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Boundedness of STFrFT on both unweighted and weighted Hardy and BMO spaces

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Abstract

The Fourier transform of a signal lacks local information in that it does not reflect the change of frequency over time. The Fourier transform method allows for the investigation of problems in either the time domain or the frequency domain, but not in both simultaneously. The use of short-time Fourier transforms enables the incorporation of both time and frequency localisation properties within a single transform function, through the utilisation of a window function. In cases where the fractional Fourier transform, which is more general than the Fourier transform, is unable to identify the frequency contents within the fractional Fourier domain, the short-time fractional Fourier transform is proposed as a potential solution. The present paper addresses the issue of boundedness of the short-time fractional Fourier transform. It is shown that the short-time fractional Fourier transform provides boundedness results in Hardy, BMO, weighted Hardy and weighted BMO spaces. This study also examines the Hardy and BMO-distance and weighted Hardy and weighted BMO-distance between two short-time fractional Fourier transforms associated with different windows and different signals.

Keywords: Short-time fractional Fourier transform, Hardy spaces, weighted Hardy spaces, unweighted and weighted BMO spaces

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1. Introduction

Fourier analysis constitutes the earliest among the various methodologies available for the analysis and synthesis of signals. In the Fourier transform (FT), the 'basis functions' are wholly concentrated in frequency and wholly distributed in time. This can be interpreted as a statement that the FT provides the greatest amount of information regarding the distribution of frequencies, but simultaneously loses information about time. The standard FT provides an averaged frequency representation over the entire time interval of the signal. In contrast, the short-time Fourier transform (STFT) is a technique that provides time-localised frequency information in situations where the frequency components of a signal vary with respect to time. By multiplying the infinitely long complex exponential with a window to localise it, the STFT adds a time dimension to the parameters of the base function. Windowed functions are functions in which the amplitude is gradually and smoothly reduced to zero at the edges. The final output of the STFT shows the spectral content of the signal at each corresponding time period, since each block occupies a different time period, (*cf.* [7, 8, 14]).

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Currently, the fractional Fourier transform (FrFT for short) has been employed in numerous fields of scientific enquiry, including signal and image processing, communications, optics and wave propagation, and so on. The FrFT has the potential to be a valuable tool in any field where the standard FT is employed. The discovery of a new application is to focus on an application that uses the ordinary FT and ask if an improvement or generalization is possible using the FrFT instead. The "fraction" parameter typically enables enhanced performance or more extensive generalization due to its capacity to provide an additional degree of freedom for optimization. As chirp signals are foundational to the FrFT, there is typically significant potential for improvement in signals exhibiting linear frequency growth. The short-time fractional Fourier transform (STFrFT) has been proposed to solve the FrFT problem when it fails to locate the FrF domain frequency contents, (*cf.* [1], [3]-[5], [10]-[13], [16]).

The present study is informed by the theoretical insights of Chuong & Duong, [6], and Verma & Gupta, [17]. In [6], the boundedness of the wavelet integral operator on the H^1, BMO, and Besov spaces was established. In a later study, Verma and Gupta (*cf.* [17]) introduced a novel class of continuous fractional wavelet transforms and undertook a comprehensive investigation into their properties within the context of Hardy and Morrey spaces.

The remainder of this paper is structured in the following manner. In Section 1.1, we introduce the terminology used in this paper. In Section 2, boundedness results for the STFrFT in Hardy and BMO spaces are obtained. In addition, in Section 3, we examine the STFrFT in the context of weighted Hardy and BMO_k spaces, with a particular focus on the role of a tempered weight function.

1.1. Preliminaries

We have put together a few basic facts for you as follows:

Let us consider a measurable function $f : \mathbb{R} \to \mathbb{C}$. It is the translation operator, defined as $T_x f(t) = f(t-x)$ and the modulation operator, defined as $M_w f(t) = \exp(2\pi i w t) f(t)$, which are the main actors in the time frequency analysis, where $x, w \in \mathbb{R}$.

As Lebesgue spaces for $p \in [1, \infty]$, we write $(L^p(\mathbb{R}), \|\ .\ \|_p)$. For $f \in L^1(\mathbb{R})$, $\widehat{f}$ (or $\mathcal{F}f$) is defined as follows

$$\widehat{f}(w) = \mathcal{F}[f(x)](w) = \int_{\mathbb{R}} f(x) \exp(-2\pi i x w) dx, \quad w \in \mathbb{R}. \tag{1.1}$$

The function $\widehat{f}$ is referred to as the Fourier transform (FT) of the function $f(x)$. The inversion formula is

$$f(x) = \mathcal{F}^{-1}[\widehat{f}(w)](x) = \int_{\mathbb{R}} \widehat{f}(w) \exp(2\pi i x w) dw \tag{1.2}$$

if f and $\widehat{f}$ are in $L^1(\mathbb{R})$. Equation (1.2) allows us to write $f(x)$ in terms of $\widehat{f}(w)$. $f(x)$ is referred to as the inverse Fourier transform (IFT) of $\widehat{f}(w)$.

