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# A new family of meromorphic multivalent functions defined by Bessel functions

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**Abstract**

In this paper, we define a subclass which consists of Bessel functions denoted by $B^{\nu}_{k,p}(\zeta,\xi)$ of the form $B^{\nu}_{k}f(z) = z^{-p}+\sum\limits_{n=1}^{\infty}\frac{(-\nu)^n}{4^n n!(k)_n}a_{n-p}z^{n-p}$ in the open unit disk $\mathbb{U}^*$. Further, we determine the coefficient estimates, radius of starlikeness, radius of convexity and distortion bounds for the class $B^{\nu}_{k,p}(\zeta,\xi)$.

*Keywords:* Bessel functions, meromorphic, analytic, $p$-valent

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## 1. Introduction

Let us denote by $\Sigma_p$ the class of functions which are meromorphic and $p$-valent and it consists of functions of the following form

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p}z^{n-p}, \quad (p \in \{1,2,\dots\}); \quad a_{n-p} \geq 0 \tag{1.1}$$

are analytic in $\mathbb{U}^* = \{z : z \in \mathbb{C} \text{ such that } 0 < |z| < 1\}$.

The various subclasses of $\Sigma_p$ have been investigated by many authors like, Uralegaddi and Somanatha [17], Akgül [1], Albehbah and Darus [2], Aouf and Hossen [3], Aouf and Srivastava [4], Frasin and Murugusundaramoorthy [10], Owa et al. [14], Yan and Liu [19].

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If $f \in \Sigma_p$ then $f$ belongs to the class of meromorphically $p$-valent starlike functions of order $\alpha$ denoted by $\bigwedge_p^*(\alpha)$, provided $f$ satisfies (1.2). And, $f$ belongs to the class of meromorphically $p$-valent convex functions of order $\alpha$ denoted by $\bigwedge_p(\alpha)$ provided $f$ satisfies (1.3).

$$\Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha \tag{1.2}$$

and

$$\Re\left\{-1-\frac{zf''(z)}{f'(z)}\right\} > \alpha \tag{1.3}$$

$(z \in \mathbb{U}^*; 0 \leq \alpha < p; p \in \{1, 2, 3, \dots\})$.

The first kind of Bessel function of order $t$ is generalized as

$$\Omega_{t,u,v}(z) = \sum_{n=0}^{\infty} \frac{(-v)^n z^{2n+t}}{2^{2n+t} n! \Gamma\left(t + n + \frac{u+1}{2}\right)},$$

where $t, u, v, z \in \mathbb{C}$ and $v \neq 0$. Let us define a map $\phi_{t,u,v} : \mathbb{C} \to \mathbb{C}$ by

$$\phi_{t,u,v}(z) = z^{\frac{-t}{2}} 2^t \Gamma\left(t + \frac{u+1}{2}\right) \Omega_{t,u,v}\left(\sqrt{z}\right).$$

Using the above function many researchers like Baricz [5], Khan et al. [11], Cho et al. [6], Mondal and Swaminathan [13], Deniz et al. [8], Deniz [7], Ezhilarasi et al. [9], Saliu et al. [16], Ramachandran et al. [15] have obtained the generalized Bessel function $\Omega_{t,u,v}(z)$ in the following representation

$$\phi_{t,u,v}(z) = \sum_{n=0}^{\infty} \frac{(-v)^n}{4^n n! (t + \frac{u+1}{2})_n} z^n,$$

where $t + \frac{u+1}{2} \neq z_0^- = \{0, -1, -2, -3, \dots\}$.

For our convenience, let us take $\phi_{t,u,v}(z) = \phi_{k,v}(z)$, $k = t + \frac{u+1}{2}$. Pochhammer symbol in the form of Euler gamma function is defined as

$$(s)_n = \frac{\Gamma(s+n)}{\Gamma(s)} = s(s+1)(s+2)\dots(s+n-1),$$

where $n \in \mathbb{N}$; $s \in \mathbb{C}$ and $(s)_0 = 1$.

