

On some inequalities for means involving the polygamma functions

Kwara Nantomah  ^a

^aDepartment of Mathematics, School of Mathematical Sciences, C. K. Tedam University of Technology and Applied Sciences, P. O. Box 24, Navrongo, Upper-East Region, Ghana

Abstract

Let $\Psi^{(r)}(z)$ be the polygamma function of order $r \in \mathbb{N}$. In this paper, we prove some inequalities for certain means involving the functions $\Psi^{(r)}(z)$ and $\Psi^{(r)}(1/z)$ for $z > 0$. The inequalities provide bounds in terms of the Riemann zeta function for harmonic, geometric and arithmetic means involving the functions $\Psi^{(r)}(z)$ and $\Psi^{(r)}(1/z)$. The techniques adopted in proving the results depend largely on monotonicity properties of some function involving the polygamma functions.

Keywords: Gamma function, polygamma function, Riemann zeta function, harmonic mean, inequality

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1. Introduction

The gamma function is one of the most important special functions due to its wide areas of application. Even though it has other representations, it is commonly defined as

$$\Gamma(z) = \int_0^\infty s^{z-1} e^{-s} ds$$


for $z > 0$. The digamma (or psi) function is defined as

$$\Psi(z) = \frac{d}{dz} [\ln \Gamma(z)] = -\gamma + \int_0^\infty \frac{e^{-s} - e^{-zs}}{1 - e^{-s}} ds \tag{1.1}$$

$$= \int_0^\infty \left(\frac{e^{-s}}{s} - \frac{e^{-zs}}{1 - e^{-s}} \right) ds \tag{1.2}$$

$$= -\gamma - \frac{1}{z} + \sum_{\kappa=1}^\infty \frac{z}{\kappa(\kappa + z)}, \tag{1.3}$$

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Email address: knantomah@cktutas.edu.gh (Kwara Nantomah )

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*Corresponding Author: Kwara Nantomah



with γ being the Euler-Mascheroni constant. The polygamma functions are defined for $z > 0$ and $r \in \mathbb{N}$ as

$$\Psi^{(r)}(z) = \frac{d^r}{dz^r} \Psi(z) = (-1)^{r+1} \int_0^\infty \frac{s^r e^{-zs}}{1 - e^{-s}} ds \tag{1.4}$$

$$= (-1)^{r+1} \sum_{k=0}^\infty \frac{r!}{(k+z)^{r+1}} \tag{1.5}$$

$$= (-1)^{r+1} r! \zeta(r+1, z), \tag{1.6}$$

with $\zeta(v, z)$ being the Hurwitz zeta function. In particular, $\zeta(v, 1) = \zeta(v)$ where $\zeta(v)$ is the Riemann zeta function. Moreover, the integral

$$\frac{k!}{z^{k+1}} = \int_0^\infty s^k e^{-zs} ds \tag{1.7}$$

is satisfied for $z > 0$ and $k \in \mathbb{N}_0$.

Gautschi [12] established that the inequality

$$\frac{2\Gamma(z)\Gamma(1/z)}{\Gamma(z) + \Gamma(1/z)} \geq 1 \tag{1.8}$$

is satisfied for $z > 0$. From (1.8), and with the same condition on z , it can be deduced that

$$\Gamma(z)\Gamma(1/z) \geq 1 \tag{1.9}$$

and

$$\Gamma(z) + \Gamma(1/z) \geq 2. \tag{1.10}$$

Because of the practicalities of these inequalities, some advances have been reported in the literature (cf. [1]-[6], [13, 14]).

Motivated by the results (1.8), (1.9) and (1.10), Alzer and Jameson [8] established that

$$\frac{2\Psi(z)\Psi(1/z)}{\Psi(z) + \Psi(1/z)} \geq -\gamma, \quad z > 0. \tag{1.11}$$

$$\Psi(z)\Psi(1/z) < \gamma^2, \quad z > 0, \quad z \neq 1, \tag{1.12}$$

$$\Psi(z) + \Psi(1/z) < -2\gamma, \quad z > 0, \quad z \neq 1. \tag{1.13}$$

Alzer further provided a refinement of (1.11) in [7]. Nantomah, Abe-I-Kpeng and Sandow [23] extended (1.11), (1.12) and (1.13) to the trigamma function by proving that

$$\frac{2\Psi'(z)\Psi'(1/z)}{\Psi'(z) + \Psi'(1/z)} \leq \frac{\pi^2}{6}, \quad z > 0, \tag{1.14}$$

$$\Psi'(z)\Psi'(1/z) \geq \left(\frac{\pi^2}{6}\right)^2, \quad z > 0 \tag{1.15}$$

and

$$\Psi'(z) + \Psi'(1/z) \geq \frac{\pi^2}{3}, \quad z > 0. \tag{1.16}$$

Das and Swaminathan [11] gave more general results by extending (1.11), (1.12) and (1.13) to the polygamma function. In particular, they established that