The FT of f is a function of frequency and therefore resides on the vertical axis, while f resides on the horizontal axis. Consequently, the representation axis is transformed from the domain of time to that of frequency. This transformation is equivalent to a counterclockwise rotation through an angle of $\pi/2$. The application of (1.1) and then (1.2) leads to

$$(\mathcal{F}^2 f)(x) = (\mathcal{F}(\mathcal{F}(f))(x) = f(-x),$$

from which it follows that

$$(\mathcal{F}^3 f)(w) = (\mathcal{F}f)(-w) \quad \text{and} \quad (\mathcal{F}^4 f)(x) = f(x),$$

so the transformation is of period 4.

The FrFT represents a generalization of the FT with a parameter α. It can be interpreted as a rotation by an angle α in the time-frequency plane. For $\alpha \in \mathbb{R}$, the FrFT of a function $f \in L^1(\mathbb{R})$ is defined by

$$(\mathcal{F}^\alpha f)(x) = \int_{\mathbb{R}} f(t) K_\alpha(t, x)) dt,$$

when $\alpha \neq n\pi$, $n \in \mathbb{N}$, where the transformation kernel is

$$K_\alpha(t, x) = A_\alpha \exp\left[2\pi i \left(\left(\frac{t^2 + x^2}{2}\right) \cot\alpha - xt \csc\alpha\right)\right],$$

and $A_\alpha = \sqrt{1 - i\cot\alpha}$. If $\alpha = 2n\pi$, $(\mathcal{F}^\alpha f)(x) = f(x)$, and if $\alpha = (2n+1)\pi$, $(\mathcal{F}^\alpha f)(x) = f(-x)$.
Given that the parameter α is employed solely as an argument of trigonometric functions, it can be deduced that the FrFT is 2π-periodic with respect to α. Accordingly, throughout this paper, we will operate with the assumption that α is a value within the range $[0, 2\pi)$, (*cf.* [1, 3, 5], [11]-[13]).

Let us consider a non-zero function g, referred to as the "window function". The STFT of a function f with respect to g is defined by

$$V_g f(x,w) = \langle f, M_w T_x g\rangle = \int_{\mathbb{R}} f(t)\overline{g(t-x)}\exp(-2\pi i t w)dt,$$

for all real numbers x and w. In essence, Vgf is defined insofar as the integral is convergent. For example, it has been demonstrated that if f and g are elements of the space $L^2(\mathbb{R})$, then $V_g f$ is an element of the space $L^2(\mathbb{R}\times\mathbb{R})$ and is uniformly continuous (*cf.* [7, 8]).

STFrFT is the multiplication of the signal with a time-shifted window before the FrFT is taken, namely

$$\begin{aligned} V_g^\alpha f(x,w) &= \mathcal{F}_t^\alpha\left\{f(t)\overline{g(t-x)}\right\} \\ &= \int_{\mathbb{R}} f(t)\overline{g(t-x)}K_\alpha(t,w)dt, \end{aligned} \tag{1.3}$$

where $\mathcal{F}_t^\alpha\{\cdot\}$ denotes a FrFT integral over the variable t, [10, 16].

The theory of Hardy Spaces exhibits substantial interconnections with numerous domains within mathematical research, including Fourier analysis, harmonic analysis, singular integrals and operator theory, signal and image processing, control theory. It is established in the literature that the Hardy space is more appropriate than the Lebesgue space for a number of questions within the discipline of harmonic analysis. It should be recalled here that an equivalent definition of $H^1(\mathbb{R})$ can be given in terms of the maximal functions, which are defined as follows: Let us fix an integrable, smooth function, denoted by ϕ on the real line with support in the unit ball. We require that ϕ integrates to 1 over the real line. We then set $\phi_t(x) = t^{-1}\phi\left(\frac{x}{t}\right)$, where t is a positive constant. The maximal operator, denoted by the symbol $\mathcal{M}_\phi$, is defined for an integrable function f as follows:

$$\mathcal{M}_\phi f(x) = \sup_{t>0}|f * \phi_t(x)|.$$

The Hardy space $H^1(\mathbb{R})$ denotes the vector space of all $f \in L^1(\mathbb{R})$ if, for some $\phi \in \mathcal{S}(\mathbb{R})$ with $\int_{\mathbb{R}}\phi = 1$, the maximal function $\mathcal{M}_\phi f$ is in $L^1(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is the set of $\mathbb{C}$-valued continuous functions on $\mathbb{R}$ rapidly decreasing at infinity. If $f \in H^1(\mathbb{R})$, then the translation operator $T_x f$ is in $H^1(\mathbb{R})$ with

$$\|T_x f\|_{H^1} = \|f\|_{H^1}. \tag{1.4}$$

The space $BMO(\mathbb{R})$, which was first proposed by John and Nirenberg in [9], is the Banach space of all locally integrable functions f on the real line such that

$$\|f\|_{BMO} = \sup_{P\subset\mathbb{R}} P(|f - P(f)|) < \infty,$$

where the supremum is taken over all the cubes P whose sides are parallel to the axes of the coordinates, the term $|P|$ represents the Lebesgue measure of P and $P(f)$ indicates the mean of f over the ball P, that is

$$P(f) = |P|^{-1}\int_P f(z)\,dz \le |P|^{-1}\int_P |f(z)|\,dz \le M < \infty. \tag{1.5}$$

