Montes Taurus J. Pure Appl. Math. 6 (3), 284–298, 2024

Montes Taurus Journal of Pure and Applied Mathematics
https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

A study of restricted divisor functions with coprime conditions

Nazli Yildiz Ikikardes [a], Daeyeoul Kim [b]

[a]Department of Mathematics and Science Education, Necatibey Faculty of Education, Balikesir University, Turkey
[b]Department of Mathematics and Institute of Pure and Applied Mathematics, Jeonbuk National University, Korea

Abstract

For a natural number N, let $E(N) := \sum_{\substack{d|N \\ d\equiv 1 \pmod 3}} 1 - \sum_{\substack{d|N \\ d\equiv -1 \pmod 3}} 1$ and $\lambda(N) := E(N) - 3E\left(\frac{N}{3}\right)$. The formula for convolution sums $\sum_{t=1}^{N-1} \lambda(t)\lambda(N-t)$ is very well known. In this article, $\sum_{\substack{t=1 \\ \gcd(t,N-t)=1}}^{N-1} \lambda(t)\lambda(N-t)$ is calculated using arithmetical inverse of λ defined by Dirichlet convolution. Furthermore, we define the homogeneous Dirichlet convolution sum of arithmetical functions and find formulas of the homogeneous Dirichlet convolution sums of the inverse divisor functions.

Keywords: Dirichlet convolution, homogeneous convolution sums, restricted divisor functions

2020 MSC: 11A05, 11A25

1. Introduction

The study of the convolution sum of arithmetical functions has been studied by many mathematicians (see [1, 3, 4, 6, 7, 9, 12, 13, 16] and the references therein). Regarding the arithmetical function f, the general convolution sum

$$\sum_{t=1}^{N-1} f(t)f(N-t)$$

has been studied a lot, but the convolution sum of the form

$$\sum_{\substack{t=1 \\ \gcd(t,N-t)=d}}^{N-1} f(t)f(N-t)$$

is not well known. Here, d and N are natural numbers.

In this article, we will mainly study to find the formula of the convolution sum

$$\sum_{\substack{t=1 \\ \gcd(t,N-t)=1}}^{N-1} f(t)f(N-t).$$

†*Article ID:* MTJPAM-D-24-00089
Email addresses: nyildizikikardes@gmail.com (Nazli Yildiz Ikikardes), kdaeyeoul@jbnu.ac.kr (Daeyeoul Kim)
Received:24 June 2024, Accepted:8 July 2024, Published:9 September 2024
⋆*Corresponding Author:* Daeyeoul Kim

To do this, some functions for this article are first introduced. $\mathbb{N}$ and $\mathbb{N}_0$ will be denoted by the set of natural numbers and the set of non-negative integers, respectively. Unless otherwise specified, prime numbers p, p_i, q, q_j, $\mathfrak{p}$ and $\mathfrak{p}_k$ will be expressed as follows.

Let $p \equiv p_i \equiv 1 \pmod 3$ and $q \equiv q_j \equiv 2 \pmod 3$ be positive prime numbers and $\mathfrak{p}$, $\mathfrak{p}_k$ be general positive prime numbers with $i = 1, \ldots, t$, $j = 1, \ldots, r$ and $k = 1, \ldots u$.

For $d, t, N \in \mathbb{N}$ and $s \in \mathbb{N}_0$, we use notations in [14] as follows:

$$\sigma_s(N) := \sum_{d|N} d^s,$$

$$\sigma(N) := \sigma_1(N),$$

$$E(N) := \sum_{\substack{d|N \\ d\equiv 1 \pmod 3}} 1 - \sum_{\substack{d|N \\ d\equiv -1 \pmod 3}} 1,$$

$$\lambda(N) := E(N) - 3E\left(\frac{N}{3}\right),$$

$$D_1(N) := \{d | d \equiv 1 \ (\mathrm{mod}\ 3), d|N\},$$

$$D_2(N) := \{d | d \equiv 2 \ (\mathrm{mod}\ 3), d|N\},$$

$$\Lambda(N) := \sum_{t=1}^{N-1} \lambda(t)\lambda(N-t),$$

$$\overline{\Lambda}(N) := \sum_{\substack{t=1 \\ \gcd(t,N-t)=1}}^{N-1} \lambda(t)\lambda(N-t).$$

Here, divisors are always assumed to be positive divisors. We also make use of the following convention:

$$\sigma_s(N) = E(N) = 0 \text{ if } N \notin \mathbb{N} \text{ and } \Lambda(1) = \overline{\Lambda}(1) := 0.$$

In order to deal with $\sum_{\substack{t=1 \\ \gcd(t,N-t)=1}}^{N-1} \lambda(t)\lambda(N-t)$, some definitions and properties related to Dirichlet convolution are needed, so they are stated below. The Dirichlet convolution sum $h_1 * h_2$ of h_1 and h_2 is defined by

$$(h_1 * h_2)(N) = \sum_{d|N} h_1(d) h_2\left(\frac{N}{d}\right).$$

Define the function δ by $\delta(1) = 1$ and $\delta(N) = 0$ for $N > 1$. Then $h_1 * \delta = \delta * h_1 = h_1$. In fact, the set of arithmetical functions, together with the binary operation of addition and convolution, is a commutative ring $\mathfrak{A}$. Let $h_1 \in \mathfrak{A}$.

An arithmetical function h_1^{-1} is called an inverse of h_1 if $h_1 * h_1^{-1} = h_1^{-1} * h_1 = \delta$. Let $U(\mathfrak{A})$ be the set of units of $\mathfrak{A}$. In fact, $h_1 \in U(\mathfrak{A})$ if and only if $h_1(1) \neq 0$. In detail,

$$h_1^{-1}(1) = \frac{1}{h_1(1)} \tag{1.1}$$

and

$$h_1^{-1}(N) = -\frac{1}{h_1(1)} \sum_{\substack{d|N \\ d>1}} h_1(d) h_1^{-1}\left(\frac{N}{d}\right) \tag{1.2}$$

for all $N > 1$. If $h_i (1 \leq i \leq l)$ are arithmetical functions then their product $h_1 \cdots h_l$ are defined in the usual way: $h_1 \cdots h_l(N) = h_1(N) \cdots h_l(N)$ and $h_1^l(N) = \{h_1(N)\}^l$ for all $l, N \in \mathbb{N}$. Originally, $(h_1^l)^{-1} * h_1^l = h_1^l * (h_1^l)^{-1} = \delta$ is correct, but in this paper, for convenience, we define $h_1^{-l}(N) := (h_1^l)^{-1}(N)$ and $h_1^{-l} * h_1^l = h_1^l * h_1^{-l} = \delta$. For more details on arithmetical functions, refer to [10, 11, 15].

The main results of this paper are as follows.

Theorem 1.1. *Let $n \in \mathbb{N}$. Then we have*

(a) $\overline{\Lambda}(3^n) = 0$.

