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New approach to construct generating function for the Apostol-Bernoulli polynomials via Nörlund sum operator

Elif Sukruoglu [a], Yilmaz Simsek [b]

[a]*Department of Mathematics Faculty of Science, University of Akdeniz, TR-0705, Antalya, Turkey*
[b]*Department of Mathematics Faculty of Science, University of Akdeniz, TR-0705, Antalya, Turkey*

Abstract

The polynomial family defined by Apostol in 1951 and known today as the Apostol-Bernoulli polynomials is known to be studied using many different methods including the Lerch zeta type functions, p-adic Volkenborn integral, spline curves, infinite series involving the Eulerian numbers, etc. The aim of this article is to construct a generating function for the Apostol-Bernoulli polynomials using a new approach technique related to the Nörlund sum operator, in contrast to these methods. The method we use here is to construct these polynomials with a different technique by applying the Nörlund sum operator to analytic function and periodic function with the Laplace transform and the difference operator approach. Furthermore, by applying the Euler differential operator (or Cauchy–Euler operator) to function $G(\lambda; z) = \lambda^z$, and using the Nörlund sum operator, we also give another novel formula involving the Apostol-Bernoulli polynomials. Finally, we give some applications of our theorems. These applications give some new formulas involving the array polynomials, the Eulerian polynomials and the Stirling numbers of the second kind. We also give an explicit computational formula for the Nörlund sum operator.

Keywords: Nörlund sum operator, Apostol-Bernoulli polynomials, generating function, difference operator, Laplace transform, array polynomials

2020 MSC: 05A15, 11B68, 39A70, 44A10, 47B37, 47E07

1. Introduction

It is known that the formal derivative operator was defined by Gottfried Wilhelm Leibniz (1 July 1646-14 November 1716), the famous German mathematician, philosopher, scientist and diplomat born in Leipzig, Germany. Operator theory has developed and deepened since 1716. Contributors to this area in terms of algebra and analysis were Leonhard Euler, Joseph-Louis Lagrange, and the Bernoulli family, Pierre-Simon Laplace, Joseph Fourier, Johann Carl Friedrich Gauss, Karl Theodor Wilhelm Weierstrass, James Joseph Sylvester, Stefan Banach, etc., who defined different operators.

The Nörlund sum operator, which is the subject of this article and whose definition is given below, was given by Niels Erik Nörlund (26 October 1885–4 July 1981), a famous Danish mathematician and astronomer who made great

†*Article ID:* MTJPAM-D-24-00099
Email addresses: elifsukruoglu@akdeniz.edu.tr (Elif Sukruoglu), ysimsek@akdeniz.edu.tr (Yilmaz Simsek)
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⋆*Corresponding Author:* Elif Sukruoglu

contributions to the theory of difference equations:

$$\mathop{\boldsymbol{S}}_{a}^{x}\phi(z)\mathop{\Delta}_{w} z=\int_{a}^{\infty}\phi(t)dt-w\sum_{n=0}^{\infty}\phi(x+nw), \tag{1.1}$$

where

$$\mathop{\Delta}_{w}=\frac{E^{w}-I}{w},$$

E^{w}, I denotes the shift and identity operator, respectively (*cf.* [4]). The above operator also known as the *Nörlund Principal Solution formula*.

In [17], we studied that the Nörlund sum operator is related to the p-adic Volkenborn integral, indefinite sum and inverse of the Δ. We also constructed the following multiplication formula for the Nörlund sum operator (*cf.* [17]):

Theorem 1.1 (Multiplication formula for the Nörlund sum operator)**.** *Let $d \in \mathbb{N}$. Then we have*

$$\mathop{\boldsymbol{S}}_{a}^{x}\phi(z)\mathop{\Delta}_{w} z=\frac{1}{d}\sum_{j=0}^{d-1}\mathop{\boldsymbol{S}}_{a}^{x+jw}\phi(z)\mathop{\Delta}_{wd} z. \tag{1.2}$$