The Hardy space $H^1(\mathbb{R})$ is a substitute for $L^1(\mathbb{R})$ and the space $BMO(\mathbb{R})$ is the corresponding natural substitute for the space $L^\infty(\mathbb{R})$.Those seeking an excellent reference for these spaces would be well advised to consult either of the following sources: [2] and [15].

We define a positive function k on $\mathbb{R}$. If there exist $C > 0$ and $N > 0$ such that

$$k(x+y) \le (1 + C|x|)^N k(y), \quad x, y \in \mathbb{R},$$

it is referred to as a temperate weight function. In this context, the set of all such functions will be denoted by the symbol $\mathcal{K}$.

The weighted L^p, $1 \leq p < \infty$, is defined as the vector space of all measurable functions on the real line such that

$$\|f\|_{p,k}^{p} = \int_{\mathbb{R}} |f(z)|^p k(z)dz < \infty,$$

and is denoted by $L_k^p(\mathbb{R})$.

The weighted Hardy space $H_k^1(\mathbb{R})$ is defined as the set of all $f \in L_k^1(\mathbb{R})$ if, for some $\phi \in \mathcal{S}(\mathbb{R})$ with $\int_{\mathbb{R}} \phi = 1$, the maximal function $\mathcal{M}_\phi f$ is in $L_k^1(\mathbb{R})$.

The weighted bounded mean oscillation space $BMO_k(\mathbb{R})$ is defined as the set of all functions $f \in L_{loc}^{1,k}(\mathbb{R})$ such that

$$\|f\|_{BMO_k} = \sup_{P \subset \mathbb{R}} |P|_k^{-1} \int_P |f(z) - P(f)| \, k(z)dz < \infty,$$

where $|P|_k = \int_P k(z)\, dz$ and the supremum is taken over balls P in $\mathbb{R}$.

2. Boundedness of STFrFT on Hardy and BMO spaces

2.1. STFrFT on $H^1(\mathbb{R})$ space

We prove the $H^1(\mathbb{R})$-boundedness of STFrFT.

Theorem 2.1. *Let $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. The operator $V_g^\alpha : H^1(\mathbb{R}) \to H^1(\mathbb{R})$ given by $f \to V_g^\alpha f(\cdot, b)$ is bounded. In particular,*

$$\left\|V_g^\alpha f(\cdot, b)\right\|_{H^1} \leq |A_\alpha| \, \|g\|_1 \, \|f\|_{H^1}.$$

Proof. By making the substitution $t - a = x$ in the definition of $V_g^\alpha f$, we get

$$V_g^\alpha f(a,b) = \int_{\mathbb{R}} f(x+a)\,\overline{g(x)} K_\alpha(x+a,b)dx. \tag{2.1}$$

Take any $f \in H^1(\mathbb{R})$. Then $f \in L^1(\mathbb{R})$ and so

$$\begin{aligned}
\int_{\mathbb{R}} |V_g^\alpha f(a,b)| da &\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x+a)| \, |g(x)| \, |K_\alpha(x+a,b)| dx \right) da \\
&= |A_\alpha| \int_{\mathbb{R}} |g(x)| \left(\int_{\mathbb{R}} |f(x+a)| \, da \right) dx \\
&= |A_\alpha| \, \|g\|_1 \, \|f\|_1 < \infty.
\end{aligned}$$

It thus follows that $V_g^\alpha f(\cdot, b) \in L^1(\mathbb{R})$. Moreover, since

$$\begin{aligned}
\left(V_g^\alpha f(\cdot, b) * \phi_t\right)(a) &= \int_{\mathbb{R}} V_g^\alpha f(a-y, b)\, \phi_t(y)\, dy \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x+a-y)\, \overline{g(x)} K_\alpha(x+a-y, b) dx \right) \phi_t(y)\, dy,
\end{aligned}$$

we get

$$\begin{aligned}
\left\|V_g^\alpha f(\cdot, b)\right\|_{H^1} &= \int_{\mathbb{R}} \sup_{t>0} \left| \left(V_g^\alpha f(\cdot, b) * \phi_t\right)(a) \right| da \\
&\leq |A_\alpha| \int_{\mathbb{R}} |g(x)| \left(\int_{\mathbb{R}} \sup_{t>0} \left(|f| * |\phi_t|\right)(x+a) da \right) dx \\
&= |A_\alpha| \, \|g\|_1 \, \|f\|_{H^1}.
\end{aligned}$$

□

By using the above theorem, we are led to the following equality.