$$\frac{(-1)^r 2\Psi^{(r)}(z)\Psi^{(r)}(1/z)}{\Psi^{(r)}(z) + \Psi^{(r)}(1/z)} \geq (-1)^r \Psi^{(r)}(1), \quad z > 0, r \in \mathbb{N}, \tag{1.17}$$

$$\Psi^{(r)}(z)\Psi^{(r)}(1/z) > \left(\Psi^{(r)}(1)\right)^2, \quad z > 0, z \neq 1, r \in \mathbb{N} \tag{1.18}$$

and

$$(-1)^r \left(\Psi^{(r)}(z) + \Psi^{(r)}(1/z)\right) < (-1)^r 2\Psi^{(r)}(1), \quad z > 0, z \neq 1, r \in \mathbb{N}. \tag{1.19}$$

Analogous results having to do with other special functions can be found in the works [10], [15]-[21], [26, 27].

The goal of this work is to establish inequalities which are equivalent to (1.17), (1.18) and (1.19) by using different techniques. The main findings are presented in Section 3.

2. Lemmas

In this section, we present some results that are required in order to prove our main results.

Lemma 2.1 (cf. [24]). Let $p * q = \int_0^s p(s-u)q(u)du$ be the convolution of the functions $p(s)$ and $q(s)$. Then

$$\mathcal{L}\{p * q\} = \mathcal{L}\{p\} \mathcal{L}\{q\},$$

where $\mathcal{L}\{h\}$ is the Laplace transform of h . That is,

$$\int_0^\infty \left[\int_0^s p(s-u)q(u)du \right] e^{-zs} ds = \int_0^\infty p(s)e^{-zs} ds \int_0^\infty q(s)e^{-zs} ds. \tag{2.1}$$

Lemma 2.2. Let $z > 0$ and $r \in \mathbb{N}$. Then the inequality

$$\Psi^{(r)}(z)\Psi^{(r+2)}(z) - 2\left(\Psi^{(r+1)}(z)\right)^2 < 0 \tag{2.2}$$

holds.

Proof. It has been established in [9] and [25] that the inequality

$$\frac{r}{r+1} < \frac{\left(\Psi^{(r+1)}(z)\right)^2}{\Psi^{(r)}(z)\Psi^{(r+2)}(z)} < \frac{r+1}{r+2}$$

holds for $z > 0$ and $r \in \mathbb{N}$. Since $\frac{1}{2} \leq \frac{r}{r+1}$ for all $r \in \mathbb{N}$, we have

$$\frac{1}{2} < \frac{\left(\Psi^{(r+1)}(z)\right)^2}{\Psi^{(r)}(z)\Psi^{(r+2)}(z)}$$

which gives the inequality (2.2). □

Lemma 2.3. Let $z > 0$ and $r \in \mathbb{N}$. Then the function

$$T(z) = \frac{z\Psi^{(r+1)}(z)}{\left(\Psi^{(r)}(z)\right)^2} \tag{2.3}$$

is strictly decreasing if r is odd and strictly increasing if r is even.

Proof. By applying the inequality (2.2), we obtain

$$\begin{aligned} [\Psi^{(r)}(z)]^3 T'(z) &= \Psi^{(r)}(z)\Psi^{(r+1)}(z) + z\Psi^{(r)}(z)\Psi^{(r+2)}(z) - 2z\left(\Psi^{(r+1)}(z)\right)^2 \\ &= \Psi^{(r)}(z)\Psi^{(r+1)}(z) + z\left[\Psi^{(r)}(z)\Psi^{(r+2)}(z) - 2\left(\Psi^{(r+1)}(z)\right)^2\right] \\ &< 0. \end{aligned}$$

By this, we have $T'(z) < 0$ if r is odd and $T'(z) > 0$ if r is even. □

Lemma 2.4 (cf. [7, Lemma 2]). Let $z > 0$ and $r \in \mathbb{N}$. Then the function

$$V(z) = z \frac{\Psi^{(r+1)}(z)}{\Psi^{(r)}(z)} \tag{2.4}$$

is strictly increasing.

Lemma 2.5. For $z > 0$ and $r \in \mathbb{N}$, let $P(z)$ be defined as

$$P(z) = z\Psi^{(r+1)}(z). \tag{2.5}$$

Then $P(z)$ is strictly increasing if r is odd and strictly decreasing if r is even.