Now we define the function $f_g(z)$ as

$$f_g(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p,g} z^{n-p}; \quad g = 1, 2$$

and when we consider two functions $f_1(z)$ and $f_2(z)$, its Hadamard product is stated in the following form

$$(f_1 * f_2)(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{n-p,1} a_{n-p,2} z^{n-p}.$$

For $f \in \Sigma_p$, we have the operator $B_k^v$ as

$$\begin{aligned} B_k^v f(z) &= \frac{1}{z^p} \phi_{k,v}(z) * f(z) \\ &= \frac{1}{z^p} + \sum_{n=1}^{\infty} \frac{(-v)^n}{4^n n! (k)_n} a_{n-p} z^{n-p}. \end{aligned} \tag{1.4}$$

By using the operator $B_k^v$, we consider a subclass of $\Sigma_p$.

**Definition 1.1.** If $f \in \Sigma_p$ and $f$ satisfies the following inequality

$$\left| \frac{z^{p+1}(B_k^v f(z))' + p}{\xi z^{p+1}(B_k^v f(z))' + \zeta p} \right| < 1, \tag{1.5}$$

where $z \in \mathbb{U}^*, -1 \le \xi < 0 < \zeta \le 1; p \in \{1, 2, \dots\}$, then $f \in B_{k,p}^v(\zeta, \xi)$.

By the work of Khan et al. [11] (see also [9, 12, 16, 18, 19]), we have obtained the coefficient bounds for the function $f(z)$ to be in the subclass of $B_{k,p}^v(\zeta, \xi)$, also we have determined the growth and distortion properties. Further the radius of starlikeness and the radius of convexity and linear combination of functions are discussed.

## 2. Main results

This section mainly focuses on deriving the important conditions which facilitate to prove the initially stated function to be in the particular class mentioned above and further it helps to determine the bounds.

**Theorem 2.1.** *The function $f(z)$ belongs to the class $B_{k,p}^v(\zeta, \xi)$ if and only if*

$$p(\zeta - \xi) \ge \sum_{n=1}^{\infty} \frac{(n-p)(1-\xi)|v|^n}{n! 4^n |(k)_n|} a_{n-p}, \tag{2.1}$$

*where $k \neq z_0^-;\ -1 \le \xi < 0 < \zeta \le 1;\ v \neq 0;\ p \in \{1, 2, \dots\}$.*

*Proof.* Let us assume that $f(z)$ satisfies the inequality (2.1), then

$$\begin{aligned}
\left| \frac{p + z^{p+1}(B_k^v f(z))'}{\zeta p + \xi z^{p+1}(B_k^v f(z))'} \right| &= \left| \frac{p + \left( z^{-p} + \sum\limits_{n=1}^{\infty} \frac{(-v)^n}{4^n n!(k)_n} a_{n-p} z^{n-p} \right)' z^{p+1}}{\zeta p + \xi z^{p+1} \left( z^{-p} + \sum\limits_{n=1}^{\infty} \frac{(-v)^n}{4^n n!(k)_n} a_{n-p} z^{n-p} \right)'} \right| \\
&= \left| \frac{\sum\limits_{n=1}^{\infty} (n-p) \frac{(-v)^n}{4^n n!(k)_n} a_{n-p} z^n}{-\xi p + \xi \sum\limits_{n=1}^{\infty} (n-p) \frac{(-v)^n}{4^n n!(k)_n} a_{n-p} z^n + \zeta p} \right| \\
&\le \frac{\sum\limits_{n=1}^{\infty} \frac{(n-p)|v|^n}{n!4^n|(k)_n|} a_{n-p} |z|^n}{p(\zeta - \xi) + \xi \sum\limits_{n=1}^{\infty} \frac{(n-p)|v|^n}{n!4^n|(k)_n|} a_{n-p} |z|^n} \\
&< \frac{\sum\limits_{n=1}^{\infty} \frac{(n-p)|v|^n}{n!4^n|(k)_n|} a_{n-p}}{p(\zeta - \xi) + \xi \sum\limits_{n=1}^{\infty} \frac{(n-p)|v|^n}{n!4^n|(k)_n|} a_{n-p}} \\
&\le 1.
\end{aligned}$$

Hence, by the implication of the famous theorem called maximum modulus, we obtain $f(z) \in B_{k,p}^v \zeta, \xi)$.

