of this type curves, see (*cf.* [7, 8], [10]-[13], [17, 22, 29]).

In the light of the procedures and methods given in the above sections, the characteristic features of the curve families mentioned in this section, which many scientists are currently working on, can perhaps be brought to light. For this reason, this section is displayed as a reminder to researchers.

6. A look at the Bernstein polynomials in terms of approach to the techniques of topological and functional analysis

To appeal to a diverse range of researchers, Bernstein polynomials have recently gained significant traction across various scientific domains. Notably, they have found applications in classical analysis, functional analysis, approximation theory, topology, geometry, numerical analysis, algebra, linear algebra, probability and statistics, generator functions, algorithm theory, geometric designs, medicine, economics, and more. For instance, Saye [21] devised an algorithm that recursively partitions topology into hyperrectangular subcells until simplified, with a particular topology test leveraging Bernstein polynomial properties. Numerous applications of this algorithm have been demonstrated. Additionally, employing uniformly continuous tests, the Schroder-Bernstein theorem, and Weierstrass approximation theorem, Bernstein polynomials yield a plethora of intriguing approximation theorems and formulas (*cf.* [7]-[23]).

Moreover, Bèzier curves, B-splines, and splines, all rooted in Bernstein basis functions, exhibit similarly essential applications. It appears that further exploration of their diverse applications will continue to enrich researchers' endeavors.

The utilization of Bèzier-type curves and spline-type curves, constructed based on the Bernstein-Stancu polynomials, raises questions about their effects, distinctions from classical curves, advantages, disadvantages, and topological properties. Additionally, inquiries into the problems where their applications may arise are pertinent. Undoubtedly, numerous queries akin to these can be posed. It is our aspiration that the unresolved issues presented in this section will beckon interested scientists to delve deeper into their respective fields of study.

7. Conclusion

We gave alternative definitions for the Bernstein-Stancu basis functions with the aid of descending and ascending factorial polynomials with their properties. By using these definitions, we drove some formulas for the Bernstein-Stancu basis functions. We constructed generating functions for the Bernstein-Stancu basis functions with the aid of the hypergeometric series. We also gave some properties of these functions. We gave derivative formulas for the Bernstein-Stancu basis functions. Some comments, suggestions and open problems about the Bèzier type curves in terms of control points and the generalized Bernstein Stancu basis functions were presented to the researchers. We also gave observation on the Bernstein-Stancu basis functions in terms of approach to the techniques of topological and functional analysis.

It is useful to explain whether formulas and methods similar to those of the studies in [6, 11, 16], [22]-[29], [32, 37] can be obtained within the Bernstein-Stancu basis functions, which could open up an important research area.

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