For proof of equation (1.2) and its application, for detail see [17]. By using (1.2) and the following formulas, which given in [17],

$$\mathop{\boldsymbol{S}}_{a}^{x}\frac{1}{(\alpha z+\beta)^{s}}\mathop{\Delta}_{w} z=\frac{1}{d}\sum_{j=0}^{d-1}\mathop{\boldsymbol{S}}_{a}^{x+jv}\frac{1}{(\alpha z+\beta)^{s}}\mathop{\Delta}_{wd} z,$$

we derived novel formulas for not only the Hurwitz zeta function, but also other special polynomials.

In 1924, Nörlund studied on the difference equation in the real and complex cases. Mahler [6] studied this equation with the multiplication formula. By the help of his formula, he gave some properties of the Bernoulli polynomials and numbers. He also constructed p-adic integral representation on the suitable interval in the p-adic rational numbers. By applying p limit to the sum of consecutive integers, he gave another definition of the Bernoulli numbers related to the p-adic integral on the related interval.

Using of the method of Riemann, which presumably depend on his famous zeta function, in 1857, German mathematician Rudolf Lipschitz and in 1887, Czech mathematician Mathias Lerch independently defined the following function and investigated on its analytical continuation:

$$\phi(u,c,z)=\sum_{m=0}^{\infty}\frac{e^{2\pi imu}}{(m+c)^{z}},$$

where $\mathrm{Re}(z)>1$, $u \in \mathbb{R}$, set of real numbers, and $c \notin \{0\}\cup\mathbb{Z}^{-}$, $\mathbb{Z}^{-} := \{-1,-2,-3,\ldots\}$ negative integers (*cf.* [1, 10, 16]).

Following this, in this field, very famous mathematicians, physicists and other researchers have examined not only the function $\phi(u,c,z)$, but also this type function in detail and hundreds of varieties have been found to date. These are now known as zeta type functions. American mathematician Tom M. Apostol investigate some properties of the function $\phi(u,c,z)$ in 1951. He gave the analytic continuation of the function $\phi(u,c,z)$ with the help of the Cauchy Residue Theorem. He also found relations between the function $\phi(u,c,z)$ and theta functions. He discovered that the following value of the function $\phi(u,c,z)$ in negative integers is the Apostol-Bernoulli polynomials, which are given below with the generating function; today these polynomials are known by his name:

$$\phi(u,c,-n)=-\frac{\mathcal{B}_{n}\left(e^{2\pi iu};\lambda\right)}{n+1},$$

where $n \in \mathbb{N}_{0}=\mathbb{N}\cup\{0\}$, $\mathbb{N}=\{1,2,3,\ldots\}$, and $\mathcal{B}_{n}\left(e^{2\pi iu};\lambda\right)$ denotes the Apostol-Bernoulli polynomials, which are defined by

$$A(t,x;\lambda)=\frac{te^{tx}}{\lambda e^{t}-1}=\sum_{n=0}^{\infty}\mathcal{B}_{n}(x;\lambda)\frac{t^{n}}{n!} \tag{1.3}$$

(*cf.* [1]).

Putting $\lambda = 1$ in equation (1.3), $\mathcal{B}_n(x; 1)$ reduces to the Bernoulli polynomials $B_n(x)$. Here, we assume that $\lambda \neq 1$. When $x = 0$, equation (1.3) reduces to the Apostol-Bernoulli numbers, which are defined by

$$\frac{t}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(\lambda) \frac{t^n}{n!} \tag{1.4}$$

(*cf.* [1]).

Substituting $x = 0$ into (1.3), for $n = 1$, we have

$$\lambda \mathcal{B}_1(1; \lambda) - \mathcal{B}_1(\lambda) = 1$$

and, for $n \geq 2$, we also have

$$\mathcal{B}_1(1; \lambda) = \frac{1}{\lambda} \mathcal{B}_n(\lambda).$$

Using (1.3) and (1.4), we have

$$\mathcal{B}_n(x; \lambda) = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} \mathcal{B}_j(\lambda) \tag{1.5}$$

(*cf.* [1]).