Conversely, assume $f(z) \in B_{k,p}^v(\zeta, \xi)$, then by using (1.4) and (1.5) we get

$$\left| \frac{p + z^{p+1}(B_k^v f(z))'}{\zeta p + \xi z^{p+1}(B_k^v f(z))'} \right| = \left| \frac{\sum\limits_{n=1}^{\infty} (n-p) \frac{(-v)^n}{4^n n!(k)_n} a_{n-p} z^n}{-\xi p + \xi \sum\limits_{n=1}^{\infty} (n-p) \frac{(-v)^n}{4^n n!(k)_n} a_{n-p} z^n + \zeta p} \right| < 1. \tag{2.2}$$

Using fact that $|z| \ge |\Re(z)|$ for all $z \in \mathbb{U}$, we have

$$\Re\left(\frac{\sum\limits_{n=1}^{\infty}(n-p)\frac{(-\nu)^n}{4^n n!(k)_n}a_{n-p}z^n}{-\xi p+\xi\sum\limits_{n=1}^{\infty}(n-p)\frac{(-\nu)^n}{4^n n!(k)_n}a_{n-p}z^n+\zeta p}\right)<1. \tag{2.3}$$

Now, for $z^n = r^n e^{i\alpha}$, $\alpha \in \mathbb{R}$ and $\frac{(-\nu)^n}{(k)_n} = \frac{|\nu|^n}{|(k)_n|}e^{i\beta}$, $\beta \in \mathbb{R}$ we choose $\alpha + \beta = 2m\pi$, $m \in \mathbb{N}$, we find from the inequality (2.3)

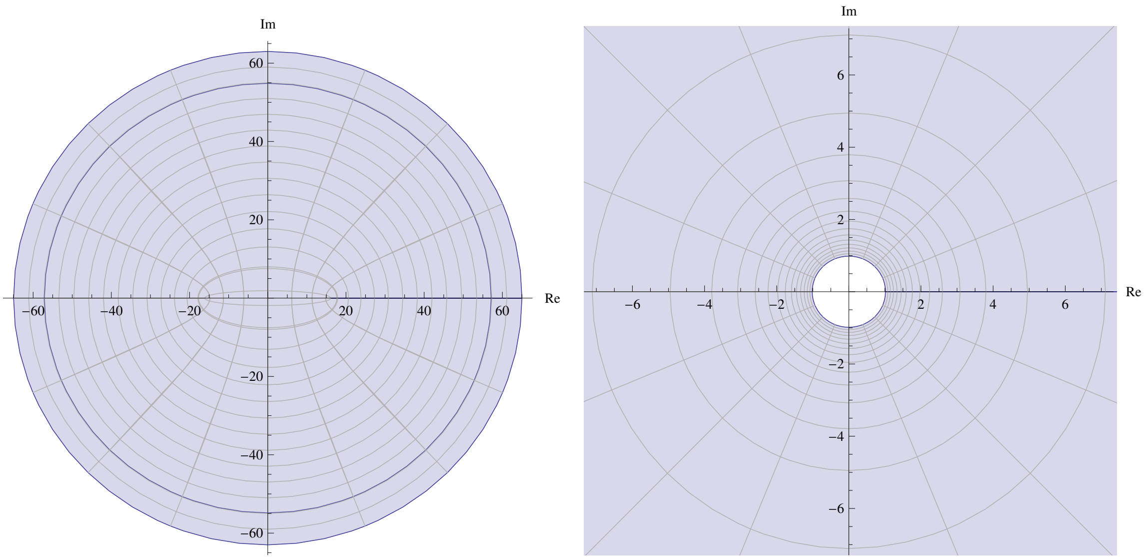
$$\frac{\sum\limits_{n=1}^{\infty}(n-p)\frac{|\nu|^n}{4^n n!|(k)_n|}a_{n-p}\, r^n}{-\xi p+\xi\sum\limits_{n=1}^{\infty}(n-p)\frac{|\nu|^n}{4^n n!|(k)_n|}a_{n-p}\, r^n+\zeta p}<1.$$