Corollary 2.2. *If $f \in H^1(\mathbb{R})$ and $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then*

$$\left\| V_g^\alpha f(\cdot, b) \right\|_{H^1} = O(|A_\alpha|).$$

Now, we will calculate the $H^1(\mathbb{R})$-distance between two STFrFT.

Theorem 2.3. *Let $g_1, g_2 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be two non-zero window functions. If $f, h \in H^1(\mathbb{R})$, the following conclusion is reached:*

$$\left\| V_{g_1}^\alpha f(\cdot, b) - V_{g_2}^\alpha h(\cdot, b) \right\|_{H^1} \leq |A_\alpha| \left(\|g_1 - g_2\|_1 \|f\|_{H^1} + \|g_2\|_1 \|f - h\|_{H^1} \right).$$

Proof. We write

$$\left\| V_{g_1}^\alpha f(\cdot, b) - V_{g_2}^\alpha h(\cdot, b) \right\|_{H^1} \leq \left\| V_{g_1}^\alpha f(\cdot, b) - V_{g_2}^\alpha f(\cdot, b) \right\|_{H^1} + \left\| V_{g_2}^\alpha f(\cdot, b) - V_{g_2}^\alpha h(\cdot, b) \right\|_{H^1}. \tag{2.2}$$

By using the equality (2.1), we write

$$\left(\left(V_{g_1}^\alpha f(\cdot, b) - V_{g_2}^\alpha f(\cdot, b) \right) * \phi_t \right)(a) = \left(V_{g_1 - g_2}^\alpha f(\cdot, b) * \phi_t \right)(a).$$

Then by Theorem 2.1, the first part on the right side of (2.2) is

$$\left\| V_{g_1}^\alpha f(\cdot, b) - V_{g_2}^\alpha f(\cdot, b) \right\|_{H^1} \leq |A_\alpha| \|g_1 - g_2\|_1 \|f\|_{H^1}. \tag{2.3}$$

Also since

$$\begin{aligned} V_{g_2}^\alpha f(a, b) - V_{g_2}^\alpha h(a, b) &= \int_{\mathbb{R}} (f - h)(x + a) \overline{g_2(x)} K_\alpha(x + a, b) dx \\ &= V_{g_2}^\alpha (f - h)(a, b), \end{aligned}$$

and so, using Theorem 2.1

$$\begin{aligned} \left\| V_{g_2}^\alpha f(\cdot, b) - V_{g_2}^\alpha h(\cdot, b) \right\|_{H^1} &= \left\| V_{g_2}^\alpha (f - h)(\cdot, b) \right\|_{H^1} \\ &\leq |A_\alpha| \|g_2\|_1 \|f - h\|_{H^1}. \end{aligned} \tag{2.4}$$

Hence, by (2.3) and (2.4), we arrive at the following conclusion:

$$\begin{aligned} \left\| V_{g_1}^\alpha f(\cdot, b) - V_{g_2}^\alpha h(\cdot, b) \right\|_{H^1} &\leq \left\| V_{g_1}^\alpha f(\cdot, b) - V_{g_2}^\alpha f(\cdot, b) \right\|_{H^1} + \left\| V_{g_2}^\alpha f(\cdot, b) - V_{g_2}^\alpha h(\cdot, b) \right\|_{H^1} \\ &\leq |A_\alpha| (\|g_1 - g_2\|_1 \|f\|_{H^1} + \|g_2\|_1 \|f - h\|_{H^1}). \end{aligned}$$

□

2.2. *Boundedness of STFrFT on BMO space*

The BMO-boundedness of the STFrFT is discussed in this part of the paper. We have to prove the following lemma concerning the space $BMO(\mathbb{R})$ to facilitate the proof of the $BMO(\mathbb{R})$-boundedness of STFrFT.

Lemma 2.4 (*cf.* [14], Lemma 2.2)**.** *If $f \in BMO(\mathbb{R})$, $M_w T_x f \in BMO(\mathbb{R})$ for all real numbers x and w, and we have*

$$\|M_w T_x f\|_{BMO} \leq \|f\|_{BMO} + 2M.$$

Now, let us define the operator M_w^α as $M_w^\alpha f(\cdot) = f(\cdot) K_\alpha(\cdot, w)$. If M_w^α is substituted for M_w in Lemma 2.4, the inequality

$$\left\| M_w^\alpha T_x f \right\|_{BMO} \leq |A_\alpha| (\|f\|_{BMO} + 2M) \tag{2.5}$$

is obtained, where M is as in (1.5).
The subsequent theorems will necessitate the application of the next lemma.