Using umbral calculus convention in (1.4), we have the following the Apostol-Bernoulli numbers:

$$t = \sum_{n=0}^{\infty} \lambda(\mathcal{B}(\lambda) + 1)^n - \mathcal{B}_n(\lambda) \frac{t^n}{n!}, \tag{1.6}$$

where after applying the binomial expansion, each index of $\mathcal{B}^n$ is to be replaced by the corresponding suffix: $\mathcal{B}_n$, which denotes the Apostol-Bernoulli numbers. Using (1.6), we have

$$\mathcal{B}_0(\lambda) = 0,$$

$$\mathcal{B}_1(\lambda) = \frac{1}{\lambda - 1},$$

$$\mathcal{B}_2(\lambda) = \frac{-2\lambda}{(\lambda - 1)^2}$$

and

$$(\mathcal{B}(\lambda) + 1)^n = \mathcal{B}_n(\lambda),$$

that is

$$\mathcal{B}_n(\lambda) = \sum_{j=0}^{n} \binom{n}{j} \mathcal{B}_j(\lambda)$$

(*cf.* [1]), for detail see also [5, 13, 14, 16]).

The array polynomials are defined by

$$\frac{e^{tx}(e^t - 1)^c}{c!} = \sum_{m=0}^{\infty} S_c^m(x) \frac{t^m}{m!}, \tag{1.7}$$

when $x = 0$, $S_c^m(0)$ reduces to the Stirling numbers of the second kind (*cf.* [12, 16]).

The Eulerian polynomials are defined by

$$\frac{1 - \lambda}{\lambda - e^{t(1-\lambda)}} = \sum_{m=0}^{\infty} A_m(\lambda) \frac{t^m}{m!}$$

(*cf.* [9]).

The motivation of this paper is to summarize as follows:

In Section 2, we construct generating function for the Apostol-Bernoulli polynomials by using the Nörlund sum operator. Applications these transformation formula involving generating function for the Apostol-Bernoulli polynomials, we present the Apostol-Bernoulli polynomials in terms of the Nörlund sum operator. We also give the Apostol-Bernoulli polynomials in terms of the Euler differential operator.

In Section 3, we give some applications of Theorem 2.7, which is given in Section 2. Using these applications, we derive some novel formulas including the array polynomials, the Eulerian polynomials and the Stirling numbers of the second kind. By applying the Euler differential operator (*cf.* [11]):

$$\theta = \lambda \frac{d}{d\lambda}$$

to the function

$$G(\lambda; z) = \lambda^z,$$

we also give some formulas involving the Apostol-Bernoulli polynomials and the Nörlund sum operator.

In Section 4, we give conclusion section.

2. Formulas for the Apostol-Bernoulli polynomials via Nörlund sum operator

In this section, we give some new formulas for the Apostol-Bernoulli polynomials by applying Nörlund sum operator and the Laplace transform to the function

$$\lambda^z \widetilde{f}(z),$$

where $\widetilde{f}(z)$ is the Laplace transform of $f(u)$.

Substituting $\phi(z) = \widetilde{f}(z)$ into (1.1), we modify (3.59) in the book of Jagerman [4] as follows:

$$\underset{a}{\overset{x}{\mathbf{S}}} \widetilde{f}(z) \underset{w}{\Delta} z = \int_0^\infty f(u) \left(\frac{e^{-ta}}{t} - \frac{we^{-tu}}{1 - e^{-wt}} \right) du, \tag{2.1}$$

where $t \geq a > 0$.

When we glance carefully to equation (2.1), we notice that the modification of the generating function for the Bernoulli polynomials springs up on the right-hand side of the equation.