Hence by letting $r \to 1^-$, we find the inequality (2.1). For a function $f$, the result is sharp when

$$f_{n-p}(z)=\frac{1}{z^p}+\frac{n!4^n|(k)_n|p(\zeta-\xi)}{(n-p)(1-\xi)|\nu|^n}z^{n-p}. \tag{2.4}$$

□

We found a visualization of above sharp function (2.4) with suitable choice of parameters, namely $f_1(z) = \frac{1}{z} + 64z$ and $f_2(z) = \frac{1}{z} + \frac{1}{64}z$ as shown in Figure 1 (a) and (b).




(a) Image of $\mathbb{U}^*$ under $f_1(z)$

(b) Image of $\mathbb{U}^*$ under $f_2(z)$

Figure 1: Image of $\mathbb{U}^*$ under sharp function

**Corollary 2.2.** *Let's describe the function $f$ as* (1.1) *and $f \in \Sigma p$. If $f \in B^{\nu}_{k,p}(\zeta,\xi)$, then*

$$a_{n-p}\le\frac{n!4^n|(k)_n|p(\zeta-\xi)}{(n-p)(1-\xi)|\nu|^n}, \tag{2.5}$$

*where $n \geq 0$, $k \in \{1, 2, \dots\}$, $-1 \leq \xi < 0 < \zeta \leq 1$.*

**Corollary 2.3.** *If $\zeta = 1$ and $\xi = -1$ in Theorem* 2.1*, as a result, we have the following inequality*

$$\sum_{n=1}^{\infty}\frac{|\nu|^n}{4^n n!|(k)_n|}\frac{(n-p)}{p}a_{n-p}\le 1.$$

The answers to many questions arise from the study of classes of functions can be obtained by determining the growth and distortion bounds as these are the bounds obtained from the modulus of the function and the modulus of derivative. These properties are examined in the following results.

**Theorem 2.4.** *If $f \in B^{v}_{k,p}(\zeta,\xi)$ and $4^n n!|(k)_n| \leq |v|^n$ then for $|z| = r$, we have*

$$\frac{1}{r^p} - \frac{p(\zeta-\xi)}{(1-p)(1-\xi)} r^{1-p} \leq |f(z)| \leq \frac{1}{r^p} + \frac{p(\zeta-\xi)}{(1-p)(1-\xi)} r^{1-p} \tag{2.6}$$

*and it is obtained as a sharp result by*

$$f(z) = \frac{1}{z^p} + \frac{p(\zeta-\xi)}{(1-p)(1-\xi)} z^p.$$

*Proof.* Using the equation (2.1), we have

$$\sum_{n=1}^{\infty} \frac{|v|^n}{4^n n!|(k)_n|} a_{n-p} \leq \frac{p(\zeta-\xi)}{(1-p)(1-\xi)}. \tag{2.7}$$

By the application of (1.1) and (2.7)

$$\begin{aligned} |f(z)| &\leq \frac{1}{r^p} + r^{1-p} \sum_{n=1}^{\infty} \frac{|v|^n}{4^n n!|(k)_n|} a_{n-p} \\ &\leq \frac{1}{r^p} + \frac{p(\zeta-\xi)}{(1-p)(1-\xi)} r^{1-p} \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\geq \frac{1}{r^p} - r^{1-p} \sum_{n=1}^{\infty} \frac{|v|^n}{4^n n!|(k)_n|} a_{n-p} \\ &\geq r^{-p} - \frac{p(\zeta-\xi)}{(1-p)(1-\xi)} r^{1-p}. \end{aligned}$$

