Montes Taurus J. Pure Appl. Math. 6 (3), 269–283, 2024

Montes Taurus Journal of Pure and Applied Mathematics
https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

On color palindrome compositions

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Abstract

In this paper, we decompose the palindrome composition set of the positive integer to construct an efficient algorithm for obtaining elements. We also develop techniques to investigate generating functions for the numbers of color palindrome compositions with respect to some coloring rules. Then we investigate the generating function, recurrence relations, and Binet formula for the n-color palindrome compositions, and so we see that the sequence of integers corresponding to the n-color palindrome compositions is the combination of even Fibonacci numbers and odd Lucas numbers. We also determine the relationships among color palindrome compositions, and the Fibonacci numbers. Moreover, by using decompositions and algorithms, we make patterns of some color palindrome compositions.

Keywords: Compositions of the positive integers, palindrome compositions, the color palindrome compositions, Binet's formulas, Fibonacci numbers, Lucas numbers, generating functions

2020 MSC: 05A15, 05A17, 05A18, 11B39,11B99

1. Introduction

Fibonacci numbers and compositions of a positive integer have made incredible contributions to almost all fields of science, and many researchers have interested in these subjects and their generalizations. These discoveries have led to the increased importance of mathematical analysis and number theory.

The Fibonacci numbers denoted by f_n are such that each number is the sum of the two preceding ones, starting from 0 and 1. That is, $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for a positive integer n greater than 1. As with the Fibonacci numbers, the Lucas number is defined to be the sum of its two immediately previous terms with $l_0 = 2$ and $l_1 = 1$. Moreover, the generalizations of Fibonacci or Lucas numbers have been used in many mathematical models and many researcher have studied these generalizations to get some combinatorial identities and generating functions (*cf.* [1, 3, 5, 10], [14]-[17], [18, 20, 21]).

The importance of the Golden ratio, the Fibonacci and Lucas numbers as effective tools in all sciences from Art to Biology or physics, with wonderful applications are well known (*cf.* [12, 15, 24]). The Golden ratio also plays a significant role in the characterization of the Fibonacci numbers and the Lucas numbers. The Golden ratio is seen in Binet's formulas for both the Fibonacci numbers and the Lucas numbers.

†*Article ID:* MTJPAM-D-24-00007

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Received:15 January 2024, Accepted:25 June 2024, Published:10 July 2024

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In 1843, Binet proved the formula for the Fibonacci numbers: f_n without using the previous the Fibonacci numbers. These formulas allow for all Fibonacci numbers f_n to be represented by using the Golden ratio.

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ is the Golden ratio and $\beta = \frac{1-\sqrt{5}}{2}$.

In [16], Levesque gave a Binet formula for the Fibonacci numbers by using generating functions. In [7], the authors considered Linear algebra method, and then gave the combinatorial representation of the Fibonacci numbers are generated by the power of the matrix. Further in [23], the authors got the analytical formulas for the generalizations of Fibonacci numbers, and then showed these formulas are similar to the Binet formulas for the Fibonacci numbers. Moreover, many authors studied these analytical formulas for the combinatorial representations of the many integer sequences like the Fibonacci and Lucas numbers (*cf.* [15, 20]).

A composition of a positive integer m is a representation of m as a sum of positive integers. In combinatorics, a classical result about the number of compositions of m is given by the coefficient of x^m of the polynomial or power series

$$\left(\sum_{i=1}^{\infty} x^i\right)^m$$

where $|x| < 1$. These coefficients exhibit fascinating mathematical properties, closely resembling binomial coefficients and have many useful applications (*cf.* [1, 2, 6, 8, 10, 11, 13, 19]).

In [13], Hoggart and Lind investigated the relationship between a composition of a positive integer and Fibonacci numbers and by using binomial properties, they got that

(i) f_m is the number of compositions of a positive integer m into odd parts,

(ii) f_{2m} is the sum of the products of the parts over all compositions of a positive integer m, i.e.

$$f_{2m} = \sum_{a_1+a_2+\cdots+a_k=m} a_1 a_2 \ldots a_k. \tag{1.1}$$

Recently, many researchers (*cf.* [1, 2, 19]) have been interested in n-color compositions of a positive integer m, defined as a composition of m for which a part of size n can take on n colors.

In [1], Agarwal was interested in n-color compositions with no parts greater than k and then Agarwal proved that the number of n-color composition of a positive integer m is the $2m$th Fibonacci number.

A palindrome composition of the positive integer m is a composition whose part-sequence is the same whether it is read from left to right to right to left. In [9], it is shown that the number of palindrome composition of m is $2^{\left\|\frac{m}{2}\right\|}$ and also in [19], Shapcott investigated the formula for the numbers of n-color of a positive integer.

There are 4 palindrome compositions of 5 as follows

$$(1, 1, \mathbf{1}, 1, 1),\ (2, \mathbf{1}, 2),\ (1, \mathbf{3}, 1),\ (\mathbf{5})$$

and there are 11 n-color palindrome compositions of 5 (we recall that 11 is 5th Lucas number);

$$(1_1, 1_1, \mathbf{1}_1, 1_1, 1_1),\ (2_1, \mathbf{1}_1, 2_1),\ (2_2, \mathbf{1}_1, 2_2),\ (1_1, \mathbf{3}_1, 1_1),\ (1_1, \mathbf{3}_2, 1_1),\ (1_1, \mathbf{3}_3, 1_1),\ (\mathbf{5}_1),\ (\mathbf{5}_2),\ (\mathbf{5}_3),\ (\mathbf{5}_4),\ (\mathbf{5}_5).$$

We assign the color for each parts like the following Figure 1.