Proof. By applying (1.4), (1.7) and Lemma 2.1, we have

$$\begin{aligned} \frac{P'(z)}{z} &= \frac{1}{z}\Psi^{(r+1)}(z) + \Psi^{(r+2)}(z) \\ &= (-1)^{r+2} \int_0^\infty e^{-zs} ds \int_0^\infty \frac{s^{r+1}e^{-zs}}{1-e^{-s}} ds + (-1)^{r+3} \int_0^\infty \frac{s^{r+2}e^{-zs}}{1-e^{-s}} ds \\ &= (-1)^{r+2} \int_0^\infty \left[\int_0^s \frac{u^{r+1}}{1-e^{-u}} du \right] e^{-zs} ds + (-1)^{r+3} \int_0^\infty \frac{s^{r+2}e^{-zs}}{1-e^{-s}} ds \\ &= Q(z). \end{aligned}$$

Suppose that r is odd. Then

$$Q(z) = \int_0^\infty K(s)e^{-zs} ds,$$

where

$$K(s) = \frac{s^{r+2}}{1-e^{-s}} - \int_0^s \frac{u^{r+1}}{1-e^{-u}} du.$$

Next,

$$K'(s) = \frac{s^{r+1}}{1-e^{-s}} \left[(r+1) - \frac{se^{-s}}{1-e^{-s}} \right] > 0$$

which shows that $K(s)$ is increasing. Hence $K(s) > \lim_{s \rightarrow 0} K(s) = 0$. This implies that $Q(z) > 0$ and consequently, $P(z)$ is increasing. Similarly, suppose that r is even. Then

$$Q(z) = - \int_0^\infty K(s)e^{-zs} ds < 0$$

which implies that $P(z)$ is decreasing. □

3. Main results

In this section, we present our main results.

Theorem 3.1. Let $r \in \mathbb{N}$ and $z > 0$. Then

$$\frac{2\Psi^{(r)}(z)\Psi^{(r)}(1/z)}{\Psi^{(r)}(z) + \Psi^{(r)}(1/z)} \leq r!\zeta(r+1) \tag{3.1}$$

holds if r is odd and

$$\frac{2\Psi^{(r)}(z)\Psi^{(r)}(1/z)}{\Psi^{(r)}(z) + \Psi^{(r)}(1/z)} \geq -r!\zeta(r+1) \tag{3.2}$$

holds if r is even. In both cases, equality is arrived at if and only if $z = 1$.

Proof. The condition for equality in (3.1) and (3.2) is obvious. Because of this, we shall consider the results for $z \in (0, 1) \cup (1, \infty)$. Now define $F(z)$ as

$$F(z) = \frac{2\Psi^{(r)}(z)\Psi^{(r)}(1/z)}{\Psi^{(r)}(z) + \Psi^{(r)}(1/z)}$$

for $r \in \mathbb{N}$ and $z \in (0, 1) \cup (1, \infty)$. Then

$$\frac{F'(z)}{F(z)} = \frac{\Psi^{(r+1)}(z)}{\Psi^{(r)}(z)} - \frac{1}{z^2} \frac{\Psi^{(r+1)}(1/z)}{\Psi^{(r)}(1/z)} - \frac{\Psi^{(r+1)}(z) - \frac{1}{z^2} \Psi^{(r+1)}(1/z)}{\Psi^{(r)}(z) + \Psi^{(r)}(1/z)}$$

which can be rearranged to obtain

$$z \left[\Psi^{(r)}(z) + \Psi^{(r)}(1/z) \right] \frac{F'(z)}{F(z)} = z \frac{\Psi^{(r+1)}(z)}{\Psi^{(r)}(z)} \Psi^{(r)}(1/z) - \frac{1}{z} \frac{\Psi^{(r+1)}(1/z)}{\Psi^{(r)}(1/z)} \Psi^{(r)}(z).$$

Furthermore, we obtain

$$\begin{aligned} z \left[\frac{1}{\Psi^{(r)}(z)} + \frac{1}{\Psi^{(r)}(1/z)} \right] \frac{F'(z)}{F(z)} &= z \frac{\Psi^{(r+1)}(z)}{(\Psi^{(r)}(z))^2} - \frac{1}{z} \frac{\Psi^{(r+1)}(1/z)}{(\Psi^{(r)}(1/z))^2} \\ &= \mathcal{X}(z). \end{aligned}$$

Suppose that r is odd. Since $T(z)$ is decreasing for odd r , we arrive at the conclusion that $\mathcal{X}(z) > 0$ if $z \in (0, 1)$ and $\mathcal{X}(z) < 0$ if $z \in (1, \infty)$. Note that if $z \in (0, 1)$, then $z < 1/z$ and if $z \in (1, \infty)$, then $z > 1/z$. Thus, $F'(z) > 0$ if $z \in (0, 1)$ and $F'(z) < 0$ if $z \in (1, \infty)$. Hence, $F(z)$ is increasing on $(0, 1)$ and decreasing on $(1, \infty)$. Therefore, for $z \in (0, 1) \cup (1, \infty)$, we have