(b) *If $q \equiv 2 \pmod 3$ is a prime then*

$$\overline{\Lambda}(q^n) = \begin{cases} -\frac{2}{3}(q+1)q^{n-1} & \text{if } n \text{ is even,} \\ \frac{1}{3}(q+1)q^{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

(c) *If $p \equiv 1 \pmod 3$ is a prime then*

$$-\frac{3}{2}\overline{\Lambda}(p^n) = \begin{cases} (p-1) & \text{if } n = 1, \\ (p-1)(p-2) & \text{if } n = 2, \\ (p-1)(p^2+6p+18) & \text{if } n = 3, \\ \sigma(p^n) - 4\sigma(p^{n-1}) + 7\sigma(p^{n-2}) - 8p\sigma_2(p^{\frac{n-4}{2}}) - 2 & \text{if } n (\geq 4) \text{ is even,} \\ \sigma(p^n) - 4\sigma(p^{n-1}) + 7\sigma(p^{n-2}) - 8\sigma_2(p^{\frac{n-1}{2}}) + 2 & \text{if } n (\geq 5) \text{ is odd.} \end{cases}$$

1.1. Arithmetical functions E and λ

By the definition of E,

$$E\left(p_i^{e_i}\right) = (e_i + 1) \tag{1.3}$$

and

$$E\left(q_j^{f_j}\right) = \begin{cases} 0 & \text{if } f_j \equiv 1 \pmod 2, \\ 1 & \text{if } f_j \equiv 0 \pmod 2 \end{cases} \tag{1.4}$$

are easily obtained. For a general natural number N, the following lemma is needed to find the formula of $E(N)$. In fact, Lemma 1.2 is well known.

Lemma 1.2. *E is a multiplicative function.*

Proof. If $d_1, d_2 \equiv 1 \pmod 3$ and $d_3, d_4 \equiv 2 \pmod 3$ are positive integers then

$$d_1 d_2 \equiv d_3 d_4 \equiv 1 \pmod 3 \text{ and } d_1 d_3 \equiv d_1 d_4 \equiv d_2 d_3 \equiv d_2 d_4 \equiv 2 \pmod 3. \tag{1.5}$$

Let $\gcd(m, n) = 1$. By (1.5), we obtain

$$\#D_1(mn) = \#D_1(m)\#D_1(n) + \#D_2(m)\#D_2(n),$$

$$\#D_2(mn) = \#D_1(m)\#D_2(n) + \#D_2(m)\#D_1(n)$$

and

$$\begin{aligned} E(mn) &= \#D_1(mn) - \#D_2(mn) \\ &= (\#D_1(m)\#D_1(n) + \#D_2(m)\#D_2(n)) - (\#D_1(m)\#D_2(n) + \#D_2(m)\#D_1(n)) \\ &= E(m)E(n). \end{aligned}$$

This completes the proof of Lemma 1.2. □

By (1.3), (1.4) and Lemma 1.2, we obtain the next corollary. In order to prove Theorem 1.1, the result of Corollary 1.3 is easy but necessary, so we state it.

Corollary 1.3. *Let N be a positive integer. Then*

$$E(N) = \begin{cases} (e_1+1)\cdots(e_t+1) & \text{if } N = p_1^{e_1}\cdots p_t^{e_t} q_1^{2f_1}\cdots q_r^{2f_r}, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1.4. *λ is a multiplicative function.*

Proof. Let $m_1 = p_1^{e_1} \cdots p_t^{e_t} q_1^{f_1} \cdots q_r^{f_r}$ and $m_2 = p_{t+1}^{e_{t+1}} \cdots p_u^{e_u} q_{r+1}^{f_{r+1}} \cdots q_v^{f_v}$ with e_i $(1 \le i \le u)$, f_j $(1 \le j \le v) \in \mathbb{N}$. That is, $\gcd(m_1, m_2) = 1$. By Lemma 1.2,

$$\lambda(m_1 m_2) = E(m_1 m_2) = E(m_1)E(m_2) = \lambda(m_1)\lambda(m_2). \tag{1.6}$$

Let $m_3 = 3^l p_1^{e_1} \cdots p_t^{e_t} q_1^{f_1} \cdots q_r^{f_r}$ with $l \in \mathbb{N}$. Then

$$\lambda(m_3) = E(m_1) - 3E(m_1) = -2E(m_1),$$

$$\lambda(m_3 m_2) = E(m_1 m_2) - 3E(m_1 m_2) = -2E(m_1 m_2),$$
$$\lambda(m_2) = E(m_2)$$

and

$$\lambda(m_3 m_2) = -2E(m_1 m_2) = (-2E(m_1))(E(m_2)) = \lambda(m_3)\lambda(m_2). \tag{1.7}$$

By (1.6) and (1.7), Lemma 1.4 is obtained. □

In fact, Lemma 1.4 is in [5, p. 79]. By Corollary 1.3 and Lemma 1.4, we obtain

Corollary 1.5. *Let N be a positive integer. Then*

$$\lambda(N) = \begin{cases} (e_1+1)\cdots(e_t+1) & \text{if } N = p_1^{e_1} \cdots p_t^{e_t} q_1^{2f_1} \cdots q_r^{2f_r}, \\ -2(e_1+1)\cdots(e_t+1) & \text{if } N = 3^l p_1^{e_1} \cdots p_t^{e_t} q_1^{2f_1} \cdots q_r^{2f_r}, \\ 0 & \text{otherwise.} \end{cases}$$

2. Arithmetical function E^{-1} and proof of Theorem 2.4

To make the results of this section easier to understand, the following definitions are introduced. Let $h_1, \ldots, h_l \in \mathfrak{A}$, $\mathfrak{p}_1, \ldots, \mathfrak{p}_l$ be distinct primes and $l(\ge 2) \in \mathbb{N}$. We define the homogeneous Dirichlet convolution sum of $h_1, \ldots, h_l$ related to $\mathfrak{p}_1, \ldots, \mathfrak{p}_l$ as

$$(h_1 \star \ldots \star h_l)(\mathfrak{p}_1, \ldots, \mathfrak{p}_l; n) := \sum_{\substack{a_1+\cdots+a_l=n \\ 0 \le a_1, \ldots, a_l \le n}} h_1(\mathfrak{p}_1^{a_1}) \cdots h_l(\mathfrak{p}_l^{a_l})$$

with $n \in \mathbb{N}_0$. For all primes $\mathfrak{p}_i, \mathfrak{p}_j$ and for all $a \in \mathbb{N}_0$, if $h_k(\mathfrak{p}_i^a) = h_k(\mathfrak{p}_j^a)$ $(1 \le k \le l)$, then we define

$$(h_1 \star \ldots \star h_l)(n) := (h_1 \star \ldots \star h_l)(\mathfrak{p}_1, \ldots, \mathfrak{p}_l; n) = \sum_{\substack{a_1+\cdots+a_l=n \\ 0 \le a_1, \ldots, a_l \le n}} h_1(\mathfrak{p}_1^{a_1}) \cdots h_l(\mathfrak{p}_l^{a_l})$$

and if $h = h_1 = \cdots = h_l$ and all of the above conditions are satisfied, then we write

$$h_\star^l(n) := (h \star \ldots \star h)(n) = \sum_{\substack{a_1+\cdots+a_l=n \\ 0 \le a_1, \ldots, a_l \le n}} h(\mathfrak{p}_1^{a_1}) \cdots h(\mathfrak{p}_l^{a_l})$$

for short.