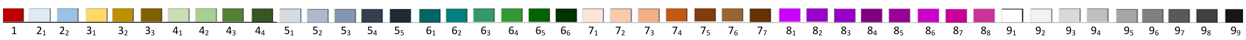

Figure 1. The charts of the color in the color compositions

Then we are able to represent the n-color palindrome compositions of 5 in rectangles with dimension 5×1 as in Figure 2.

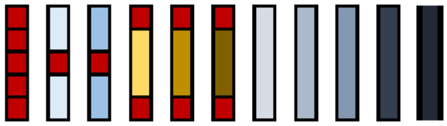

Figure 2. The n-color palindrome compositions of 5

In this paper, we study the set of palindrome compositions of the positive integers, and so set up an algorithm for obtaining elements in the palindrome composition sets of the positive integers. Then we investigate generating functions for the numbers of the elements of some palindrome compositions with respect to some coloring rules. We also investigate the relationships among the Fibonacci numbers, the Lucas numbers, and color palindrome compositions with respect to some coloring rules.

Firstly, we decompose the palindrome composition sets of the positive integers and focus on getting an efficient algorithm for obtaining elements in these sets (Algorithm 2). Hence, by using the sorting algorithm and Algorithm 2, we attain patterns of the n-color palindrome compositions of the positive integers (Figure 3). Then we develop techniques to investigate the generating function in Theorem 2.2 for the numbers of palindrome compositions with respect to some coloring rules (mainly, the numbers of palindrome compositions, the numbers of n-color palindrome compositions). Then in the subsections 2.1 and 2.2, we investigate the generating function, recurrence relations and the Binet formula for the n-color palindrome compositions of the positive integers and so we seen the odd terms of the sequence integer V of n-color palindrome compositions correspond to the Lucas numbers terms and the even terms of V to the Fibonacci numbers terms to the sequences (not exist in the OnLine Encyclopedia of Integer Sequences (OEIS)). The end of the paper, we work out the relationships. between the number of the color palindrome compositions and Fibonacci numbers in Theorem 2.8 and Theorem 2.9. We observe that the corresponding sequences do not exist in the on-line Encyclopedia of Integer Sequences (OEIS [22]).

2. Palindrome composition sets

In this section, we decompose the set of the palindrome compositions of the positive integers to explore the generating function for the numbers of palindrome compositions and construct an efficient algorithm for obtaining elements. We also investigate the generating function, recurrence relations, and Binet formula for the n-color palindrome compositions, and so we see the sequence of integers corresponding to the n-color palindrome composition is the combination of even Fibonacci numbers and odd Lucas numbers. Moreover, we make the pattern of the n-color palindrome compositions of a positive integer with respect to the palindrome color.

Let C_n be a set of palindrome compositions of a positive integer n. Then we define two set

$$\begin{aligned}
(i \cdot P_m) &= \{(x, i, x) : x \in P_m\} = \{(x_1, \ldots, x_n, i, x_n, \ldots, x_1) : x_1 + \cdots + x_n = m\}, \\
(i \circ P_m) &= \{(x, i, i, x) : x \in P_m\} = \{(x_1, \ldots, x_n, i, i, x_n, \ldots, x_1) : x_1 + \cdots + x_n = m\}.
\end{aligned}$$

Then for different integers i, j, it is easy to see that $(i \cdot P_m)$ and $(j \cdot P_m)$ are disjoint.

Now we try to characterize the set of a palindrome composition of a positive integer and investigate the numbers of the elements in the set.

Theorem 2.1. *The set of a palindrome composition of a positive integer m is*

$$\begin{aligned}
C_{2m+1} &= \bigcup_{i=0}^{m} ((2i+1) \cdot P_{m-i}), \\
C_{2m} &= \left(\bigcup_{i=1}^{m} (2i \cdot P_{m-i}) \right) \cup \left(\bigcup_{i=1}^{m} (i \circ P_{m-i}) \right).
\end{aligned}$$

Proof. i) It is enough to show that the left side is in the right's. Let $x \in C_{2m+1}$ and so $x = (x_1, \dots, x_n, y, x_n, \dots, x_1)$ where $2(x_1 + \cdots + x_n) + y = 2m + 1$ and x_j are positive integers. Therefore $y = 2i + 1$ for some integer i and so $x_1 + \cdots + x_n = m - i$. Then $x \in (y \cdot P_{m-i})$.

ii) Let $x = (t_1, \dots, t_l) \in C_{2m}$. If $l = 2i+1$ then $x = (t_1, \dots, t_i, 2j, t_i, \dots, t_1)$ for some j and $t_1 + \cdots + t_i = m - j$. Hence $x \in (2j \cdot P_{m-j})$. On the other hand, if $l = 2i$ then $x = (t_1, \dots, t_{i-1}, j, j, t_{i-1}, \dots, t_1)$ for some j and $t_1 + \cdots + t_{i-1} + j = m$. Hence $x \in (j \circ P_{m-j})$. □

We are able to set a sorting algorithm for the elements of C_{2m} and C_{2m+1} for a positive integer m. Firstly, we apply the lexical order to the elements of P_m such as $\{x_{m,1}, \dots, x_{m,2^{m-1}}\}$.

Assume that $x \in C_{2m+1}$. Then there are positive integers i, j and an element $x_{m-i,j} \in P_{m-i}$ such that $x = (x_{m-i,j}, i, x_{m-i,j})$ and now we index the element x such that $x_{2m+1,i,j}$.

Now assume that $x \in C_{2m}$, and so there are positive integers i, j and the element $x_{m-i,j} \in P_{m-i}$ such that the form of x is either $x = (x_{m-i,j}, i, i, x_{m-i})$ or $x = (x_{m-i,j}, 2i, x_{m-i,j})$. And then we index the element x such that $x_{2m+1,i,i,j}$ and $x_{2m+1,i,2i,j}$ respectively. While sorting the index according to lexical order, we sort all elements of C_{2m} and C_{2m+1}. Therefore, we will be able to make some patterns in Figures 3, 4, 5 and 6.