By using (2.1), we have

$$F(x; w) = \underset{a}{\overset{x}{\mathbf{S}}} e^{-\delta z} \underset{w}{\Delta} z = \frac{1}{\delta} e^{-\delta a} + w \frac{e^{-\delta x}}{e^{-\delta w} - 1}, \tag{2.2}$$

where $\delta > 0$ (*cf.* [4, Eq. (3.7)]).

Theorem 2.1. *Let $\lambda \neq 1$.Then we have*

$$\underset{a}{\overset{x}{\mathbf{S}}} \lambda^z \widetilde{f}(z) \underset{w}{\Delta} z = \int_0^\infty f(u) \left(\frac{e^{-a(u - \ln \lambda)}}{u - \ln \lambda} - \frac{w\lambda^x e^{-xu}}{1 - \lambda^w e^{-wt}} \right) du. \tag{2.3}$$

Proof. By substituting

$$\phi(z) = \lambda^z \widetilde{f}(z)$$

into (1.1), we get

$$\underset{a}{\overset{x}{\mathbf{S}}} \lambda^z \widetilde{f}(z) \underset{w}{\Delta} z = \int_a^\infty \lambda^t \widetilde{f}(t) dt - w \sum_{n=0}^\infty \lambda^{x+nw} \widetilde{f}(x + nw).$$

By using definition of the Laplace transform for $\widetilde{f}(t)$, we obtain

$$\mathop{\mathbf{S}}_{a}^{x}\lambda^{z}\widetilde{f}(z)\mathop{\Delta z}_{w} = \int_{a}^{\infty}\lambda^{t}\int_{0}^{\infty}e^{-tu}f(u)dudt - w\lambda^{x}\sum_{n=0}^{\infty}\int_{0}^{\infty}\lambda^{nw}e^{-u(x+nw)}f(u)du.$$

After some calculations in the above equation yield

$$\mathop{\mathbf{S}}_{a}^{x}\lambda^{z}\widetilde{f}(z)\mathop{\Delta z}_{w} = \int_{0}^{\infty}f(u)du\int_{a}^{\infty}e^{-t(u-\ln\lambda)}dt - w\lambda^{x}\int_{0}^{\infty}\frac{e^{-xu}f(u)du}{1-\lambda^{w}e^{-wu}},$$

which yields desired result. □

Remark 2.2. Glancing carefully to equation (2.3), we notice that the modification of the generating function for the Bernoulli polynomials springs up on the right-hand side of the equation. Thus we modify this equation as follows:

$$\mathop{\mathbf{S}}_{a}^{x}\lambda^{z}\widetilde{f}(z)\mathop{\Delta z}_{w} = \int_{0}^{\infty}f(u)\left(\frac{e^{-a(u-\ln\lambda)}}{u-\ln\lambda}+\frac{\lambda^{x}}{u}A\left(-u,\frac{x}{w};\lambda^{w}\right)\right)du.$$

Remark 2.3. When $\lambda = 1$, equation (2.3) reduces to (2.1).

We now give application of equation (2.3).

Substituting

$$\widetilde{f}(z) = (z+\ln\lambda)^{-v},$$

where $v \in \mathbb{N}$, into (2.1), we obtain

$$\mathop{\mathbf{S}}_{a}^{x}(z+\ln\lambda)^{-v}\mathop{\Delta z}_{w} = \int_{0}^{\infty}\left(\frac{e^{-a(u-\ln\lambda)}}{u-\ln\lambda}-\frac{w\lambda^{x}e^{-xu}}{1-\lambda^{w}e^{-wu}}\right)du.$$

Combining (2.3) with (1.3), we get the following theorem:

Theorem 2.4. *Let $\lambda \neq 1$. Then we have*

$$\mathop{\mathbf{S}}_{a}^{x}\lambda^{z}\widetilde{f}(z)\mathop{\Delta z}_{w} = \sum_{n=0}^{\infty}\frac{a^{n}(-1)^{n}}{n!}\int_{0}^{\infty}(u-\ln\lambda)^{n-1}f(u)du + w\lambda^{x}\sum_{n=0}^{\infty}\frac{(-1)^{n}\mathcal{B}_{n}\left(\frac{x}{w};\lambda\right)}{n!}\int_{0}^{\infty}u^{n}w^{n}f(u)du.$$