Lemma 2.1. *If $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$ then*

$$E^{-k}(3^n) = \begin{cases} 1 & \text{if } n = 0, \\ -1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since the divisors of 3^n are $1, 3, 3^2, \ldots, 3^n$, $E(3^n) = 1$ with $n \in \mathbb{N}_0$. Using (1.1) and (1.2),

$$E^{-k}(1) = 1,$$
$$E^{-k}(3) = -\frac{1}{E^k(1)} \sum_{\substack{d|3 \\ d>1}} E^k(d) E^{-k}(\frac{3}{d}) = -E^k(3)E^{-k}(1) = -1$$

and

$$E^{-k}(3^2) = -\frac{1}{E^k(1)} \sum_{\substack{d|9 \\ d>1}} E^k(d) E^{-k}\left(\frac{9}{d}\right) = -\left(E^k(3)E^{-k}(3) + E^k(9)E^{-k}(1)\right) = -(1 \cdot (-1) + 1 \cdot 1) = 0.$$

To complete the proof of Lemma 2.1, we have to show that

$$E^{-k}(3^n) = 0 \tag{2.1}$$

with $n \geq 2$. Let's use mathematical induction for n. In the case of $n = 2$, it was shown that (2.1) was satisfied. Assume that $E^{-k}(3^t) = 0$ with $2 \leq t \leq n$. Now, finally, consider the case of $t = n + 1$. That is,

$$\begin{aligned} E^{-k}(3^{n+1}) &= -\frac{1}{E^k(1)} \sum_{\substack{d|3^{n+1} \\ d>1}} E^k(d) E^{-k}\left(\frac{3^{n+1}}{d}\right) \\ &= -\left(E^{-k}(1)E(3^{n+1}) + E^{-k}(3)E^k(3^n) + E^{-k}(3^2)E^k(3^{n-1}) + \cdots + E^{-k}(3^n)E^k(3)\right). \end{aligned}$$

Using $E^{-k}(3^t) = 0$ with $2 \leq t \leq n$, we get $E^{-k}(3^{n+1}) = -(1 \cdot 1 + (-1) \cdot 1 + 0) = 0$. □

Lemma 2.2. *Let $p \equiv 1 \pmod 3$ be a prime.*

(a)

$$E^{-1}(p^n) = \begin{cases} 1 & \text{if } n = 0, \\ -2 & \text{if } n = 1, \\ 1 & \text{if } n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

(b)

$$E^{-2}(p^n) = \begin{cases} 1 & \text{if } n = 0, \\ -4 & \text{if } n = 1, \\ 7 & \text{if } n = 2, \\ (-1)^n 8 & \text{if } n \geq 3. \end{cases}$$

Proof. (a) If $p \equiv 1 \pmod 3$ is a prime and $n \in \mathbb{N}_0$ then $E(p^n) = n + 1$ by (1.3). Using (1.1) and (1.2),

$$E^{-1}(1) = 1,$$
$$E^{-1}(p) = -\frac{1}{E(1)} \sum_{\substack{d|p \\ d>1}} E(d) E^{-1}\left(\frac{p}{d}\right) = -E(p)E^{-1}(1) = -2,$$
$$E^{-1}(p^2) = -\frac{1}{E(1)} \sum_{\substack{d|p^2 \\ d>1}} E(d) E^{-1}\left(\frac{p^2}{d}\right) = -\{E(p)E^{-1}(p) + E(p^2)E^{-1}(1)\} = -\{2 \cdot (-2) + 3 \cdot 1\} = 1$$

and

$$E^{-1}(p^3) = -\left\{E(p)E^{-1}(p^2) + E(p^2)E^{-1}(p) + E(p^3)E^{-1}(1)\right\} = -(2 - 6 + 4) = 0.$$

Using mathematical induction as in the case of $E^{-1}(3^n)$, we can show that $E^{-1}(p^n) = 0$ with $n \geq 3$. (a) is obtained.

(b) It is proved in the same way as (a).

□

Corollary 2.3. *Let $s, t, u \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $\mathfrak{p}_1, \ldots, \mathfrak{p}_t, \mathfrak{q}_1, \ldots, \mathfrak{q}_s, \mathfrak{r}_1, \ldots, \mathfrak{r}_u$ be distinct primes.*

(a)

$$\sigma_0^{-1}(N) = \begin{cases} (-2)^t & \text{if } N = \mathfrak{p}_1 \cdots \mathfrak{p}_t \mathfrak{q}_1^2 \cdots \mathfrak{q}_s^2, \\ 0 & \text{otherwise.} \end{cases}$$

(b) *If $n_1, \ldots, n_u$ are natural numbers greater than 2 and $N = \mathfrak{p}_1 \cdots \mathfrak{p}_t \mathfrak{q}_1^2 \cdots \mathfrak{q}_s^2 \mathfrak{r}_1^{n_1} \cdots \mathfrak{r}_u^{n_u}$ then*

$$\sigma_0^{-2}(N) = (-1)^{t+n_1+\cdots+n_u} 2^{2t+3u} 7^s.$$

Proof. Since $\sigma_0(\mathfrak{p}^n) = n + 1$, using Lemma 2.2, we obtain

$$\sigma_0^{-1}(\mathfrak{p}^n) = \begin{cases} 1 & \text{if } n = 0, \\ -2 & \text{if } n = 1, \\ 1 & \text{if } n = 2, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \sigma_0^{-2}(\mathfrak{p}^n) = \begin{cases} 1 & \text{if } n = 0, \\ -4 & \text{if } n = 1, \\ 7 & \text{if } n = 2, \\ (-1)^n 8 & \text{if } n \geq 3 \end{cases}$$

for all prime numbers $\mathfrak{p}$. Using these, (a) and (b) are obtained. □

Theorem 2.4. *Let $n \in \mathbb{N}_0$ and $l(\geq 2) \in \mathbb{N}$.*

(a) *If $n \geq 2l + 1$ is a positive integer then*

$$(\sigma_0^{-1})^l_\star(n) = 0.$$

(b) *If $n(0 \leq n \leq l)$ is an integer then*

$$(\sigma_0^{-1})^l_\star(n) = (\sigma_0^{-1})^l_\star(2l - n).$$

(c) *If $n(0 \leq n \leq l)$ is an integer then*

$$(\sigma_0^{-1})^l_\star(n) = \begin{cases} \sum_{i=0}^{k} \binom{l}{2i, k-i, l-k-i} (-2)^{2i} & \text{if } n = 2k, \\ \sum_{i=1}^{k+1} \binom{l}{2i-1, k-i+1, l-k-i} (-2)^{2i-1} & \text{if } n = 2k + 1 \end{cases}$$