Now our aim is to develop an efficient algorithm for the palindrome compositions. Since we also want to enumerate the elements of the palindrome composition of a positive integer n, we will determine the compositions of all these integers up to $n/2$ in a matrix of strings $C[-][-]$ where $CC[n][i]$ is the ith compositions of n. For this reason, we will first give an algorithm for compositions and then we will then develop the algorithm for palindrome compositions. In fact, Algorithm 1 is also developed using matrix of string $CC[-]$ and one should also develop algorithm for n-color palindrome compositions.

Algorithm 1 The following procedure returns the set $CC[-][-]$ of the combination set of a positive integer number n where $CC[n][i]$ is the ith compositions of n.

```
procedure COMP(CC[-][-]: matrix of strings; n: nonnegative integer)
    Globol variable: CC[−][−]: matrix of string;
    Local variables: integer u ← power(2, n − 2);
    Local variables: string q, q1;
    Local variables: integer l, i;
    for all j in {0, 1, . . . , u − 1} do
        C ← CC[n − 1][j];
        l ← length(C);
        if l = 1 then
            CC[n][power(2, n − 1) − 1] ←Convert-to-String(Convert-to-Integer(C) + 1);
            CC[n][u − 1] ←" 1,"+C;
        else
            integer i ← 0;
            while C[i] ≠"," do
                q ← q + C[i];
                i + +;
            end while
            while i <= l do
                q1 ← q1 + C[i];
                i + +;
            end while
            CC[n][j] ←" 1," +C;
            CC[n][j + u] ←q1+Convert-to-String(Convert-to-Integer(q) + 1);
        end if
    end for
end procedure
```

Algorithm 2 The following procedure returns the set $PP[-]$ of the palindrome composition of an positive integer n where $P[i]$ is ith palindrome composition of n.

```
procedure PAL(PP[−]: matrix of string; n integer)
    function INVERSE(string p)
        Local variables: string q ←′′′′;
        for all i in {0, 1, . . . , length(p)} do
            q ← p[i] + q;
        end for
        return q
    end function
    Globol variable: CC[−][−]: matrix of string;
    Local variables: integer j, z;
    Local variables: integer l ← 0;
    for all j in {1, . . . , (n/2)} do
        COMP(CC, j);
    end for
    if mod(n, 2) = 0 then
        for all j in {2, 4, 6, . . . , n} do
            z ← (x − j)/2;
            y ← Convert-to-String(j);
            y1 ← Convert-to-String(j/2);
            k ← power(2, z − 1);
            for all t in {0, 1, 2 . . . , k} do
                l + +;
                p ← CC[z − 1][t];
                q ← INVERSE(p);
                PP[l − 1] ← P+′′,′′ +y1+′′,′′ +y1+′′,′′ +Q;
                PP[l + k − 1] ← P+′′,′′ +y+′′,′′ +Q;
            end for
            l ← l + k;
        end for
        PP[l] ← Convert-to-String(n/2)+′′,′′ +Convert-to-String(n/2);
        PP[l + 1] ← Convert-to-String(n);
    else
        z ← (x − j − 1)/2;
        j ← 1;
        for all j in {z, z − 1, . . . , 1} do
            for all t in {0, 1, . . . , power(2, z − 1)} do
                p ← CC[z − 1][t];
                q ← INVERSE(p);
                PP[l] ← P+′′,′′ +Convert-to-String(j)+′′,′′ +Q;
                l++;
            end for
            j ← j + 2;
        end for
        PP[l] ←Convert-to-String(n);
    end if
end procedure
```

We want to also investigate the generating function for the numbers of both the palindrome compositions and color palindrome compositions with respect to coloring rules and so we need the following notations for the general form;

Let $\chi(t)$ be the generating function for the numbers showing the change of the axis in the palindrome type, i.e. $\chi(t) = \sum_{i=0}^{\infty} a_i t^i$ and so a_i is the number of palindrome compositions whose center i.

Let $P(t)$ be the generating function for the numbers showing the change of the wings in a palindrome type, i.e. $P(t) = 1 + \sum_{i=1}^{\infty} b_n t^n$ and so b_n is the numbers of the wings in a palindrome type of the integer n.

We need to revise the functions $\chi(t)$ and $P(t)$ to obtain the general form of the generating functions for the numbers of the palindrome type as the following;

$$\begin{gathered} Pb(t) = \sum_{i=0}^{\infty} b_i t^{2i}, \quad \chi b(t) = \sum_{i=1}^{\infty} a_i t^{2i}, \\ \chi o(t) = \sum_{i=0}^{\infty} a_{2i+1} t^{2i}, \quad \chi e(t) = \sum_{i=1}^{\infty} a_{2i} t^{2i}. \end{gathered} \tag{2.1}$$

Then by the Cauchy products of series, we get the following products

$$\chi o(t) Pb(t) = \sum_{i=0}^{\infty} a_{2i+1} t^{2i} \sum_{m=0}^{\infty} b_m t^{2m} = \sum_{m=0}^{\infty} \left(\sum_{i=0}^{m} a_{2i+1} b_{m-i} \right) t^{2m}, \tag{2.2}$$

$$\chi e(t) Pb(t) = \sum_{i=1}^{\infty} a_{2i} t^{2i} \sum_{m=0}^{\infty} b_m t^{2m} = \sum_{m=1}^{\infty} \left(\sum_{i=1}^{m} a_{2i} b_{m-i} \right) t^{2m}, \tag{2.3}$$

$$\chi b(t) Pb(t) = \sum_{i=1}^{\infty} a_i t^{2i} \sum_{m=0}^{\infty} b_m t^{2m} = \sum_{m=1}^{\infty} \left(\sum_{i=1}^{m} a_i b_{m-i} \right) t^{2m}. \tag{2.4}$$

Now we are ready to find the generating function for palindrome compositions with respect to $\chi(t)$ and $P(t)$, which will be used frequently.