Lemma 2.5. *Let us take a function $g \in L^1(\mathbb{R})$ with compact support. If $f \in L^1_{loc}(\mathbb{R})$, then $V_g^\alpha f(\cdot, b)$ is in $L^1_{loc}(\mathbb{R})$.*

Proof. Let $f \in L^1_{loc}(\mathbb{R})$. Recall that $V_g^\alpha f(a,b)$ is a function of a. Then we write for any compact set $P \subset \mathbb{R}$

$$\begin{aligned}\int_P \left|V_g^\alpha f(a,b)\right| da &\leq \int_P \left(\int_{\mathbb{R}} |f(x+a) K_\alpha(x+a,b)| \, |g(x)| \, dx\right) da \\ &\leq |A_\alpha| \int_{\mathbb{R}} |g(x)| \left(\int_P |f(x+a)| \, da\right) dx \\ &= |A_\alpha| \int_{\mathbb{R}} |g(x)| \left(\int_B |f(y)| \, dy\right) dx,\end{aligned}$$

where $B = x + P$. Since $B \subset \sup pg + P$ is a compact set in $\mathbb{R}$ and $f \in L^1_{loc}(\mathbb{R})$, then we may write

$$\int_P \left|V_g^\alpha f(a,b)\right| da \leq |A_\alpha| \, L \, \|g\|_1 < \infty.$$

Hence $V_g^\alpha f(\cdot, b) \in L^1_{loc}(\mathbb{R})$. □

We now prove the BMO-boundedness of the STFrFT.

Theorem 2.6. *Let $g \in L^1(\mathbb{R})$ be a compactly supported. Then $V_g^\alpha : BMO(\mathbb{R}) \to BMO(\mathbb{R})$ defined by $f \to V_g^\alpha f(\cdot, b)$ is bounded. In particular,*

$$\left\|V_g^\alpha f(\cdot,b)\right\|_{BMO} \leq |A_\alpha| \, \|g\|_1 \left(\|f\|_{BMO} + 2M\right).$$

Proof. Let P be an any ball in $\mathbb{R}$ and $f \in BMO(\mathbb{R})$. Then $f \in L^1_{loc}(\mathbb{R})$ and so $V_g^\alpha f(\cdot, b) \in L^1_{loc}(\mathbb{R})$ by Lemma 2.5. By employing the Fubini theorem, we obtain the following result:

$$\begin{aligned}P\left(V_g^\alpha f\right) &= |P|^{-1} \int_P V_g^\alpha f(z,b) \, dz \\ &= |P|^{-1} \int_P \left(\int_{\mathbb{R}} f(x+z) \, \overline{g(x)} K_\alpha(x+z,b) dx\right) dz \\ &= \int_{\mathbb{R}} \overline{g(x)} \left(|P|^{-1} \int_P T_{-x} M_b^\alpha f(z) \, dz\right) dx \\ &= \int_{\mathbb{R}} \overline{g(x)} P\left(T_{-x} M_b^\alpha f\right) dx.\end{aligned}$$

Hence we write

$$\begin{aligned}\left\|V_g^\alpha f(\cdot,b)\right\|_{BMO} &= \sup_{P \subset \mathbb{R}} P(\left|V_g^\alpha f - P\left(V_g^\alpha f\right)\right|) \\ &= \sup_{P \subset \mathbb{R}} |P|^{-1} \int_P \left|V_g^\alpha f(a,b) - P\left(V_g^\alpha f\right)\right| da \\ &= \sup_{P \subset \mathbb{R}} |P|^{-1} \int_P \left|\int_{\mathbb{R}} f(x+a) \, \overline{g(x)} K_\alpha(x+a,b) dx - \int_{\mathbb{R}} \overline{g(x)} P\left(T_{-x} M_b^\alpha f\right) dx\right| da \\ &\leq \int_{\mathbb{R}} |g(x)| \left(\sup_{P \subset \mathbb{R}} |P|^{-1} \int_P \left|T_{-x} M_b^\alpha f(a) - P\left(T_{-x} M_b^\alpha f\right)\right| da\right) dx \\ &= \int_{\mathbb{R}} |g(x)| \left\|T_{-x} M_b^\alpha f\right\|_{BMO} dx\end{aligned}$$

and also by (2.5), we obtain

$$\left\|V_g^\alpha f(\cdot,b)\right\|_{BMO} \leq |A_\alpha| \, \|g\|_1 \left(\|f\|_{BMO} + 2M\right).$$

□

Corollary 2.7. *If $f \in BMO(\mathbb{R})$ and $g \in L^1(\mathbb{R})$ is a compactly supported, then we have*

$$\left\|V_g^\alpha f(\cdot, b)\right\|_{BMO} = O(|A_\alpha|).$$

The $BMO(\mathbb{R})$ distance between two STFrFTs is now to be calculated.