with $\binom{l}{a_1, \ldots, a_s} = \frac{l!}{(a_1)! \cdots (a_s)!}$, $a_1 + \cdots + a_s = l$ and $0 \leq a_1, \ldots, a_s \leq l$.
In particular,

$$(\sigma_0^{-1})^l_\star(0) = (\sigma_0^{-1})^l_\star(2l) = 1$$

and

$$(\sigma_0^{-1})^l_\star(1) = (\sigma_0^{-1})^l_\star(2l - 1) = -2l.$$

(d) *Let $S_{(i,j)}(n) := ((\sigma_0^{-1})^i \star (\sigma_0^{-1})^j)(n) = ((\sigma_0^{-1})^j \star (\sigma_0^{-1})^i)(n) = \sum_{\substack{a_1+a_2=n \\ 0 \leq a_1, a_2 \leq n}} (\sigma_0^{-1})^i(\mathfrak{p}_1^{a_1})(\sigma_0^{-1})^j(\mathfrak{p}_2^{a_2})$ with $i, j \in \{1, 2\}$. Then*

n	0	1	2	3	4	$n(\geq 5)$
$S_{(1,2)}(n)$	1	−6	16	−26	31	$(-1)^n \cdot 32$
$S_{(2,2)}(n)$	1	−8	30	−72	129	$(-1)^n(n-2) \cdot 64$

Proof. (a) By the pigeonhole principle, it is trivial.

(b) Let $0 \le n \le l$. It is the same as the problem of thinking of the number of ways of arranging $0, 1, 2$ overlapping in l positions. In addition, it is a problem to obtain a result by multiplying by 1 when a value of 0 or 2 is obtained, and by multiplying by -2 when a value of 1 is obtained. For convenience, $a_0 :=$ the number of 0, $a_1 :=$ the number of 1, $a_2 :=$ the number of 2 and $a_0 + a_1 + a_2 = l$. If $n = 1 \cdot a_1 + 2 \cdot a_2 + 0 \cdot a_3$ and $a_1 + a_2 + a_3 = l$ then $2l = 2a_1 + 2a_2 + 2a_3$ and $2l - n = 1 \cdot a_1 + 0 \cdot a_2 + 2 \cdot a_3$.

Let $\mathfrak{p}, \mathfrak{p}_1, \ldots, \mathfrak{p}_{a_1+a_2+a_3}$ be distinct primes. Using $\sigma_0^{-1}(\mathfrak{p}^0) = \sigma_0^{-1}(\mathfrak{p}^2)$ and

$$\begin{aligned}
&\sigma_0^{-1}(\mathfrak{p}_1)\cdots\sigma_0^{-1}(\mathfrak{p}_{a_1})\sigma_0^{-1}(\mathfrak{p}^2_{a_1+1})\cdots\sigma_0^{-1}(\mathfrak{p}^2_{a_1+a_2})\sigma_0^{-1}(\mathfrak{p}^0_{a_1+a_2+1})\cdots\sigma_0^{-1}(\mathfrak{p}^0_{a_1+a_2+a_3})\\
&= \sigma_0^{-1}(\mathfrak{p}_1)\cdots\sigma_0^{-1}(\mathfrak{p}_{a_1})\sigma_0^{-1}(\mathfrak{p}^0_{a_1+1})\cdots\sigma_0^{-1}(\mathfrak{p}^0_{a_1+a_2})\sigma_0^{-1}(\mathfrak{p}^2_{a_1+a_2+1})\cdots\sigma_0^{-1}(\mathfrak{p}^2_{a_1+a_2+a_3})\\
&= (-2)^{a_1},
\end{aligned}$$

we obtain (b).

(c) Let $0 \le n \le l$ and let $n = 2k$. If $a_2 = k$, then $a_0 = l - k$ and $a_1 = 0$. Hence, the value in this case is $\frac{l!}{k!(l-k)!} \cdot (-2)^0$. If $a_2 = k - 1$, then $a_1 = 2$ and $a_0 = l - k - 1$. Therefore, the value in this case is $\frac{l!}{(k-1)!2!(l-k-1)!} \cdot (-2)^2$. In this way, if we find the value while decreasing a_2, finally, let's find the case where $a_2 = 0$. That is, if $a_2 = 0$ then $a_1 = 2k$, $a_0 = l - 2k$ and the value of this case is $\frac{l!}{0!(2k)!(l-2k)!} \cdot (-2)^{2k}$. Thus,

$$(\sigma_0^{-1})^l_\star(2k) = \sum_{i=0}^{k} \frac{l!}{(k-i)!(2i)!(l-k+i)!}(-2)^{2i}.$$

In the same way as in the case of $n = 2k$,

$$(\sigma_0^{-1})^l_\star(2k+1) = \sum_{i=1}^{k+1} \frac{l!}{(k-i+1)!(2i-1)!(l-k-i)!}(-2)^{2i-1}$$

can be obtained.

(d) By (1.1), (1.2) and Corollary 2.3 (b),

$$\begin{gathered}
S_{(1,2)}(0) = \sigma_0^{-1}(1)(\sigma_0^{-1})^2(1) = 1,\\
S_{(1,2)}(1) = \sigma_0^{-1}(\mathfrak{p}_1^0)(\sigma_0^{-1})^2(\mathfrak{p}_2) + \sigma_0^{-1}(\mathfrak{p}_1)(\sigma_0^{-1})^2(\mathfrak{p}_2^0) = -6,\\
S_{(1,2)}(2) = 16,\\
S_{(1,2)}(3) = -26,\\
S_{(1,2)}(4) = 31,\\
S_{(1,2)}(5) = -32
\end{gathered}$$

and

$$S_{(1,2)}(6) = 32.$$

Using $1 \cdot (-8) + (-2) \cdot 8 + 1 \cdot (-8) = -32$ and $1 \cdot (8) + (-2) \cdot (-8) + 1 \cdot (8) = 32$, we obtain $S_{(1,2)}(n) = (-1)^n \cdot 32$ with $n(\ge 5) \in \mathbb{N}$. In the same way as for finding $S_{(1,2)}(n)$, we can find the formula for $S_{(2,2)}(n)$ with $n \in \mathbb{N}$.

□

Example 2.5. It is well-known that $\sigma_0(\mathfrak{p}^n) = n + 1$ with $n \in \mathbb{N}_0$. Using the above result and Theorem 2.4 to find the values for $l = 2$ and 3,

$$(\sigma_0)^2_\star(n) = Pyr_3(n+1) = \binom{n+3}{3},$$

$$(\sigma_0^{-1})^2_\star(n) = \begin{cases} 1 & \text{if } n = 0, \\ -4 & \text{if } n = 1, \\ 6 & \text{if } n = 2, \\ -4 & \text{if } n = 3, \\ 1 & \text{if } n = 4, \\ 0 & \text{if } n \geq 5 \end{cases}$$

and

$$(\sigma_0^{-1})^3_\star(n) := (\sigma_0^{-1} \star \sigma_0^{-1} \star \sigma_0^{-1})(n) = \sum_{\substack{a_1+a_2+a_3=n \\ 0\leq a_1,a_2,a_3\leq n}} \sigma_0^{-1}(\mathfrak{p}_1^{a_1})\sigma_0^{-1}(\mathfrak{p}_2^{a_2})\sigma_0^{-1}(\mathfrak{p}_3^{a_3}) = \begin{cases} 1 & \text{if } n = 0, \\ -6 & \text{if } n = 1, \\ 15 & \text{if } n = 2, \\ -20 & \text{if } n = 3, \\ 15 & \text{if } n = 4, \\ -6 & \text{if } n = 5, \\ 1 & \text{if } n = 6, \\ 0 & \text{if } n \geq 7. \end{cases}$$

Here, $\mathrm{Pyr}_\alpha(x) = \frac{1}{6}x(x+1)((\alpha-2)x + 5 - \alpha)$ denote the αth order pyramid number (*cf.* [2]).