Theorem 2.2. *The generating function for palindrome compositions with respect to the generating functions in Equation* (2.1) *is*

$$pal(t) = Pb(t) \left[\chi(t) + \chi b(t) \right].$$

Proof. Let

$$pal(t) = \sum_{n=1}^{\infty} \alpha_n t^n$$

be a generating function for palindrome compositions. By using the generating functions $P(t)$ and $\chi(t)$, when we count the odd and even indices respectively, for a positive integer m, we get

$$\alpha_{2m} = \sum_{i=1}^{m} a_{2i} b_{m-i} + \sum_{j=1}^{m} a_j b_{m-j}, \tag{2.5}$$

$$\alpha_{2m+1} = \sum_{i=0}^{m} a_{2i+1} b_{m-i}. \tag{2.6}$$

Then we separate $pal(t)$ within even and odd index

$$pal(t) = \sum_{n=1}^{\infty} \alpha_{2n} t^{2n} + t \sum_{n=0}^{\infty} \alpha_{2n+1} t^{2n}. \tag{2.7}$$

By combining Equations (2.5), (2.6) with Equation (2.7) and using Equations (2.2), (2.3) and (2.4), we have that

$$\begin{aligned} pal(t) &= \sum_{m=1}^{\infty} \left(\sum_{i=1}^{m} a_{2i} b_{m-i} + \sum_{j=1}^{m} a_j b_{m-j} \right) t^{2m} + t \sum_{m=0}^{\infty} \left(\sum_{i=0}^{m} a_{2i+1} b_{m-i} \right) t^{2m} \\ &= \chi e(t) Pb(t) + \chi b(t) Pb(t) + t \left[\chi o(t) Pb(t) \right] \\ &= Pb(t) \left[\chi e(t) + t \chi o(t) + \chi b(t) \right] \\ &= Pb(t) \left[\chi(t) + \chi b(t) \right]. \end{aligned}$$

Then we have completed the proof. □

By Theorem 2.2, we reckon the generating function for the numbers of the palindrome compositions or the n-color palindrome compositions of a positive integer.

Theorem 2.3. *The generating function for the numbers of the palindrome compositions is*

$$np(t) = t\frac{2t+1}{1-2t^2} = t + 2t^2 + 2t^3 + 4t^4 + 4t^5 + \cdots .$$

Proof. The generating function for the numbers of the wings in the palindrome compositions is

$$P(t) = \frac{t-1}{2t-1} = 1 + t + 2t^2 + 4t^3 + 8t^4 + \cdots .$$

The number showing the change of the axis in the palindrome composition is 1 and so the generating function is

$$\chi(t) = \frac{t}{1-t} = t + t^2 + t^3 + t^4 + t^5 + \cdots .$$

Then it is easy to compute the generating functions in Equation (2.1) and so by substituting Theorem 2.2, we reckon the generating function for the numbers of the palindrome compositions

$$np(t) = t\frac{2t+1}{1-2t^2}.$$

□

2.1. *The wings are n-color compositions*

Now our aim is to correlate the color palindrome compositions of the positive integers with the Fibonacci and Lucas numbers. Firstly we focus on the number of n-color palindrome compositions and get the Binet formula for these numbers and then we find the relations between the Fibonacci numbers and the color palindrome compositions. Then we color the palindrome compositions of the integer such that the middle part is one color and the others part with size n take on n colors, i.e. the wings are n-color compositions and we also get formulas the number of them.

By using the colors in Figure 1 and sorting algorithm on palindromes, we obtain the pattern of n-color palindrome compositions of a positive integer. We make the patterns in Figures 3 and 4.

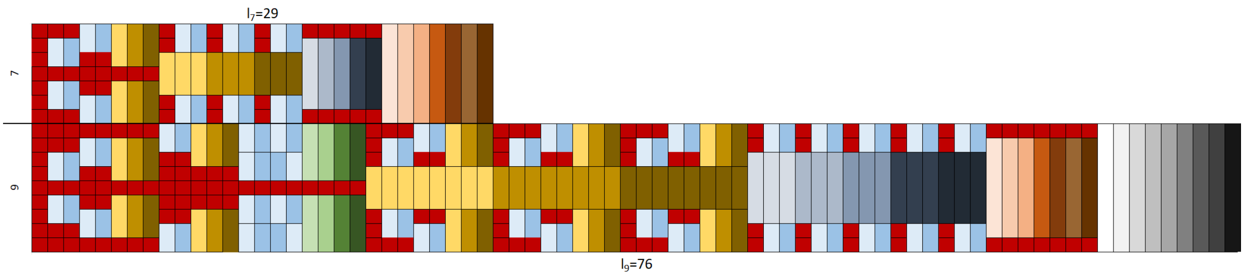

Figure 3. The patterns of n-color palindrome compositions of 7 and 9 such that the middle part is n-color

From Theorem 2.1 we observe that the pattern of 9 in Figure 3 (or Figure 4) contains the previous patterns of 7.