Hence the proof. □

**Theorem 2.5.** *If $f \in B^{v}_{k,p}(\zeta,\xi)$ and $4^n n!|(k)_n| \leq |v|^n$ then for the setting of $|z| = r$, we have*

$$pr^{-(p+1)} - \frac{p(\zeta-\xi)}{(1-\xi)} r^{-p} \leq \left|f'(z)\right| \leq pr^{-(p+1)} + \frac{p(\zeta-\xi)}{(1-\xi)} r^{-p}. \tag{2.8}$$

*Proof.* From (1.1) and Theorem 2.1,we obtain

$$\begin{aligned} |f'(z)| &\leq \frac{p}{r^{(p+1)}} + \frac{1}{r^p} \sum_{n=1}^{\infty} \frac{|v|^n}{4^n n!|(k)_n|} (n-p) a_{n-p} \\ &\leq pr^{-(p+1)} + \frac{p(\zeta-\xi)}{(1-\xi)} r^{-p} \end{aligned}$$

and

$$\begin{aligned} |f'(z)| &\geq p|z|^{-(p+1)} - \sum_{n=1}^{\infty} \frac{|v|^n}{4^n n!|(k)_n|} (n-p) a_{n-p} |z|^{n-p-1} \\ &\geq \frac{p}{r^{(p+1)}} - \frac{1}{r^p} \sum_{n=1}^{\infty} \frac{|v|^n}{4^n n!|(k)_n|} (n-p) a_{n-p} \\ &\geq pr^{-(p+1)} - \frac{p(\zeta-\xi)}{(1-\xi)} r^{-p}. \end{aligned}$$

As a consequence, the Theorem 2.5 proof is complete. □

In the following two theorems, we are able to prove the radius of star likeness and the radius of convexity.

**Theorem 2.6.** *The function $f$ is called meromorphically starlike of order $\gamma$ if $f \in B^{v}_{k,p}(\zeta,\xi)$, $\gamma \in [0,1]$ in the disk $|z| < r_1$, where*

$$\begin{aligned} r_1 &= r_1(\zeta,\xi,\gamma) \\ &= \inf_{n\geq 1}\left\{\frac{p(\zeta-\xi)(p-\gamma)}{(n-p)(1-\xi)(n+p-\gamma)}\right\}^{\frac{1}{n}}. \end{aligned}$$

*Further, the above result is sharp provided by* (2.2).

*Proof.* For proving the above, it is sufficient to show that

$$\left|p+\frac{z\left(B^{v}_{k}f(z)\right)'}{B^{v}_{k}f(z)}\right| \leq p-\gamma,$$

for $|z| < r_1$. We have

$$\begin{aligned} \left|\frac{z\left(B^{v}_{k}f(z)\right)'}{B^{v}_{k}f(z)}+p\right| &= \left|\frac{\sum\limits_{n=1}^{\infty}\psi n a_{n-p}z^{n-p}}{1/z^{p}+\sum\limits_{n=1}^{\infty}\psi a_{n-p}z^{n-p}}\right|, \quad \text{where } \psi=\frac{(-v)^{n}}{4^{n}n!(k)_{n}} \\ &\leq \frac{\sum\limits_{n=1}^{\infty}|\psi| n a_{n-p}|z|^{n}}{1-\sum\limits_{n=1}^{\infty}|\psi| a_{n-p}|z|^{n}}. \end{aligned} \tag{2.9}$$

Hence (2.9) holds good if

$$\sum_{n=1}^{\infty}|\psi| n a_{n-p}|z|^{n} \leq (p-\gamma)\left(1-\sum_{n=1}^{\infty}|\psi| a_{n-p}|z|^{n}\right)$$