Remark 2.6. By Lemma 2.2 (a) and Theorem 2.4,

$$(E^{-1} \star \ldots \star E^{-1})(p_1, \ldots, p_l; n) := \sum_{\substack{a_1+\cdots+a_l=n \\ 0\leq a_1,\ldots,a_l\leq n}} E^{-1}(p_1^{a_1}) \cdots E^{-1}(p_l^{a_l}) = (\sigma_0^{-1})^l_\star(n)$$

with $n \in \mathbb{N}_0$.

Lemma 2.7. *Let $q \equiv 2 (\mathrm{mod}\ 3)$ be a prime. If $k \in \mathbb{N}$ then*

$$E^{-k}(q^n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n = 1, \\ -1 & \text{if } n = 2, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad (E^{-k})^2_\star(n) = \begin{cases} 1 & \text{if } n = 0, 4, \\ 0 & \text{if } n = 1, 3, \\ -2 & \text{if } n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By (1.4),

$$E(q^n) = \begin{cases} 0 & \text{if } n \equiv 1 \ (\mathrm{mod}\ 2), \\ 1 & \text{if } n \equiv 0 \ (\mathrm{mod}\ 2). \end{cases} \tag{2.2}$$

Using (1.1) and (1.2),

$$E^{-k}(1) = 1,$$

$$E^{-k}(q) = -E^k(q)E^{-k}(1) = 0,$$

$$E^{-k}(q^2) = -(E^k(q)E^{-k}(q) + E^k(q^2)E^{-k}(1)) = -(0 + 1 \cdot 1) = -1,$$

$$E^{-k}(q^3) = -(E^k(q)E^{-k}(q^2) + E^k(q^2)E^{-k}(q) + E^k(q^3)E^{-k}(1)) = 0$$

and

$$E^{-k}(q^4) = -(E^k(q)E^{-k}(q^3) + E^k(q^2)E^{-k}(q^2) + E^k(q^3)E^{-k}(q) + E^k(q^4)E^{-k}(1)) = -(1 + 0 - 1 + 0) = 0.$$

Using mathematical induction, we can show that $E^{-k}(q^n) = 0$ if $n \geq 3$.
If $n = a + b$ is an odd integer then a or b is odd. So, $E^{-k}(q_1^a) = 0$ or $E^{-k}(q_2^b) = 0$. Thus, if n is odd then

$$(E^{-k})^2_\star(n) = \sum_{a+b=n} E^{-k}(q_1^a)E^{-k}(q_2^b) = 0.$$

On the other hand,

$$(E^{-k})^2_\star(0) = E^{-k}(q_1^0)E^{-k}(q_2^0) = 1,$$

$$(E^{-k})^2_\star(2) = E^{-k}(q_1^0)E^{-k}(q_2^2) + E^{-k}(q_1)E^{-k}(q_2) + E^{-k}(q_1^2)E^{-k}(q_2^0) = -2,$$

and

$$(E^{-k})^2_\star(4) = E^{-k}(q_1^2)E^{-k}(q_2^2) = 1.$$

If $n = a + b \geq 5$ then $a \geq 3$ or $b \geq 3$. Hence,

$$(E^{-k})^2_\star(n) = \sum_{a+b=n} E^{-k}(q_1^a)E^{-k}(q_2^b) = 0.$$

Therefore, Lemma 2.7 is obtained. □

To find $E^{-1}(N)$ and $\lambda^{-1}(N)$ for a general positive integer N, the following proposition is necessary.

Proposition 2.8 (*cf.* [10, p. 8]). *If h_1 is a multiplicative function then h_1^{-1} is a multiplicative function.*

Using Lemma 1.2, Lemma 1.4 and Proposition 2.8, we get the following corollary:

Corollary 2.9. *If k is a positive integer then E^{-k} and λ^{-k} are multiplicative functions.*

$E^{-1}(3^{m_1}\prod p_i^{e_i}\prod q_j^{f_j})$	3^{m_1}	$\prod p_i^{e_i}$	$\prod q_j^{f_j}$
1	1	1	1
	-1	1	-1
-1	1	1	-1
	-1	1	1
2^{2n+1}	-1	$(-2)^{2n+1}$	1
	1	$(-2)^{2n+1}$	-1
-2^{2n+1}	1	$(-2)^{2n+1}$	1
	-1	$(-2)^{2n+1}$	-1
2^{2n}	1	$(-2)^{2n}$	1
	-1	$(-2)^{2n}$	-1
-2^{2n}	1	$(-2)^{2n}$	-1
	-1	$(-2)^{2n}$	1
0	0	$\cdot$	$\cdot$
	$\cdot$	0	$\cdot$
	$\cdot$	$\cdot$	0

Table 1. Values of $E^{-1}\left(3^{m_1}\prod p_i^{e_i}\prod q_j^{f_j}\right)$

Using Lemma 2.2, Lemma 2.7, Corollary 2.9 and Table 1, we get the following theorem:

Theorem 2.10. *Let* $p_i, g_j \equiv 1 \pmod 3$*,* $q_k \equiv 2 \pmod 3$ *be distinct primes,* $r, s, t \in \mathbb{N}_0$ *and* $N \in \mathbb{N}$*. Then*

$$E^{-1}(N) = \begin{cases} 2^{2r+1} & \text{if } N = \prod_{k=1}^{2r+1} g_k \prod_{i=1}^{t} p_i^2 \prod_{j=1}^{2s+1} q_j^2 \text{ or } 3\prod_{k=1}^{2r+1} g_k \prod_{i=1}^{t} p_i^2 \prod_{j=1}^{2s} q_j^2, \\ -2^{2r+1} & \text{if } N = \prod_{k=1}^{2r+1} g_k \prod_{i=1}^{t} p_i^2 \prod_{j=1}^{2s} q_j^2 \text{ or } 3\prod_{k=1}^{2r+1} g_k \prod_{i=1}^{t} p_i^2 \prod_{j=1}^{2s+1} q_j^2, \\ 2^{2r} & \text{if } N = \prod_{k=1}^{2r} g_k \prod_{i=1}^{t} p_i^2 \prod_{j=1}^{2s} q_j^2 \text{ or } 3\prod_{k=1}^{2r} g_k \prod_{i=1}^{t} p_i^2 \prod_{j=1}^{2s+1} q_j^2, \\ -2^{2r} & \text{if } N = \prod_{k=1}^{2r} g_k \prod_{i=1}^{t} p_i^2 \prod_{j=1}^{2s+1} q_j^2 \text{ or } 3\prod_{k=1}^{2r} g_k \prod_{i=1}^{t} p_i^2 \prod_{j=1}^{2s} q_j^2, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 2.11. *The followings hold true:*