Theorem 2.5. *Let $\delta > 0$. Then we have*

$$\mathop{\mathbf{S}}_{a}^{x}\lambda^{z}e^{-\delta z}\mathop{\Delta z}_{w} = \frac{e^{-a(\delta-\ln\lambda)}}{\delta-\ln\lambda}-\frac{w\lambda^{x}e^{-\delta x}}{1-\lambda^{w}e^{-\delta w}}. \tag{2.4}$$

Proof. Substituting

$$\phi(z) = \lambda^{z}e^{-\delta z}$$

into (1.1), we get

$$\mathop{\mathbf{S}}_{a}^{x}\lambda^{z}e^{-\delta z}\mathop{\Delta z}_{w} = \int_{a}^{\infty}\lambda^{t}e^{-\delta t}dt - w\sum_{n=0}^{\infty}\lambda^{nw+x}e^{-\delta(x+nw)}.$$

Therefore,

$$\mathop{\mathbf{S}}_{a}^{x}\lambda^{z}e^{-\delta z}\mathop{\Delta z}_{w} = \int_{a}^{\infty}e^{-t(\delta-\ln\lambda)}dt - w\lambda^{x}e^{-\delta x}\frac{1}{1-\lambda^{w}e^{-\delta w}}.$$

After some calculations, we arrive at the desired result. □

Remark 2.6. When $\lambda = 1$, equation (2.4) reduces to equation (2.2).

Theorem 2.7. *Let $\lambda \neq 1$ and $m \in \mathbb{N}_0$. Then we have*

$$\underset{a}{\overset{x}{\mathbf{S}}}\lambda^z z^m \underset{w}{\Delta} z = \sum_{k=0}^{m} \frac{(-1)^{k-m+1}\lambda^a a^k}{k!(\ln\lambda)^{m-k+1}} + \frac{\lambda^x w^{m+1}\mathcal{B}_m\left(\frac{x}{w};\lambda\right)}{m+1}. \tag{2.5}$$

Proof. By using (2.4), with aid of (1.3), we get

$$\sum_{m=0}^{\infty}\underset{a}{\overset{x}{\mathbf{S}}}\lambda^z(-\delta)^m\frac{z^m}{m!}\underset{w}{\Delta} z = \frac{1}{\delta-\ln\lambda}\sum_{m=0}^{\infty}\frac{(-a)^n(\delta-\ln\lambda)^m}{m!} - \frac{\lambda^x}{\delta}\frac{(-\delta w)e^{-\delta x}}{\lambda e^{-\delta w}-1}.$$

Combining the above equation with (1.4) yields

$$\sum_{m=0}^{\infty}\underset{a}{\overset{x}{\mathbf{S}}}\lambda^z(-1)^m\delta^m\frac{z^m}{m!}\underset{w}{\Delta} z = \sum_{m=0}^{\infty}\frac{(-1)^{m+1}a^{m+1}(\delta-\ln\lambda)^m}{(m+1)m!} - \lambda^x\sum_{m=0}^{\infty}\mathcal{B}_{m+1}\left(\frac{x}{w};\lambda\right)\frac{(-1)^{m+1}\delta^m w^{m+1}}{(m+1)m!}.$$

Therefore,

$$\sum_{m=0}^{\infty}\underset{a}{\overset{x}{\mathbf{S}}}\lambda^z(-1)^m\delta^m\frac{z^m}{m!}\underset{w}{\Delta} z = -\lambda^a\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{(-1)^k a^k\delta^m}{k!(\ln\lambda)^{m-k+1}} - \lambda^x\sum_{m=0}^{\infty}\frac{\mathcal{B}_m\left(\frac{x}{w};\lambda\right)}{m+1}w^{m+1}(-1)^{m+1}\frac{\delta^m}{m!}.$$