Theorem 2.8. *Let $g_1, g_2 \in L^1(\mathbb{R})$ be two functions with compact support. If $f, h \in BMO(\mathbb{R})$, then*

$$\left\|V_{g_1}^\alpha f(\cdot, b) - V_{g_2}^\alpha h(\cdot, b)\right\|_{BMO} \leq |A_\alpha|\Big(\|g_1 - g_2\|_1 \left(\|f\|_{BMO} + 2M\right) + \|g_2\|_1 \left(\|f - h\|_{BMO} + 2M\right) \Big).$$

Proof. Let $g_1, g_2 \in L^1(\mathbb{R})$ be two functions with compact support and, f and h are in the space $BMO(\mathbb{R})$. Then f and h are in the space $L^1_{loc}(\mathbb{R})$ and so, $V_{g_1}^\alpha f(\cdot, b)$, $V_{g_2}^\alpha f(\cdot, b)$ and $V_{g_2}^\alpha h(\cdot, b) \in L^1_{loc}(\mathbb{R})$ by Lemma 2.5. Moreover, since $V_{g_1}^\alpha f(a, b) - V_{g_2}^\alpha h(a, b) = V_{g_1 - g_2}^\alpha f(a, b)$ and $V_{g_2}^\alpha f(a, b) - V_{g_2}^\alpha h(a, b) = V_{g_2}^\alpha (f - h)(a, b)$, hence by using Theorem 2.6

$$\begin{aligned} \left\|V_{g_1}^\alpha f(\cdot, b) - V_{g_2}^\alpha h(\cdot, b)\right\|_{BMO} &\leq \left\|V_{g_1}^\alpha f(\cdot, b) - V_{g_2}^\alpha f(\cdot, b)\right\|_{BMO} + \left\|V_{g_2}^\alpha f(\cdot, b) - V_{g_2}^\alpha h(\cdot, b)\right\|_{BMO} \\ &= \left\|V_{g_1 - g_2}^\alpha f(\cdot, b)\right\|_{BMO} + \left\|V_{g_2}^\alpha (f - h)(\cdot, b)\right\|_{BMO} \\ &\leq |A_\alpha|\Big(\|g_1 - g_2\|_1 \left(\|f\|_{BMO} + 2M\right) + \|g_2\|_1 \left(\|f - h\|_{BMO} + 2M\right) \Big), \end{aligned}$$

which proves the theorem. □

3. Boundedness of STFrFT on $H^1_k(\mathbb{R})$ and $BMO_k(\mathbb{R})$

Estimating the FT of a function in function space is a crucial aspect of harmonic analysis and numerous applications. For example, the FT of an L^1 function is bounded. Again, Plancherel's theorem shows that the FT is an isometry of L^2 onto itself. Similarly, if $f, g \in L^2$ and $\|g\|_2 = 1$, the STFT is an isometry from L^2 into L^2. Weighted spaces occur naturally both in the theory of functions and in the study of ordinary (unweighted) spaces. Weighted inequalities also emerge naturally in Fourier analysis, but their use is most effectively supported by the diverse range of applications in which they are employed. For example, the theory of weights is a crucial component of the investigation of boundary value problems pertaining to the Laplace equation on Lipschitz domains. The use of the inequalities with weighted extends beyond the aforementioned applications to encompass vector-valued inequalities, the extrapolation of operators, and the analysis of specific classes of integral equations. It is for this reason that we intend to study weighted spaces.

Throughout this section we will assume that the window functions g, g_1 and g_2 are compactly supported and that the support belongs to a ball of radius r centered on the origin.

3.1. Boundedness of STFrFT on $H^1_k(\mathbb{R})$

In this section, we demonstrate $H^1_k(\mathbb{R})$-boundedness of STFrFT.

Theorem 3.1. *Let $k \in \mathcal{K}$. If $g \in L^1(\mathbb{R})$, then the operator $V_g^\alpha : H^1_k(\mathbb{R}) \to H^1_k(\mathbb{R})$ defined by $f \to V_g^\alpha f(\cdot, b)$ is bounded. In particular,*

$$\left\|V_g^\alpha f(\cdot, b)\right\|_{H^1_k} \leq |A_\alpha|(1 + Cr)^N \|g\|_1 \|f\|_{H^1_k}.$$

Proof. Let $f \in H^1_k(\mathbb{R})$. Then $f \in L^1_k(\mathbb{R})$. For a fixed $b \in \mathbb{R}$, $V_g^\alpha f(a, b)$ is a function of a. Hence

$$\begin{aligned}
\left\|V_g^\alpha f(\cdot,b)\right\|_{1,k} &= \int_{\mathbb{R}} \left|V_g^\alpha f(a,b)\right| k(a)da \\
&= \int_{\mathbb{R}} \left|\int_{\mathbb{R}} f(x+a)g(x)K_\alpha(x+a,b)dx\right| k(a)da \\
&= |A_\alpha| \int_{\mathbb{R}} |g(x)| \left(\int_{\mathbb{R}} |f(y)|\, k(y-x)dy\right) dx \\
&\le |A_\alpha| \left(\int_{|x|\le r} |g(x)|\,(1+C\,|x|)^N dx\right)\left(\int_{\mathbb{R}} |f(y)|\, k(y)dy\right) \\
&= |A_\alpha|(1+Cr)^N \|g\|_1 \|f\|_{1,k}
\end{aligned}$$

and $V_g^\alpha f(\cdot,b) \in L_k^1(\mathbb{R})$. Moreover, by using the inequality