- **(a)** *If* $N \equiv 2 \pmod 3$ *is a positive integer then* $E^{-1}(N) = 0$*.*
- **(b)** *If* r *is a prime with* $r^3|N$ *then* $E^{-1}(N) = 0$*.*
- **(c)** *If* $q \equiv 2 \pmod 3$ *is a prime with* $q|N$ *and* $q^2 \nmid N$ *then* $E^{-1}(N) = 0$*.*
- **(d)** *If* $9|N$ *then* $E^{-1}(N) = 0$*.*

Proposition 2.12. *Let* $m \in \mathbb{N}_0$ *and* $k, n \in \mathbb{N}$*. Then we have*

(a)

$$\sum_{i=0}^{m} E^k(3^i)E^k(3^{m-i}) = m + 1,$$

$$\sum_{i=0}^{m} E^k(3^i)E^{-k}(3^{m-i}) = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m \geq 1 \end{cases}$$

and

$$\sum_{i=0}^{m} E^{-k}(3^i)E^{-k}(3^{m-i}) = \begin{cases} 1 & \text{if } m = 0, \\ -2 & \text{if } m = 1, \\ 1 & \text{if } m = 2, \\ 0 & \text{if } m \geq 3. \end{cases}$$

(b)

$$\mathfrak{E}^k(3^n) := \sum_{i=1}^{3^n-1} E^k(i)E^k(3^n - i) = 0.$$

(c)

$$\mathfrak{E}^{-1}(3^n) := \sum_{i=1}^{3^n-1} E^{-1}(i)E^{-1}(3^n - i) = 0.$$

Proof. (a) By (1.1), (1.2), (1.3), (1.4), Corollary 1.3, Corollary 2.11 and Lemma 2.1, we obtain (a).

(b) Let $l \equiv 2 \pmod 3$ be a positive integer. It is easily checked that

$$\#D_1(l) = \#D_2(l)$$

and

$$E(l) = E^k(l) = 0. \tag{2.3}$$

By (2.3),

$$S_a(3^n) := \sum_{\substack{i=1 \\ i\equiv a (\mathrm{mod} 3)}}^{3^n-1} E^k(i)E^k(3^n-i) = 0$$

with $a = 1, 2$. On the other hand,

$$\mathfrak{E}^k(3) = \sum_{i=1}^{3-1} E^k(i)E^k(3-i) = S_1(3) + S_2(3) = 0$$

and

$$\mathfrak{E}^k(3^2) = \sum_{i=1}^{3^2-1} E^k(i)E^k(3^2-i) = S_1(3^2) + S_2(3^2) + (E^k(3))^2\mathfrak{E}^k(3) = 0.$$

In the same way, we can assume that $\mathfrak{E}^k(3^u) = 0$ for $1 \le u \le n$. Therefore, we get

$$\mathfrak{E}^k(3^{n+1}) = S_1(3^{n+1}) + S_2(3^{n+1}) + (E^k(3))^2\mathfrak{E}^k(3^n) = 0.$$

(c) By Corollary 2.11 (a),

$$\mathfrak{E}^{-1}(3) := 2E^{-1}(1)E^{-1}(2) = 0$$

and

$$\mathfrak{E}^{-1}(9) := 2(E^{-1}(1)E^{-1}(8) + E^{-1}(2)E^{-1}(7)) + (E^{-1}(3))^2\mathfrak{E}^{-1}(3) = 0.$$

Let $n \ge 3$. By Corollary 2.11 (a) and (d),

$$\begin{aligned}\mathfrak{E}^{-1}(3^n) :&= \sum_{\substack{i=1 \\ i\equiv 1,2 \ (\mathrm{mod} 3)}}^{3^n-1} E^{-1}(i)E^{-1}(3^n-i) + \sum_{i=1}^{3^{n-1}-1} E^{-1}(3i)E^{-1}(3^n-3i) \\ &= (E^{-1}(3))^2 \sum_{\substack{i=1 \\ i\equiv 1,2 \ (\mathrm{mod} 3)}}^{3^{n-1}-1} E^{-1}(i)E^{-1}(3^{n-1}-i) + \sum_{i=1}^{3^{n-2}-1} E^{-1}(9i)E^{-1}(3^n-9i) \\ &= 0.\end{aligned}$$

□

3. Arithmetical function λ^{-1}

Using the definition of $\lambda(3^n)$, we get $\lambda(3^0) = 1$ and $\lambda(3^n) = E(3^n) - 3E(3^{n-1}) = -2$ with $n \in \mathbb{N}$. So,

$$\lambda(3^n) = \begin{cases} 1 & \text{if } n = 0, \\ -2 & \text{otherwise.} \end{cases} \tag{3.1}$$

By (1.3) and (1.4),

$$\lambda\left(p_i^{e_i}\right) = E\left(p_i^{e_i}\right) = (e_i + 1) \tag{3.2}$$

and

$$\lambda\left(q_j^{f_j}\right) = E\left(q_j^{f_j}\right) = \begin{cases} 0 & \text{if } f_j \equiv 1 \ (\mathrm{mod}\ 2), \\ 1 & \text{if } f_j \equiv 0 \ (\mathrm{mod}\ 2). \end{cases} \tag{3.3}$$

Theorem 3.1. *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then*

$$\lambda^{-1}(3^n) = \begin{cases} 1 & \text{if } n = 0, \\ 2 \cdot 3^{n-1} & \text{if } n \geq 1, \end{cases}$$

$$\lambda^{-2}(3^n) = \begin{cases} 1 & \text{if } n = 0, \\ (-1)^n 4 \cdot 3^{n-1} & \text{if } n \geq 1, \end{cases}$$

$$\lambda^{-1}(p^n) = \begin{cases} 1 & \text{if } n = 0, \\ -2 & \text{if } n = 1, \\ 1 & \text{if } n = 2, \\ 0 & \text{if } n \geq 3, \end{cases}$$

$$\lambda^{-2}(p^n) = \begin{cases} 1 & \text{if } n = 0, \\ -4 & \text{if } n = 1, \\ 7 & \text{if } n = 2, \\ (-1)^n 8 & \text{if } n \geq 3, \end{cases}$$

and

$$\lambda^{-k}(q^n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n = 1, \\ -1 & \text{if } n = 2, \\ 0 & \text{if } n \geq 3. \end{cases}$$