(or)

$$\sum_{n=1}^{\infty}|\psi|\frac{p+n-\gamma}{p-\gamma}a_{n-p}|z|^{n} \leq 1 \tag{2.10}$$

which is true when (2.1) and (2.10) are used and if

$$\frac{p+n-\gamma}{p-\gamma}|z|^{n} \leq \frac{p(\zeta-\xi)}{(1-\xi)(n-p)}, \tag{2.11}$$

where $n \geq 0$. Further we get

$$|z| < \left\{\frac{p(\zeta-\xi)(p-\gamma)}{(n-p)(1-\xi)(n+p-\gamma)}\right\}^{\frac{1}{n}}. \tag{2.12}$$

As a result, the proof is obtained. □

**Theorem 2.7.** *The function $f$ is called meromorphically convex of order $\gamma$ if $f \in B^{v}_{k,p}(\zeta,\xi)$, $\gamma \in [0,1]$ in the disk $|z| < r_1$, where*

$$\begin{aligned} r_2 &= r_2(\zeta,\xi,\gamma) \\ &= inf_{n\geq 1}\left\{\frac{p(p+\zeta-\xi-\gamma)}{(1-\xi)(n+p-\gamma)}\right\}^{\frac{1}{n}} \end{aligned}$$

*provided the sharp result is obtained by* (2.4).

*Proof.* To demonstrate the above, it is adequate to show that

$$\left|\frac{z\left(B_k^v f(z)\right)''}{\left(B_k^v f(z)\right)'}\right| \leq p-\gamma,$$

for $|z| < r_2$, we have

$$\begin{aligned}\left|p+1+\frac{z\left(B_k^v f(z)\right)''}{\left(B_k^v f(z)\right)'}\right| &= \left|\frac{\sum\limits_{n=1}^{\infty}\psi n(n-p)a_{n-p}z^{n-p-1}}{-p/z^{p+1}+\sum\limits_{n=1}^{\infty}\psi(n-p)a_{n-p}z^{n-p-1}}\right| \\ &\leq \frac{\sum\limits_{n=1}^{\infty}|\psi|n(n-p)a_{n-p}|z|^n}{-p-\sum\limits_{n=1}^{\infty}|\psi|(n-p)a_{n-p}|z|^n}. \end{aligned} \tag{2.13}$$

Hence (2.13) holds good if

$$\sum_{n=1}^{\infty}|\psi|n(n-p)a_{n-p}|z|^n \leq -p(p-\gamma)-\sum_{n=1}^{\infty}|\psi|(n-p)(p-\gamma)a_{n-p}|z|^n$$

(or)

$$\sum_{n=1}^{\infty}|\psi|\frac{(n-p)(p-\gamma+n)}{p(p-\gamma)}a_{n-p}|z|^n \leq 1. \tag{2.14}$$

It is true when (2.1) and (2.14) are used and if

$$\frac{(n+p-\gamma)}{p-\gamma}|z|^n \leq \frac{p(\zeta-\xi)}{(1-\xi)}, \tag{2.15}$$

where $n \geq 0$. And solving (2.11) on the basis of $|z|$, we get

$$|z| \leq \left\{\frac{p(p+\zeta-\xi-\gamma)}{(1-\xi)(n+p-\gamma)}\right\}^{\frac{1}{n}} \tag{2.16}$$

which proves the theorem. □

The following result is a linear combination of multiple functions of the form given in (2.4).

**Theorem 2.8.** *Let*

$$f_{p-1}(z) = z^{-p} \tag{2.17}$$

*and*

$$f_{n-p}(z) = \frac{1}{z^p}+\frac{n!4^n|(k)_n|p(\zeta-\xi)}{(n-p)(1-\xi)|v|^n}z^{n-p} \tag{2.18}$$

*for* $n = \{0,1,2,\dots\}$ *and* $k \in \{1,2,3,\dots\}$. *Then* $f \in B_{k,p}^v(\zeta,\xi)$ *if and only if*

$$f(z) = \sum_{n=0}^{\infty}\lambda_{n-p-1}f_{n-p-1}(z), \tag{2.19}$$

*where* $\lambda_{n-p-1} \geq 0$ *and* $\sum\limits_{n=0}^{\infty}\lambda_{n-p-1} = 1$.