$$F(z) < \lim_{z \rightarrow 1} F(z) = \Psi^{(r)}(1) = r! \zeta(r+1)$$

which gives (3.1). Next, suppose that r is even. Since $T(z)$ is increasing for even r , we arrive at the conclusion that $\mathcal{X}(z) < 0$ if $z \in (0, 1)$ and $\mathcal{X}(z) > 0$ if $z \in (1, \infty)$. By this, $F'(z) < 0$ if $z \in (0, 1)$ and $F'(z) > 0$ if $z \in (1, \infty)$. Hence, $F(z)$ is decreasing on $(0, 1)$ and increasing on $(1, \infty)$. Therefore, for $z \in (0, 1) \cup (1, \infty)$, we have

$$F(z) > \lim_{z \rightarrow 1} F(z) = \Psi^{(r)}(1) = -r! \zeta(r+1)$$

which gives (3.2). □

Theorem 3.2. Let $r \in \mathbb{N}$. Then the inequality

$$\Psi^{(r)}(z) \Psi^{(r)}(1/z) \geq (r! \zeta(r+1))^2 \tag{3.3}$$

holds for $z > 0$. Equality is arrived at if and only if $z = 1$.

Proof. The condition for equality is obvious. So let $G(z) = \Psi^{(r)}(z) \Psi^{(r)}(1/z)$ for $r \in \mathbb{N}$ and $z \in (0, 1) \cup (1, \infty)$. Then

$$\begin{aligned} z \frac{G'(z)}{G(z)} &= z \frac{\Psi^{(r+1)}(z)}{\Psi^{(r)}(z)} - \frac{1}{z} \frac{\Psi^{(r+1)}(1/z)}{\Psi^{(r)}(1/z)} \\ &= \mathcal{D}(z). \end{aligned}$$

Since $V(z)$ is increasing, we arrive at the conclusion that $\mathcal{D}(z) < 0$ if $z \in (0, 1)$ and $\mathcal{D}(z) > 0$ if $z \in (1, \infty)$. Thus, $G'(z) < 0$ if $z \in (0, 1)$ and $G'(z) > 0$ if $z \in (1, \infty)$. Hence, $G(z)$ is decreasing on $(0, 1)$ and increasing on $(1, \infty)$. Therefore, for $z \in (0, 1) \cup (1, \infty)$, we have

$$G(z) > \lim_{z \rightarrow 1} G(z) = \left(\Psi^{(r)}(1) \right)^2 = (r! \zeta(r+1))^2$$

which gives (3.3). □

Theorem 3.3. Let $r \in \mathbb{N}$ and $z > 0$. Then

$$\Psi^{(r)}(z) + \Psi^{(r)}(1/z) \geq 2(r! \zeta(r+1)) \tag{3.4}$$

holds if r is odd and

$$\Psi^{(r)}(z) + \Psi^{(r)}(1/z) \leq -2(r! \zeta(r+1)) \tag{3.5}$$

holds if r is even. In both cases, equality is arrived at if and only if $z = 1$.

Proof. The condition for equality is obvious. Due to this, let $H(z) = \Psi^{(r)}(z) + \Psi^{(r)}(1/z)$ for $r \in \mathbb{N}$ and $z \in (0, 1) \cup (1, \infty)$. Then

$$\begin{aligned} zH'(z) &= z\Psi^{(r+1)}(z) - \frac{1}{z}\Psi^{(r+1)}(1/z) \\ &= \mathcal{U}(z). \end{aligned}$$

Now suppose that r is odd. Since $P(z)$ is increasing for odd r , we arrive at the conclusion that $\mathcal{U}(z) < 0$ if $z \in (0, 1)$ and $\mathcal{U}(z) > 0$ if $z \in (1, \infty)$. Hence, $H(z)$ is decreasing on $(0, 1)$ and increasing on $(1, \infty)$. Therefore, for $z \in (0, 1) \cup (1, \infty)$, we have

$$H(z) > \lim_{z \rightarrow 1} H(z) = 2\Psi^{(r)}(1) = 2r!\zeta(r+1)$$

which gives (3.4). Similarly, suppose that r is even. Since $P(z)$ is decreasing for even r , we arrive at the conclusion that $\mathcal{U}(z) > 0$ if $z \in (0, 1)$ and $\mathcal{U}(z) < 0$ if $z \in (1, \infty)$. Hence, $H(z)$ is increasing on $(0, 1)$ and decreasing on $(1, \infty)$. Therefore, for $z \in (0, 1) \cup (1, \infty)$, we have

$$H(z) < \lim_{z \rightarrow 1} H(z) = 2\Psi^{(r)}(1) = -2r!\zeta(r+1)$$

which gives (3.5). □

4. Conclusion

In this paper, we have established arithmetic mean, geometric mean and harmonic mean inequalities involving the polygamma functions. These inequalities provide bounds for the means in terms of the classical Riemann zeta function. The findings lay a good foundation for further research on the subject matter.

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