Proof. By (3.1),

$$\lambda^{-1}(1) = \lambda^{-2}(1) = 1,$$
$$\lambda^{-1}(3) = -\lambda^{-1}(1)\lambda(3) = 2$$

and

$$\lambda^{-2}(3) = -\lambda^{-2}(1)\lambda^2(3) = -4.$$

May assume that

$$\lambda^{-1}(3^n) = 2 \cdot 3^{n-1} \quad \text{and} \quad \lambda^{-1}(3^n) = (-1)^n 4 \cdot 3^{n-1} \tag{3.4}$$

with $n \geq 1$. By (1.2), (3.1) and (3.4),

$$\begin{aligned} \lambda^{-1}(3^n) &= -\left(\lambda^{-1}(1)\lambda(3^{n+1}) + \lambda^{-1}(3)\lambda(3^n) + \cdots + \lambda^{-1}(3^n)\lambda(3)\right) \\ &= 2\left(\lambda^{-1}(1) + \lambda^{-1}(3) + \cdots + \lambda^{-1}(3^n)\right) \\ &= 2\left(1 + 2 \cdot 3^0 + \cdots + 2 \cdot 3^{n-1}\right) \end{aligned}$$

and

$$\begin{aligned} \lambda^{-2}(3^n) &= -\left(\lambda^{-2}(1)\lambda^2(3^{n+1}) + \lambda^{-2}(3)\lambda^2(3^n) + \cdots + \lambda^{-2}(3^n)\lambda^2(3)\right) \\ &= -4\left(\lambda^{-2}(1) + \lambda^{-2}(3) + \cdots + \lambda^{-2}(3^n)\right) \\ &= -4\left(1 - 4 \cdot 3^0 + \cdots + (-1)^n 4 \cdot 3^{n-1}\right). \end{aligned}$$

Therefore,

$$\lambda^{-1}(3^n) = \begin{cases} 1 & \text{if } n = 0, \\ 2 \cdot 3^{n-1} & \text{if } n \geq 1 \end{cases} \quad \text{and} \quad \lambda^{-2}(3^n) = \begin{cases} 1 & \text{if } n = 0, \\ (-1)^n 4 \cdot 3^{n-1} & \text{if } n \geq 1. \end{cases}$$

Proofs for the remaining cases are obtained from Lemma 2.2 and Lemma 2.7. □

4. Proof of Theorem 1.1

We state the result related to the convolution sum of the arithmetical function λ and necessary to prove Theorem 1.1 below.

Proposition 4.1 (*cf.* [3, Theorem 1], [14]). *If $N \geq 2$, then*

$$\Lambda(N) = \sum_{t=1}^{N-1} \lambda(t)\lambda(N-t) = \begin{cases} \frac{2}{3}\left(E(N) - \sigma(N)\right) & \text{if } N \equiv 1 \pmod 3, \\ \frac{1}{3}\sigma(N) & \text{if } N \equiv 2 \pmod 3, \\ \frac{4}{3}\left(-E(n) + \sigma(n)\right) & \text{if } N = 3^e n,\ e \geq 1,\ \gcd(3,n) = 1. \end{cases}$$

Proof of Theorem 1.1. (a) If $k, n \in \mathbb{N}$ with $k \not\equiv 0 \pmod 3$ then $\lambda(k) = 0$ or $\lambda(3^n - k) = 0$ by Corollary 1.5. Thus,

$$\overline{\Lambda}(3^n) = \sum_{\substack{k=1 \\ \gcd(k, 3^n - k) = 1}} \lambda(k)\lambda(3^n - k) = 0.$$

(b) By Proposition 4.1, $\overline{\Lambda}(q) = \Lambda(q) = \frac{1}{3}(q+1)$.

Let $n(\geq 2)$ be a positive integer. If $k \in \{1, \ldots, q^n - 1\}$ then $\gcd(k, q^n - k) = q^i$ with $i \in \{0, 1, \ldots, n-1\}$. Using Lemma 1.4,

$$\Lambda(q^n) = \lambda(1)^2\overline{\Lambda}(q^n) + \lambda(q)^2\overline{\Lambda}(q^{n-1}) + \cdots + \lambda(q^{n-1})^2\overline{\Lambda}(q) + \lambda(q^n)^2\overline{\Lambda}(1)$$

and

$$\overline{\Lambda}(q^n) = (\lambda^{-2} * \Lambda)(q^n). \tag{4.1}$$

It is easy, but remember, if $n \equiv 1$(resp., 0) (mod 2) then $q^n \equiv 2$(resp., 1) (mod 2).

Putting $n = 2$ in (4.1),

$$\overline{\Lambda}(q^2) = \lambda^{-2}(1)\Lambda(q^2) + \lambda^{-2}(q)\Lambda(q) = \frac{2}{3}\left(E(q^2) - \sigma(q^2)\right) + 0 = -\frac{2}{3}q(q+1)$$

by Theorem 3.1. If $n(\geq 3)$ is a positive integer then

$$\begin{aligned} \overline{\Lambda}(q^n) &= \lambda^{-2}(1)\Lambda(q^n) + \lambda^{-2}(q^2)\Lambda(q^{n-2}) \\ &= \begin{cases} \frac{1}{3}\left(\sigma(q^n) - \sigma(q^{n-2})\right) = \frac{1}{3}q^{n-1}(q+1) & \text{if } n \equiv 1 \pmod 2, \\ \frac{2}{3}\left((E(q^n) - \sigma(q^n)) - \left(E(q^{n-2}) - \sigma(q^{n-2})\right)\right) = -\frac{2}{3}q^{n-1}(q+1) & \text{if } n \equiv 0 \pmod 2 \end{cases} \end{aligned}$$

by (4.1) and Theorem 3.1. Therefore, (b) is obtained.

(c) By Proposition 4.1,

$$\overline{\Lambda}(p) = \frac{2}{3}(E(p) - \sigma(p)) = -\frac{2}{3}(p-1).$$

If we replace q with p in (4.1) and put $n = 2$ and 3, we get

$$\overline{\Lambda}(p^2) = \lambda^{-2}(1)\Lambda(p^2) + \lambda^{-2}(p)\Lambda(p) = -\frac{2}{3}(p-1)(p-2)$$

and

$$\overline{\Lambda}(p^3) = \lambda^{-2}(1)\Lambda(p^3) + \lambda^{-2}(p)\Lambda(p^2) + \lambda^{-2}(p^2)\Lambda(p) = -\frac{2}{3}(p-1)(p^2 + 6p + 18)$$

by Theorem 3.1 and Proposition 4.1.