*Proof.* From (2.17), (2.18) and (2.19) it can be easily proven that

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \lambda_{n-p-1} f_{n-p-1}(z) \\ &= \frac{1}{z^p} + \frac{\lambda_{n-p} p(\zeta-\xi) n!|(k)_n| 4^n}{(n-p)(1-\xi)|v|^n} z^{n-p} \\ &= z^{-p} + \sum_{n=1}^{\infty} b_{n-p} z^{n-p}. \end{aligned}$$

Since

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(n-p)(1-\xi)|v|^n}{p(\zeta-\xi) n!(k)_n 4^n} \lambda_{n-p} \times \frac{p(\zeta-\xi) n!(k)_n 4^n}{(n-p)(1-\xi)|v|^n} &= \sum_{n=1}^{\infty} \lambda_{n-p} \\ &= 1 - \lambda_{-(p+1)} \\ &\leq 1. \end{aligned}$$

The function $f \in B^v_{k,p}(\zeta, \xi)$ follows from Theorem 2.1. On the other hand, if suppose that we have $f \in B^v_{k,p}(\zeta, \xi)$. Since then,

$$\left|a_{n-p}\right| \leq \frac{4^n n!(k)_n p(\zeta-\xi)}{(n-p)|v|^n(1-\xi)}, \quad n \geq 0 \tag{2.20}$$

$$\lambda_{n-p} = \frac{(n-p)(1-\xi)|v|^n}{p 4^n n!(k)_n(\zeta-\xi)} a_{n-p}, \quad n = \{0, 1, 2, \dots\}, \quad p \in \mathbb{N} \tag{2.21}$$

and $\lambda_{-p-1} = 1 - \sum\limits_{n=1}^{\infty} \lambda_{n-p}$. Hence, we have (2.19). □

**Theorem 2.9.** *Under convex linear combination, the class $B^v_{k,p}(\zeta, \xi)$ is closed.*

*Proof.* Assume that $f_1$ and $f_2$ are defined by

$$f_j(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{n-p,j} z^{n-p}, \quad (j = 1, 2) \tag{2.22}$$

and belong to the class $B^v_{k,p}(\zeta, \xi)$. Now set $f(z) = (1-\mu) f_2(z) + \mu f_1(z),\ \mu \in [0, 1]$ and using (2.22) we get

$$f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} \left\{ \mu a_{n-p,1} + (1-\mu) a_{n-p,2} \right\} z^{n-p},$$

where $\mu \in [0, 1], z \in \mathbb{U}^*,\ p \in \{1, 2, \dots\}$.

In light of Theorem 2.1, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{|v|^n}{4^n n!|(k)_n|} (n-p)(1-\xi) \left\{ \mu a_{n-p,1} + (1-\mu) a_{n-p,2} \right\} &= \sum_{n=1}^{\infty} (n-p)\mu(1-\xi) \frac{|v|^n}{4^n n!|(k)_n|} a_{n-p,1} \\ &\quad + (1-\mu) \sum_{n=1}^{\infty} (n-p)(1-\xi) \frac{|v|^n}{4^n n!|(k)_n|} a_{n-p,2} \\ &\leq p(\zeta - \xi) \end{aligned}$$

which demonstrates that $f \in B^v_{k,p}(\zeta, \xi)$. □

## 3. Conclusion

During our investigation, we have extensively studied a particular type of meromorphic multivalent function known as $B_k^\nu f(z)$ in $\mathbb{U}^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\}$. Our analysis involved deriving coefficient inequality, which helped us gain insights into the behavior and structure of functions within this subclass. Moreover, we have established distortion bounds that provide valuable information on the preservation of geometric properties under mapping. Additionally, our research delves into inclusion and neighborhood results, which have helped us understand the subclass $B_k^\nu f(z)$. This deeper understanding of its mathematical characteristics and implications will go a long way in advancing the field.

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***Conflict of Interest:*** There is no conflict of interest.

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