Comparing the coefficients of $\frac{\delta^m}{m!}$ on both sides of the above equation, after some calculations, we get the desired result. □

3. Applications of Theorem 2.7

In this section, we give some applications of Theorem 2.7. These applications give us many new formulas involving the array polynomials, the Eulerian polynomials and the Stirling numbers of the second kind. We also give an explicit computational formula for the operator

$$\underset{a}{\overset{x}{\mathbf{S}}}\lambda^z z^m \underset{w}{\Delta} z.$$

Finally, by applying the Euler differential operator

$$\theta = \lambda\frac{d}{d\lambda}$$

to the function

$$G(\lambda; z) = \lambda^z$$

and Theorem 2.7 with the help of the Nörlund sum operator, we also derive the following new formulas.

Theorem 3.1. *Let $\lambda \neq 1$ and $m \in \mathbb{N}_0$. Then we have*

$$\underset{a}{\overset{x}{\mathbf{S}}}\lambda^z z^m \underset{w}{\Delta} z = \sum_{k=0}^{m}\frac{(-1)^{k-m+1}\lambda^a a^k}{k!(\ln\lambda)^{m-k+1}} + \frac{m\lambda^x w^{m+1}}{(m+1)(\lambda-1)}\left(\left(\frac{x}{w}\right)^{m-1} + \sum_{n=1}^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^n n! S_n^{m-1}\left(\frac{x}{w}\right)\right).$$

Proof. Combining equation (2.5) with the following formula:

$$\mathcal{B}_m\left(\frac{x}{w};\lambda\right) = \frac{m}{\lambda-1}\left(\left(\frac{x}{w}\right)^{m-1} + \sum_{n=1}^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^n n! S_n^{m-1}\left(\frac{x}{w}\right)\right),$$

(*cf.* [15]), we arrive at the desired result. □

Theorem 3.2. *Let $\lambda \neq 1$. Then we have*

$$\underset{a}{\overset{x}{\mathbf{S}}}\lambda^z z^m \underset{w}{\Delta} z = \sum_{k=0}^{m} \frac{(-1)^{k-m+1}\lambda^a a^k}{k!(\ln \lambda)^{m-k+1}} - \frac{\lambda^{x+1} w^{m+1}}{m+1} \sum_{n=0}^{m} \binom{m}{n} \left(\frac{x}{w}\right)^{m-n} \frac{nA_{n-1}(\lambda)}{(1-\lambda)^n}. \tag{3.1}$$

Proof. By combining equation (2.5) with the following known formula

$$\mathcal{B}_m\left(\frac{x}{w};\lambda\right) = \sum_{n=0}^{m} \binom{m}{n} \left(\frac{x}{w}\right)^{m-n} \mathcal{B}_n(\lambda),$$

we have

$$\underset{a}{\overset{x}{\mathbf{S}}}\lambda^z z^m \underset{w}{\Delta} z = \sum_{k=0}^{m} \frac{(-1)^{k-m+1}\lambda^a a^k}{k!(\ln \lambda)^{m-k+1}} + \frac{\lambda^{x} w^{m+1}}{m+1} \sum_{n=0}^{m} \binom{m}{n} \left(\frac{x}{w}\right)^{m-n} \mathcal{B}_n(\lambda). \tag{3.2}$$