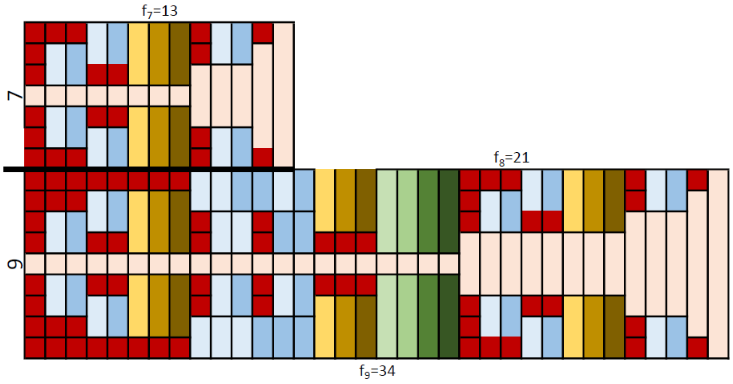

Figure 4. The patterns of n-color palindrome compositions of 7 and 9 such that the middle part is one color

Theorem 2.4. *The generating function for the numbers of the n-color palindrome compositions is*

$$cp_1(t) = \sum_{m=1}^{\infty} v_m t^m = \frac{t^3+3t^2+t}{t^4-3t^2+1} = t + 3t^2 + 4t^3 + 9t^4 + 11t^5 + 24t^6 + 29t^7 + 63t^8 + \cdots .$$

Proof. To compute the generating function for the numbers of the n-color palindrome compositions, we need to determine the generating functions for the numbers of the wings in the n-color palindrome compositions and the numbers showing the change of the axis in the n-color palindrome composition by Theorem 2.2.

By the definition of the n-color palindrome composition, the axis changes with respect to positive integer and so the generating function for the positive integers is

$$\chi(t) = \frac{t}{(1-t)^2} = t + 2t^2 + 3t^3 + 4t^4 + 5t^5 + 6t^6 + 7t^7 + 8t^8 + \cdots .$$

On the other hand, the generating function for the numbers of the wings in the n-color palindrome compositions is ones of the even Fibonacci numbers. Then it follows that

$$Pb(t) = 1 + \frac{t^2}{1-3t^2+t^4}.$$

By Theorem 2.2, we compute the generating function for the numbers of the n-color palindrome compositions

$$cp_1(t) = Pb(t)\left[\chi(t) + \chi b(t)\right] = \frac{t^3+3t^2+t}{t^4-3t^2+1}.$$

□

Theorem 2.5. *Let m be an positive integer. Then the recurrence relation for the numbers of the n-color palindrome compositions is*

$$v_m = 3v_{m-2} - v_{m-4}, \qquad m = 5, 6, \ldots$$

with the initial conditions $v_1 = 1, v_2 = 3, v_3 = 4$ *and* $v_4 = 9$.

Proof. By Theorem 2.4, the generating function for the numbers of the n-color palindrome compositions

$$\frac{t^3+3t^2+t}{t^4-3t^2+1} = \sum_{m=1}^{\infty} v_m t^m$$

and it follows that

$$\begin{aligned} t^3+3t^2+t &= \sum_{m=1}^{\infty} v_m t^{4+m} - \sum_{m=1}^{\infty} 3v_m t^{2+n} + \sum_{m=1}^{\infty} v_m t^m \\ &= -3v_1t^3 - 3v_2t^4 + \sum_{m=1}^{4} v_m t^m + \sum_{m=5}^{\infty} (v_{m-4} - 3v_{m-2} + v_m)\, t^m. \end{aligned}$$

By the coefficient of t^m, we decude that $v_1 = 1, v_2 = 3$, $v_3 = 4$ and $v_4 = 9$. Therefore, we get that

$$v_m = 3v_{m-2} - v_{m-4}.$$

□

Consider the matrix

$$A = \begin{bmatrix} 0 & 3 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } v = \begin{bmatrix} v_4 \\ v_3 \\ v_2 \\ v_1 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ 3 \\ 1 \end{bmatrix}.$$

Then for a positive integer m, it follows that

$$A^m v = \begin{bmatrix} v_{m+4} \\ v_{m+3} \\ v_{m+2} \\ v_{m+1} \end{bmatrix}.$$

Theorem 2.6 (Binet formula for the n-color palindrome compositions)**.** *Let m be a positive integer. Then we get the numbers v_{m+1} of the n-color palindrome compositions of m;*

$$v_{m+1} = \begin{cases} \frac{\sqrt{5}}{5}\left(3\left(\alpha^m - \beta^m\right) + \left(\alpha^{m-1} - \beta^{m-1}\right)\right); & m \text{ is even} \\ \frac{3\sqrt{5}}{5}\left(\left(\alpha^m - \beta^m\right) + \left(\alpha^{m-1} - \beta^{m-1}\right)\right); & m \text{ is odd} \end{cases},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$.

Proof. The eigenvalues of the matrix A are related to Golden proportion and they are

$$\alpha_1 = \frac{1+\sqrt{5}}{2},\ \alpha_2 = \frac{1-\sqrt{5}}{2},\ \alpha_3 = \frac{-1+\sqrt{5}}{2} \text{ and } \alpha_4 = \frac{-1-\sqrt{5}}{2}.$$

This means that A is diagonalizable matrix and now we consider the matrix

$$P = \begin{bmatrix} 2\alpha_1+1 & 2\alpha_2+1 & -2\alpha_2-1 & -2\alpha_1-1 \\ \alpha_1+1 & \alpha_2+1 & \alpha_2+1 & \alpha_1+1 \\ \alpha_1 & \alpha_2 & -\alpha_2 & -\alpha_1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

is invertible and so we compute that

$$P^{-1} = \frac{-\sqrt{5}}{10}\begin{bmatrix} \alpha_2 & -1 & -2\alpha_2-1 & \alpha_2+1 \\ -\alpha_1 & 1 & 2\alpha_1+1 & -\alpha_1-1 \\ \alpha_1 & 1 & -2\alpha_1-1 & -\alpha_1-1 \\ -\alpha_2 & -1 & 2\alpha_2+1 & \alpha_2+1 \end{bmatrix}.$$