$$|\left(V_g^\alpha f(\cdot,b) * \phi_t\right)(a)| \le |A_\alpha| \int_{\mathbb{R}} |g(x)|\, ((|f| * |\phi_t|)(x+a))dx,$$

we obtain

$$\begin{aligned}
\left\|V_g^\alpha f(\cdot,b)\right\|_{H_k^1} &= \int_{\mathbb{R}} \sup_{t>0} \left|\left(V_g^\alpha f(\cdot,b) * \phi_t\right)(a)\right| k(a)da \\
&\le |A_\alpha| \int_{\mathbb{R}} |g(x)| \left(\int_{\mathbb{R}} \sup_{t>0} (|f| * |\phi_t|)(x+a)k(a)da\right) dx \\
&= |A_\alpha| \int_{\mathbb{R}} |g(x)| \left(\int_{\mathbb{R}} \sup_{t>0} (|f| * |\phi_t|)(y)k(y-x)dy\right) dx \\
&\le |A_\alpha| \left(\int_{|x|\le r} |g(x)|\,(1+C\,|x|)^N dx\right)\left(\int_{\mathbb{R}} \sup_{t>0} (|f| * |\phi_t|)(y)\, k(y)dy\right) \\
&\le |A_\alpha|(1+Cr)^N \|g\|_1 \|f\|_{H_k^1}.
\end{aligned}$$

□

We now give the $H_k^1(\mathbb{R})$-distance between two STFrFTs. The proof follows a similar structure to that of Theorem 2.3.

Theorem 3.2. *Let $k \in \mathcal{K}$. If $g_1, g_2 \in L^1(\mathbb{R})$ and $f, h \in H_k^1(\mathbb{R})$, then we have*

$$\left\|V_{g_1}^\alpha f(\cdot,b) - V_{g_2}^\alpha h(\cdot,b)\right\|_{H_k^1} \le |A_\alpha|(1+Cr)^N \left(\|g_1 - g_2\|_1 \|f\|_{H_k^1} + \|g_2\|_1 \|f-h\|_{H_k^1}\right).$$

3.2. Boundedness of STFrFT on BMO_k space

The objective of this part is to examine the BMO_k-boundedness of STFrFT. To facilitate the proof that STFrFT is $BMO_k(\mathbb{R})$ bounded, it is necessary to consider the following lemmas pertaining to the space $BMO_k(\mathbb{R})$.

Lemma 3.3. *For every $f \in BMO_k(\mathbb{R})$, we have the inequality*

$$\left\|T_x M_w^\alpha f\right\|_{BMO_k} \le (1+C|x|)^{2N} |A_\alpha| (\|f\|_{BMO_k} + 2M),$$

for $x, w \in \mathbb{R}$, where M_w^α is as in (2.5).

Proof. Let $f \in BMO_k(\mathbb{R})$, P be an arbitrary ball in $\mathbb{R}$. We calculate

$$\begin{aligned}
\left\|T_x M_w^\alpha f\right\|_{BMO_k} &= \sup_{P\subset\mathbb{R}} |P|_k^{-1} \int_P \left|T_x M_w^\alpha f(t) - P\left(T_x M_w^\alpha f\right)\right| k(t)dt \\
&= \sup_{P\subset\mathbb{R}} |P|_k^{-1} \int_P \left|K_\alpha(t-x,w) f(t-x) - |P|^{-1} \int_P K_\alpha(z-x,w) f(z-x)\, dz\right| k(t)dt \\
&\le \sup_{P\subset\mathbb{R}} |P|_k^{-1} \int_P \left|K_\alpha(t-x,w) f(t-x) - |P|^{-1} K_\alpha(t-x,w) \int_P f(z-x)\, dz\right| k(t)dt \\
&\quad + \sup_{P\subset\mathbb{R}} |P|_k^{-1} \int_P \left||P|^{-1} K_\alpha(t-x,w) \int_P f(z-x)\, dz\right| k(t)dt \\
&\quad + \sup_{P\subset\mathbb{R}} |P|_k^{-1} \int_P \left||P|^{-1} \int_P K_\alpha(z-x,w) f(z-x)\, dz\right| k(t)dt \\
&\le |A_\alpha| \Bigg(\sup_{P\subset\mathbb{R}} |P|_k^{-1} \int_P \left|f(t-x) - |P|^{-1} \int_P f(z-x)\, dz\right| k(t)dt \\
&\quad + 2\sup_{P\subset\mathbb{R}} |P|_k^{-1} \int_P \left(|P|^{-1} \int_P |f(z-x)|\, dz\right) k(t)dt \Bigg).
\end{aligned}$$

If we say $B = P - x$ for $x \in \mathbb{R}$, we write

$$|B|_k = \int_B k(u)du = \int_P k(t-x)dt \le (1 + C|x|)^N |P|_k.$$

From here,

$$\left\|T_x M_w^\alpha f\right\|_{BMO_k} \le (1 + C|x|)^{2N} |A_\alpha| \left(\|f\|_{BMO_k} + 2M \right).$$

□

To prove the following theorems, it is first necessary to state the next lemma.