Substituting the following formula

$$\mathcal{B}_n(\lambda) = -\frac{n\lambda A_{n-1}(\lambda)}{(1-\lambda)^n}$$

(*cf.* [15]) into (3.2), we arrive at the desired result. □

Theorem 3.3. *Let $\lambda \neq 1$. Then we have*

$$\underset{a}{\overset{x}{\mathbf{S}}}\lambda^z z^m \underset{w}{\Delta} z = \sum_{k=0}^{m} \frac{(-1)^{k-m+1}\lambda^a a^k}{k!(\ln \lambda)^{m-k+1}} + \frac{m\lambda^x w^{m+1}}{(m+1)(\lambda-1)} \left(\frac{x}{w}\right)^{m-1} + \frac{m\lambda^x w^{m+1}}{(m+1)(\lambda-1)} \sum_{n=1}^{m-1} \left(\frac{\lambda}{1-\lambda}\right)^n \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \left(\frac{x}{w}+j\right)^{m-1}.$$

Proof. Substituting the following formula

$$S_n^{m-1}\left(\frac{x}{w}\right) = \frac{1}{n!} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \left(\frac{x}{w}+j\right)^{m-1},$$

(*cf.* [3, 12]) into Theorem 3.1, we arrive at the desired result. □

By applying the Euler differential operator, which is also known as the Cauchy–Euler operator

$$\theta = \lambda \frac{d}{d\lambda}$$

to the function

$$G(\lambda; z) = \lambda^z$$

n-times, the left side of equation (2.5) transforms into the following differential form:

$$\underset{a}{\overset{x}{\mathbf{S}}}\theta^m \left\{\lambda^z\right\} \underset{w}{\Delta} z = \lambda^a \sum_{k=0}^{m} \frac{(-1)^{k-m+1} a^k}{k!(\ln \lambda)^{m-k+1}} + \frac{\lambda^x w^{m+1} \mathcal{B}_m(\frac{x}{w};\lambda)}{m+1},$$

where

$$\theta^m = \left(\theta^{m-1}\right)\theta.$$

After some elementary calculations, we arrive at the following theorem:

Theorem 3.4. *Let $\lambda \neq 1$ and $m \in \mathbb{N}$. Then we have*

$$\underset{a}{\overset{x}{\mathbf{S}}}\theta^m \left\{\lambda^z\right\} \underset{w}{\Delta} z = \sum_{k=0}^{m} \frac{(-1)^{k-m+1}}{k!(\ln \lambda)^{m-k+1}} \theta^k \left\{\lambda^a\right\} + \frac{\lambda^x w^{m+1} \mathcal{B}_m\left(\frac{x}{w};\lambda\right)}{m+1}. \tag{3.3}$$

Theorem 3.5. *Let $\lambda \neq 1$ and $m \in \mathbb{N}$. Then we have*

$$m\underset{a}{\overset{x}{\mathbf{S}}}\lambda^z z^{m-1}\underset{w}{\Delta} z + (\ln\lambda)\underset{a}{\overset{x}{\mathbf{S}}}\lambda^z z^m\underset{w}{\Delta} z = -\lambda^a a^m + \lambda^x w^m \mathcal{B}_m\left(\frac{x}{w};\lambda\right) + \frac{\lambda^x w^{m+1}\ln\lambda}{m+1}\mathcal{B}_{m+1}\left(\frac{x}{w};\lambda\right). \tag{3.4}$$

Proof. By using (2.4), we obtain

$$\sum_{m=0}^{\infty}\frac{(-1)^m\delta^m}{m!}\underset{a}{\overset{x}{\mathbf{S}}}\lambda^z z^m\underset{w}{\Delta} z = \frac{\lambda^a}{\delta-\ln\lambda}\sum_{m=0}^{\infty}\frac{\delta^m(-a)^m}{m!} - \frac{\lambda^x}{\delta}\sum_{m=0}^{\infty}\mathcal{B}_m\left(\frac{x}{w};\lambda\right)\frac{(-w)^m\delta^m}{m!}.$$

By multiplying each side of the above equation by $(\delta - \ln\lambda)$, after some calculations, we get