Let $n = 2l + 4$ with $l \geq 0$. For ease of calculations, let

$$\begin{aligned} \mathfrak{B}_1 &:= \lambda^{-2}(1)\Lambda(p^{2l+4}) + \lambda^{-2}(p)\Lambda(p^{2l+3}) + \lambda^{-2}(p^2)\Lambda(p^{2l+2}), \\ \mathfrak{B}_2 &:= \lambda^{-2}(p^3)\Lambda(p^{2l+1}) + \cdots + \lambda^{-2}(p^{2l+2})\Lambda(p^2), \\ \mathfrak{B}_3 &:= \lambda^{-2}(p^{2l+3})\Lambda(p). \end{aligned}$$

By (1.3), Theorem 3.1 and Proposition 4.1,

$$\begin{aligned}\mathfrak{B}_1 &= -\frac{2}{3}\left((\sigma(p^{2l+4}) - 4\sigma(p^{2l+3}) + 7\sigma(p^{2l+2})) - (E(p^{2l+4}) - 4E(p^{2l+3}) + 7E(p^{2l+2}))\right)\\ &= -\frac{2}{3}\left((\sigma(p^{2l+4}) - 4\sigma(p^{2l+3}) + 7\sigma(p^{2l+2})) - (8l+10)\right),\\ \mathfrak{B}_2 &= -8\left((\Lambda(p^{2l+1}) - \Lambda(p^{2l})) + \cdots + (\Lambda(p^3) - \Lambda(p^2))\right)\\ &= -\frac{16}{3}\left((1-p^{2l+1}) + \cdots + (1-p^3)\right)\end{aligned}$$

and

$$\mathfrak{B}_3 = -8\cdot\frac{2}{3}(E(p) - \sigma(p)) = \frac{16}{3}(p-1).$$

Using $\overline{\Lambda}(p^{2l+4}) = \mathfrak{B}_1 + \mathfrak{B}_2 + \mathfrak{B}_3$ and $n = 2l+4$, we obtain

$$\overline{\Lambda}(p^n) = -\frac{2}{3}\left(\sigma(p^n) - 4\sigma(p^{n-1}) + 7\sigma(p^{n-2}) - 8p\sigma_2(p^{\frac{n-4}{2}}) - 2\right).$$

Let $n = 2l+5$ with $l \geq 1$. Similar to the even case, let

$$\begin{aligned}\mathfrak{B}'_1 &:= \lambda^{-2}(1)\Lambda(p^{2l+5}) + \lambda^{-2}(p)\Lambda(p^{2l+4}) + \lambda^{-2}(p^2)\Lambda(p^{2l+3}),\\ \mathfrak{B}'_2 &:= \lambda^{-2}(p^3)\Lambda(p^{2l+2}) + \cdots + \lambda^{-2}(p^{2l+4})\Lambda(p).\end{aligned}$$

By (1.3), Theorem 3.1 and Proposition 4.1,

$$\begin{aligned}\mathfrak{B}'_1 &= -\frac{2}{3}\left((\sigma(p^{2l+5}) - 4\sigma(p^{2l+4}) + 7\sigma(p^{2l+3})) - (E(p^{2l+5}) - 4E(p^{2l+4}) + 7E(p^{2l+3}))\right)\\ &= -\frac{2}{3}\left((\sigma(p^{2l+5}) - 4\sigma(p^{2l+4}) + 7\sigma(p^{2l+3})) - (8l+14)\right),\\ \mathfrak{B}'_2 &= -8\left((\Lambda(p^{2l+2}) - \Lambda(p^{2l+1})) + \cdots + (\Lambda(p^2) - \Lambda(p))\right)\\ &= -\frac{16}{3}\left((1-p^{2l+2}) + \cdots + (1-p^2)\right).\end{aligned}$$

From $n = 2l+5$, we deduce

$$\begin{aligned}\overline{\Lambda}(p^n) &= \mathfrak{B}'_1 + \mathfrak{B}'_2\\ &= -\frac{2}{3}\left(\sigma(p^n) - 4\sigma(p^{n-1}) + 7\sigma(p^{n-2}) - 8\sigma_2\left(p^{\frac{n-1}{2}}\right) + 2\right).\end{aligned}$$

□

5. Conclusion

There are several proofs of the Lambda function that use modular equations or theta series. Moreover, the theory of convolution sum of divisor functions is well studied and is a great help in the study of number theory. However, it is well known that it is somewhat difficult to obtain a formula for the convolution sums of divisor functions under the condition of relatively prime.

The Dirichlet convolution is used to obtain these formulas. See [8] for a recent result that uses the inverse of a rational function to find the convolution of a rational function. This article gives these formulae, which are meaningful and effective for finding the convolution sums of various restricted divisor functions.

Acknowledgments

This paper is dedicated to Professor Yilmaz Simsek on the occasion of his 60th Anniversary.

Author Contributions: All authors are equally contributed.

Conflict of Interest: There are no conflict of interest.

Funding (Financial Disclosure): There is no funding for this work.

References

[1] B. Cho, *Convolution sums of divisor functions for prime levels*, Int. J. Number Theory **16**, 537–546, 2020.
[2] A. Dujella, K. Gyory and A. Pinter, *On power values of pyramidal numbers, I*, Acta Arith. **155 (2)**, 217–226, 2012.
[3] H. M. Farkas, *On an arithmetical function*, Ramanujan J. **8**, 309–315, 2004.
[4] H. M. Farkas, *On an arithmetical function II*, Contemp. Math. **382**, 121–130, 2005.
[5] N. J. Fine, *Basic hypergeometric series and applications*, American Mathematical Society, Providence, Rhode Island, USA, 1988.
[6] G. H. Hardy, P. V. S. Aiyar and B. M. Wilson, *Collected papers of Srinivasa Ramanujan*, AMS Chelsea Publishing, Providence, Rhode Island, USA, 2000.
[7] D. Kim and Y. K. Park, *Certain combinatorial convolution sums involving divisor functions product formula*, Taiwanese J. Math. **18**, 973–988, 2014.
[8] S. Kong and D. Kim, *Inverse divisor functions using Dirichlet convolution*, Montes Taurus J. Pure Appl. Math. **6 (3)**, 169–185, 2024.
[9] K. Matsuda, *Note on a theorem of Farkas and Kra*, Ramanjan J. **53**, 319–356, 2020.
[10] P. J. McCarthy, *Introduction to arithmetical functions*, Springer Science & Business Media, New York, 2012.
[11] M. B. Nathanson, *Additive number theory the classical bases*, In: Graduate Texts in Mathematics, Springer, New York, 1996.
[12] E. Ntienjem, *Elementary evaluation of convolution sums involving the divisor function for a class of levels*, North-West. Eur. J. Math. **5**, 101–165, 2019.
[13] E. Royer, *Evaluating convolution sums of the divisor function by quasimodular forms*, Int. J. Number Theory **3**, 231–261, 2007.
[14] J. So, J. Hwang and D. Kim, *A study of coefficients derived from eta functions*, J. Appl. Math. Inform. **39 (3-4)**, 359–380, 2021.
[15] R. Vaidyanathaswamy, *The theory of multiplicative arithmetic functions*, Trans. Amer. Math. Soc. **33 (2)**, 579–662, 1931.
[16] K. S. Williams, *Number theory in the Spirit of Liouville*, London Mathematical Society, Student Texts 76, Cambridge, 2011.

How to cite this article: N. Y. Ikikardes and D. Kim, *A study of restricted divisor functions with coprime conditions*, Montes Taurus J. Pure Appl. Math. **6 (3)**, 284–298, 2024; Article ID: MTJPAM-D-24-00089.