Then we have $A = PDP^{-1}$ and $A^m v = PD^m P^{-1} v$ for an integer m where $D = Diag\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Therefore by the matrix equalities, we get

$$v_{m+1} = \frac{1}{5}\left(3\sqrt{5}\alpha_1^m - 3\sqrt{5}\alpha_2^m + 2\sqrt{5}\alpha_1\alpha_2^m - 2\sqrt{5}\alpha_1^m\alpha_2 - \sqrt{5}\alpha_1\alpha_3^m + \sqrt{5}\alpha_2\alpha_4^m\right)$$

and then by using the equality $\alpha_1 = -\alpha_4,\ \alpha_2 = -\alpha_3,\ \alpha_1\alpha_2 = -1$ and $\alpha_1\alpha_3 = 1$, we compute that

$$v_{m+1} = \frac{\sqrt{5}}{5}\left(3\left(\alpha_1^m - \alpha_2^m\right) + 2\left(\alpha_1^{m-1} - \alpha_2^{m-1}\right) - \left(\alpha_3^{m-1} - \alpha_4^{m-1}\right)\right).$$

If m is even numbers, then we get

$$\begin{aligned} v_{m+1} &= \frac{\sqrt{5}}{5}\left(3\left(\alpha_1^m - \alpha_2^m\right) + 2\left(\alpha_1^{m-1} - \alpha_2^{m-1}\right) + \left(\alpha_2^{m-1} - \alpha_1^{m-1}\right)\right) \\ &= \frac{\sqrt{5}}{5}\left(3\left(\alpha_1^m - \alpha_2^m\right) + \left(\alpha_1^{m-1} - \alpha_2^{m-1}\right)\right). \end{aligned}$$

If m is odd numbers, then we get

$$\begin{aligned} v_{m+1} &= \frac{\sqrt{5}}{5}\left(3\left(\alpha_1^m - \alpha_2^m\right) + 2\left(\alpha_1^{m-1} - \alpha_2^{m-1}\right) - \left(\alpha_2^{m-1} - \alpha_1^{m-1}\right)\right) \\ &= \frac{3\sqrt{5}}{5}\left(\left(\alpha_1^m - \alpha_2^m\right) + \left(\alpha_1^{m-1} - \alpha_2^{m-1}\right)\right). \end{aligned}$$

Therefore, we have completed the proof. □

Corollary 2.7. *Let m be a positive integer. Then the numbers v_m of the n-color palindrome compositions*

$$v_m = \begin{cases} l_m & ; \quad m \text{ is odd} \\ 3f_m & ; \quad m \text{ is even} \end{cases}.$$

Proof. By the Binet formula and Theorem 2.6, we get

$$v_{m+1} = \begin{cases} \frac{\sqrt{5}}{5}\left(3\left(\alpha^m - \beta^m\right) + \left(\alpha^{m-1} - \beta^{m-1}\right)\right) & ; \quad m \text{ is even} \\ \frac{3\sqrt{5}}{5}\left(\left(\alpha^m - \beta^m\right) + \left(\alpha^{m-1} - \beta^{m-1}\right)\right) & ; \quad m \text{ is odd} \end{cases}$$

and

$$v_{m+1} = \begin{cases} 3f_m + f_{m-1} = 2f_m + f_{m+1} = l_{m+1} & ; \quad m \text{ is even} \\ 3f_m + 3f_{m-1} = 3f_{m+1} & ; \quad m \text{ is odd} \end{cases}.$$

□

We have seen the odd terms of the sequence integer V of n-color palindrome compositions correspond to Lucas numbers terms and the even terms of V to the Fibonacci numbers terms to the sequences and we note that the corresponding sequences does not exist in the OnLine Encyclopedia of Integer Sequences (OEIS [22]). We recall the following identities used later. By induction method, it is easy to see the following well known identities (*cf.* [15]):

$$\begin{aligned} \sum_{i=1}^{m} f_{2i} + 1 &= f_{2m+1}, \\ \sum_{i=1}^{m} f_{2i-1} &= f_{2m}. \end{aligned}$$

Then we correlate the color palindrome compositions of the positive integers with the Fibonacci numbers as follows;

Theorem 2.8. *Let b_m be the number of the color palindrome compositions of the positive integer m such that the middle part is one color and the others with size n take on n colors. Then we have that the number*

(a) $b_{2m+1} = f_{2m+1}$,

(b) $b_{2m} = 2f_{2m-1}$.

The generating function for these numbers is

$$\begin{aligned}\sum_{m=2}^{\infty} b_m t^m &= 2\sum_{m=1}^{\infty} f_{2m-1}t^{2m} + \sum_{m=1}^{\infty} f_{2m+1}t^{2m+1}\\ &= \frac{(2t+1)(t-t^3)}{(t^2+t-1)(t^2-t-1)}.\end{aligned}$$

Proof. Let m be a positive integer. Then we complete the proof in the two parts;

(a) First, we compute the numbers. We have

$$C_{2m+1} = \bigcup_{i=0}^{m-1}((2i+1)\cdot P_{m-i}) \cup \{(2m+1)\}$$

and so the number of the color palindrome compositions of the set $((2i+1)\cdot P_{m-i})$ with respect to the middle part is one color and the others part with size n take on k colors is the $f_{2(m-i)}$. Then the number b_{2m+1} of the palindrome compositions of $2m+1$ such that the middle part is one color and the others part with size n take on n colors is the sum of the $2(m-i)$ th Fibonacci number. Therefore we get that

$$b_{2m+1} = \sum_{i=1}^{m} f_{2(m-i)} + 1 = \sum_{i=1}^{m} f_{2i} + 1 = f_{2m+1}.$$

Similarly, we get that

$$C_{2m} = \left[\bigcup_{i=1}^{m-1}(2i\cdot P_{m-i}) \cup (i\circ P_{m-i})\right] \cup \{(2m)\} \cup \{(m,m)\}$$

and so

$$b_{2m} = 2\left(\sum_{i=1}^{m-1} f_{2(m-i)} + 1\right) = 2\left(\sum_{i=1}^{m-1} f_{2i} + 1\right) = 2f_{2m-1}.$$

Then we have the following

$$b_{2m} = \sum_{i=1}^{m} f_{2(m-i)} + \sum_{i=0}^{m} f_{2(m-i)} = 2\sum_{i=0}^{m} f_{2i} = 2f_{2m-1}.$$

Therefore, we have computed the numbers.