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{(-1)^{m-1}m\delta^m}{m!}\underset{a}{\overset{x}{\mathbf{S}}}\lambda^z z^{m-1}\underset{w}{\Delta} z - (\ln\lambda)\sum_{m=0}^{\infty}\frac{(-1)^m\delta^m}{m!}\underset{a}{\overset{x}{\mathbf{S}}}\lambda^z z^m\underset{w}{\Delta} z \quad &= \quad \lambda^a\sum_{m=0}^{\infty}\frac{\delta^m(-a)^m}{m!} - \lambda^x\sum_{m=0}^{\infty}\mathcal{B}_m\left(\frac{x}{w};\lambda\right)\frac{(-w)^m\delta^m}{m!} \\ &\quad +\lambda^x(\ln\lambda)\sum_{m=0}^{\infty}\frac{\mathcal{B}_{m+1}\left(\frac{x}{w};\lambda\right)}{m+1}\frac{(-w)^{m+1}\delta^m}{m!}.\end{aligned}$$

Comparing the coefficients of $\frac{\delta^m}{m!}$ on both sides of the above equation yields

$$(-1)^{m-1}\underset{a}{\overset{x}{\mathbf{S}}}\lambda^z z^{m-1}\underset{w}{\Delta} z - \ln\lambda(-1)^m\underset{a}{\overset{x}{\mathbf{S}}}\lambda^z z^m\underset{w}{\Delta} z = \lambda^a(-a)^m - \lambda^x(-w)^m\mathcal{B}_m(\frac{x}{w};\lambda) + \lambda^x\ln\lambda\frac{\mathcal{B}_{m+1}(\frac{x}{w};\lambda)}{m+1}(-w)^{m+1}.$$

After some calculations in the above equation, we get the desired result. □

Putting $\lambda = 1$ into (3.4), we get the following corollary:

Corollary 3.6. *Let $m \in \mathbb{N}$. Then we have*

$$\underset{a}{\overset{x}{\mathbf{S}}} z^{m-1}\underset{w}{\Delta} z = \frac{-a^m + w^m B_m(\frac{x}{w})}{m}, \tag{3.5}$$

where $B_m\left(\frac{x}{w}\right)$ denotes the Bernoulli polynomials.

Remark 3.7. Putting $a = 0$ into (3.5), we get

$$\underset{0}{\overset{x}{\mathbf{S}}} z^{m-1}\underset{w}{\Delta} z = \frac{w^m B_m(\frac{x}{w})}{m}, \tag{3.6}$$

where $m \in \mathbb{N}$ (*cf.* [4, Eq. (3.132)]).

4. Conclusion

In this article, we gave some properties of not only the Nörlund sum operator, but also the Apostol-Bernoulli polynomials. The motivation of this article is to come up with new formulas for the Apostol-Bernoulli polynomials by applying the Nörlund sum operator to analytic function and periodic function with the aid of the Laplace transform and the difference operator.

Using equation (2.3), we gave the modification of the generating function for the Bernoulli polynomials appears on the right-hand side of the equation.

We gave some applications of Theorem 2.7. The results of these applications give us some formulas involving the array polynomials, the Eulerian polynomials and the Stirling numbers of the second kind.

Moreover, by applying the Euler differential operator

$$\theta = \lambda\frac{d}{d\lambda}$$

to the function

$$G(\lambda; z) = \lambda^z$$

n-times and also using the Nörlund sum operator, we also gave the following new formula for the Apostol-Bernoulli polynomials:

$$\mathcal{B}_m\left(\frac{x}{w};\lambda\right) = \frac{m+1}{\lambda^x w^{m+1}}\left(\sum_{k=0}^{m}\frac{(-1)^{k-m+1}\theta^k\{\lambda^a\}}{k!(\ln\lambda)^{m-k+1}} - \underset{a}{\overset{x}{\mathbf{S}}}\theta^m\{\lambda^z\}\underset{w}{\Delta}z\right).$$

Throughout this article, special cases of the new formulas were given. These special cases of these formulas were also linked to known formulas with which they are related.

The new method given for the Apostol-Bernoulli polynomials in this article has a high potential to be a resource for many researchers.

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