Lemma 3.4. *Let $k \in \mathcal{K}$ and $g \in L^1(\mathbb{R})$. If $f \in L^{1,k}_{loc}(\mathbb{R})$, then $V_g^\alpha f(\cdot, b)$ is in $L^{1,k}_{loc}(\mathbb{R})$.*

Proof. For any compact set $P \subset \mathbb{R}$, from proof of Lemma 2.5, we can write

$$\int_P \left|V_g^\alpha f(a,b)\right| k(a)da \le |A_\alpha| \int_{\mathbb{R}} |g(x)| \left(\int_P |f(x+a)|\, k(a)da \right) dx.$$

Let $U = x + P$. Since $U \subset \sup pg + P$ is a compact set in $\mathbb{R}$ and $f \in L^{1,k}_{loc}(\mathbb{R})$, we get

$$\begin{aligned}
\int_P \left|V_g^\alpha f(a,b)\right| k(a)da &\le |A_\alpha| \int_{\mathbb{R}} |g(x)| \left(\int_U |f(y)|\, k(y-x)dy \right) dx \\
&\le |A_\alpha| \left(\int_{|x|\le r} |g(x)|\, (1 + C|x|)^N dx \right) \left(\int_U |f(y)|\, k(y)dy \right) \\
&= |A_\alpha| L (1 + Cr)^N \|g\|_1 < \infty.
\end{aligned}$$

Hence $V_g^\alpha f(\cdot, b) \in L^{1,k}_{loc}(\mathbb{R})$. □

Now we will state that the STFrFT is BMO_k-bounded.

Theorem 3.5. *Let $g \in L^1(\mathbb{R})$ and $k \in \mathcal{K}$. Then the operator $V_g^\alpha : BMO_k(\mathbb{R}) \to BMO_k(\mathbb{R})$ defined by $f \to V_g^\alpha f(\cdot, b)$ is bounded. Moreover,*

$$\left\|V_g^\alpha f(\cdot, b)\right\|_{BMO_k} \le |A_\alpha| (1 + Cr)^{2N} \|g\|_1 \left(\|f\|_{BMO_k} + 2M \right).$$

Proof. Let P be an arbitrary ball in the real line and $f \in BMO_k(\mathbb{R})$. Then $f \in L^{1,k}_{loc}(\mathbb{R})$ and so $V^\alpha_g f(\cdot, b) \in L^{1,k}_{loc}(\mathbb{R})$ by Lemma 3.4. Thus similar to Theorem 2.6, we have

$$\begin{aligned} \left\|V^\alpha_g f(\cdot, b)\right\|_{BMO_k} &= \sup_{P\subset\mathbb{R}} |P|^{-1}_k \int_P \left|V^\alpha_g f(a,b) - P\left(V^\alpha_g f\right)\right| k(a)da \\ &\leq \int_{\mathbb{R}} |g(x)| \left(\sup_{P\subset\mathbb{R}} |P|^{-1}_k \int_P \left|T_{-x}M^\alpha_b f(a) - P\left(T_{-x}M^\alpha_b f\right)\right| k(a)da\right) dx \\ &= \int_{\mathbb{R}} |g(x)| \left\|T_{-x}M^\alpha_b f\right\|_{BMO_k} dx, \end{aligned}$$

also by Lemma 3.3, we obtain

$$\begin{aligned} \left\|V^\alpha_g f(\cdot, b)\right\|_{BMO_k} &\leq \int_{|x|\leq r} |g(x)| (1 + C|x|)^{2N} |A_\alpha| \left(\|f\|_{BMO_k} + 2M\right) dx \\ &\leq |A_\alpha|(1 + Cr)^{2N} \|g\|_1 \left(\|f\|_{BMO_k} + 2M\right). \end{aligned}$$

□

The proof of the subsequent result can be conducted in a manner analogous to that of Theorem 2.3, utilising the tenets of Lemma 3.3 and Theorem 3.5.

Theorem 3.6. *Let $g_1, g_2 \in L^1(\mathbb{R})$ and $k \in \mathcal{K}$. If f and h are in $BMO_k(\mathbb{R})$, then*

$$\left\|V^\alpha_{g_1} f(\cdot, b) - V^\alpha_{g_2} h(\cdot, b)\right\|_{BMO_k} \leq |A_\alpha|(1 + Cr)^{2N} \Big(\|g_1 - g_2\|_1 \left(\|f\|_{BMO_k} + 2M\right) + \|g_2\|_1 \left(\|f - h\|_{BMO_k} + 2M\right)\Big).$$

4. Conclusion

STFrFT has been developed to better represent strongly nonlinear chirp signals by Capus and Brown in [4]. Later, it has been studied in a number of papers [10, 16]. The present study investigates the boundedness properties of STFrFT on Hardy, weighted Hardy, BMO and weighted BMO spaces, which have significant connections with numerous mathematical disciplines.

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