(b) To investigate the generating function, it is enough to investigate the generating function in Equation (2.1), and then we reckon the generating function by using Theorem 2.2.

Then by the rule of coloring, the generating function in Equation (2.1) are

$$\chi(t) = \frac{t}{1-t} \text{ and so } \chi b(t) = \frac{t^2}{1-t^2}.$$

The generating function for the numbers of the n-color palindrome compositions in the wings is ones of the even Fibonacci numbers. Then it follows that

$$Pb(t) = 1 + \frac{t^2}{1-3t^2+t^4}.$$

By Theorem 2.2, we compute the generating function for the numbers of the n-color palindrome compositions

$$\begin{aligned}\sum_{m=2}^{\infty} b_m t^m &= \frac{(2t+1)(t-t^3)}{(t^2+t-1)(t^2-t-1)}\\ &= t + 2t^2 + 2t^3 + 4t^4 + 5t^5 + 10t^6 + 13t^7 + 26t^8 + 34t^9 + \cdots.\end{aligned}$$

Therefore, we have completed the proof. □

2.2. The wings are the compositions whose first parts take on white color and the other parts are n-color

In this subsection, we continue to find relations between the Fibonacci numbers and the color palindrome compositions. By using Theorem 2.1, we compute the numbers of the color palindrome compositions of the integer such that in the wings, their first parts take on white color and the other parts with size n can take on n colors and also due to Theorem 2.2, we investigate the generating function for these numbers.

By using the colors in Figure 1, we make patterns of them in the following:

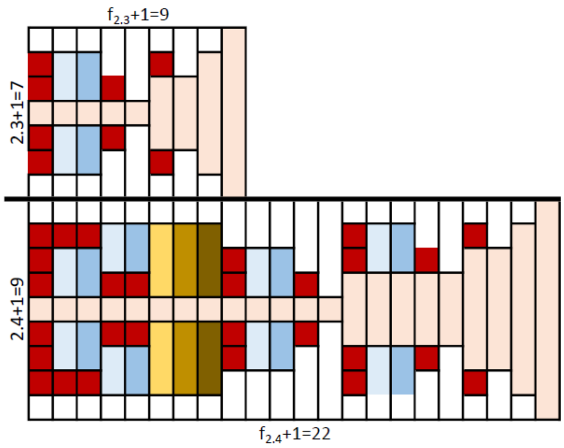

Figure 5. The patterns of color palindrome compositions of 7, 9 such that their first parts take on white color, the other parts with size n can take on n colors and the middle parts take on one color

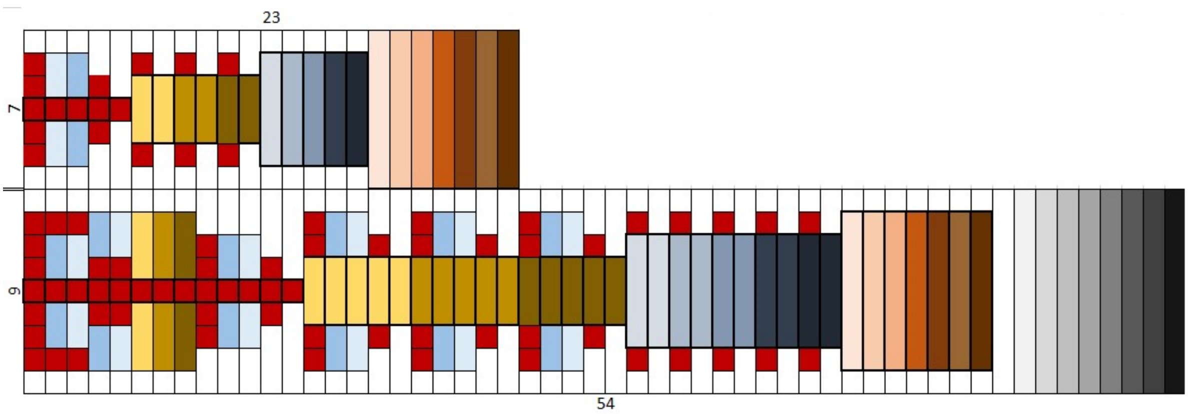

Figure 6. The patterns of palindrome compositions of 7, 9 with respect to the first part is white and the others with size n take on n colors

Theorem 2.9. *Let c_m be the number of the color palindrome compositions of the positive integer m such that in the wings, their first parts take on white color, the other parts with size n can take on n colors and the middle parts take on one color. Then we have the numbers*

(a) $c_{2m+1} = f_{2m} + 1,$

(b) $c_{2m} = 2f_{2m-2} + 2.$

The generating function for these numbers is

$$\sum_{m=1}^{\infty} c_m t^m = 2\sum_{m=1}^{\infty}(f_{2m-2}+1)t^{2m} + \sum_{m=1}^{\infty}(f_{2m}+1)t^{2m+1} = \frac{4t^4+2t^3-2t^2-t}{t^6-4t^4+4t^2-1}.$$

Proof. Let m be a positive integer. Then we complete the proof in the two parts:

(a) First, we compute the numbers. We have

$$C_{2m+1} = \bigcup_{i=0}^{m-1}((2i+1)\cdot P_{m-i}) \cup \{(2m+1)\}$$

and so the number of the color palindrome compositions of the set $((2i+1)\cdot P_{m-i})$ with respect to the coloring rule is the $f_{2(m-i)-1}$. Then we have

$$c_{2m+1} = \sum_{i=0}^{m-1} f_{2(m-i)-1} + 1 = \sum_{i=1}^{m} f_{2i-1} + 1 = f_{2m} + 1.$$

Similarly, we get that

$$C_{2m} = \left[\bigcup_{i=1}^{m-1}(2i\cdot P_{m-i}) \cup (i\circ P_{m-i})\right] \cup \{(2m)\} \cup \{(m,m)\}$$

and so

$$c_{2m} = 2\left(\sum_{i=1}^{m-1} f_{2(m-i)-1} + 1\right) = 2\left(\sum_{i=1}^{m-1} f_{2i-1} + 1\right) = 2f_{2m-2} + 2.$$

Therefore, we have computed the numbers.

(b) To investigate the generating function, it is enough to investigate the generating function in Equation (2.1), and then we reckon the generating function by using Theorem 2.2.

By the rule of coloring, the generating function in Theorem 2.2 is

$$\chi(t) = \frac{t}{1-t} \text{ and so } \chi b(t) = \frac{t^2}{1-t^2}.$$

The generating function for the numbers of the wings in the n-color palindrome compositions is ones of the odd Fibonacci numbers from [4]. Then by [15], it follows that

$$P(t) = 1 + \tfrac{t-t^2}{1-3t+t^2} \text{ and so } Pb(t) = 1 + \tfrac{t^2-t^4}{1-3t^2+t^4}.$$

Then using Theorem 2.2, we compute the generating function for the numbers

$$\begin{aligned}\sum_{m=1}^{\infty} c_m t^m &= Pb(t)\left[\chi(t) + \chi b(t)\right] \\ &= 2\sum_{m=1}^{\infty}(f_{2m-2}+1)t^{2m} + \sum_{m=1}^{\infty}(f_{2m}+1)t^{2m+1} \\ &= \frac{4t^4+2t^3-2t^2-t}{t^6-4t^4+4t^2-1}.\end{aligned}$$

Therefore, we have completed the proof. □

Theorem 2.10. *The generating function for the numbers of the color palindrome compositions whose the first part is white and the others with size n take on n colors is*

$$cp_2(t) = \frac{t - 2t^5 - 6t^4 - t^3 + 3t^2}{t^8 - 5t^6 + 8t^4 - 5t^2 + 1} = t + 3t^2 + 4t^3 + 9t^4 + 10t^5 + 21t^6 + 23t^7 + \cdots .$$

Proof. It is enough to find the generating functions Equation (2.1), for the numbers of the wings in the palindrome compositions whose the first part after the axes is white and the others with size n take on n colors and the numbers showing the change of the axis in the palindrome composition whose parts with size n take on n colors. Then by the definition of the n-color composition, the generating function for the numbers showing the change of the axis in the palindrome composition is the generating function for positive integers and so we have

$$\chi(t) = \tfrac{t}{(1-t)^2}, \quad \chi b(t) = \tfrac{t^2}{(1-t^2)^2}.$$

By [4], the numbers showing the change of the wings in a palindrome type is the odd Fibonacci numbers and so it follows that

$$P(t) = 1 + \tfrac{t-t^2}{1-3t+t^2}, \quad Pb(t) = 1 + \tfrac{t^2-t^4}{1-3t^2+t^4}.$$

By Theorem 2.2, we work out the generating function

$$cp_2(t) = Pb(t)\left[\chi(t) + \chi b(t)\right] = \frac{t\left(3t + t^2 + 1\right)\left(1 - 2t^2\right)}{(t^2 + t - 1)\,(t^2 - t - 1)\,(t^2 - 1)^2}.$$

□

3. Conclusions and future direction

In this paper, we have derived efficient algorithms from Theorem 2.1 for the compositions and palindrome compositions of a positive integer. By using the algorithms, one may improve the algorithms for n-color palindrome composition, and then any computer program to make the patterns of the positive integers for any color sets would be compiled.

We have also obtained the generating function, recurrence relations and the Binet formula for the numbers of n-color palindrome compositions of the positive integers. We have seen the odd terms of the sequence integer V of n-color palindrome compositions correspond to the Lucas number terms and the even terms of V to the Fibonacci numbers terms to the sequences, which does not exits in the OnLine Encyclopedia of Integer Sequences (OEIS [22]). By using the matrix A in Theorem 2.6, one may get new combinatorics identities containing the Fibonacci numbers and the Lucas numbers.

We have proved that the terms of the integer sequences in Theorems 2.8 and 2.9 are related to the Fibonacci numbers and so their Binet formula for these sequences are clear, and one may get new combinatorics identities for the Fibonacci numbers. On the other hand, since Goldie Ratio is one of the roots of the polynomial $t^8-5t^6+8t^4-5t^2+1$, the integer sequence in Theorem 2.8 may be related to the well known integer sequences (such as Fibonacci numbers, Lucas numbers, Jacobstal numbers or Pell numbers) and also one may obtain recurrence relations, Binet formula or new combinatorics identities for these sequences and again we note that the corresponding sequences does not exist in the OnLine Encyclopedia of Integer Sequences (OEIS [22]).

Acknowledgments

This paper is dedicated to Professor Yilmaz Simsek on the occasion of his 60th anniversary.

We would also like to express our endless gratitude to the referees for their valuable suggestions.

Author Contributions: All the authors contributed equally to this work.

Conflict of Interest: The authors declare no conflict of interest.

Funding (Financial Disclosure): The authors have no relevant financial or non-financial interests to disclose.

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How to cite this article: B. Al and M. Alkan, *On color palindrome compositions*, Montes Taurus J. Pure Appl. Math. **6 (3)**, 269–283, 2024; Article ID: MTJPAM-D-24-00007.