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Some identities for harmonic numbers based on composition of generating functions

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Abstract

In the paper we study the harmonic numbers and their properties. Applying a method of generating functions and a composition operation, we get a series of identities that are related to the harmonic, central binomial, triangular, Catalan and Mersenne numbers. We find a new generating function for the sum of products of the harmonic numbers and inverse binomial coefficients. Its integral representation is written. In addition, we provide examples of calculating infinite sums and obtaining identities.

Keywords: Generating function, harmonic numbers, central binomial numbers, Catalan numbers

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1. Introduction

Harmonic numbers are an important class of special numbers that have been the focus of research for many prominent mathematicians [4]-[7], [10, 19]. There exist an immense number of various identities related to the harmonic numbers. For example, in [20] author discusses over 40 various identities obtained through manipulations with the generating function

$$H(x) = \frac{-\log(1-x)}{(1-x)^m}.$$

In [2], authors obtained two remarkable generating functions for the product of the central binomial numbers and the harmonic numbers:

$$\binom{2n}{n} H(n) = [x^n] \frac{2}{\sqrt{1-4x}} \log\left(\frac{\sqrt{1-4x}+1}{2\sqrt{1-4x}}\right), \tag{1.1}$$

$$\binom{2n}{n} H(2n) = [x^n] \frac{1}{\sqrt{1-4x}} \log\left(\frac{\sqrt{1-4x}+1}{2(1-4x)}\right),$$

where H_n denotes the harmonic numbers and $H_0 = 0$.

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A large number of different identities for the harmonic and the inverse binomial coefficients is known and has been the subject of notable works, among them [15, 21]. Recent papers have discussed identities that provide a relation between the harmonic numbers and binomial coefficients, for example, in [1] there is the following identity:

$$\sum_{k=1}^{n}\frac{H(k)}{\binom{n}{k}}=\frac{1+(n+1)H(n)}{2^{n+1}}\sum_{k=1}^{n+1}\frac{2^k}{k}-\frac{n+1}{2^{n+2}}\sum_{k=1}^{n+1}\frac{H(k)2^k}{k}-\frac{n+1}{2^{n+2}}\sum_{k=1}^{n+1}\frac{2^k}{k^2}.$$

Let us show the left-hand side of this identity. The exponential version of it has the following expression:

$$a(n)=n!\sum_{k=1}^{n}\frac{H(k)}{\binom{n}{k}}=\sum_{k=1}^{n}H(k)k!(n-k)!.$$

This formula defines the sequence A091530 in the Online Encyclopedia of Integer Sequences [16]. It allows for a combinatorial interpretation: let there be two sets M_1 and M_2, having $n+1$ elements in total. Let the cardinality of the set M_1 be $k+1$, elements of this set create all permutations, each having exactly two cycles. Elements of the set M_2 create usual permutations. The total size of the set of permutation pairs for all possible values k is $a(n)$.

Many interesting identities connecting the squares of the binomial numbers, central binomial numbers and different classes of the harmonic numbers are presented in [3] (see formulas 3.45, 3.46, 3.47, 3.48). Recently, in [18], based on the formula

$$y(n,\lambda)=\sum_{j=0}^{n}\frac{(-1)^n}{(j+1)\lambda^{j+1}(\lambda-1)^{n+1-j}},$$

it is obtained

$$y\left(n,\frac{1}{2}\right)=2^{n+2}\left(H\left(\left[\frac{n}{2}\right]\right)-H(n)+\frac{(-1)^{n+1}}{n+1}\right).$$

Further development of this approach was used in [17], where it is obtained the following identity:

$$\sum_{c=1}^{m}\frac{(c-1)(8dc-c-d-1)}{c}=$$
$$\frac{12d}{d-1}\sum_{c=1}^{m}(f(c,d)-v(c,d))+\frac{m(m+1)(8d^2-d9d+1)+18dm(1-d)c+2(d^2-1)H(m)}{2(d-1)}$$

for $d,m\in\mathbb{N}$.

Other related identities one can find in the reference list of the cited literature.

The aim of the paper is to show a way of applying a method of generating functions and a composition operation for getting a series of results that are related to the harmonic numbers. In Section 2 we give a few new identities that are related to the harmonic, central binomial, triangular, Catalan and Mersenne numbers. In Section 3 we construct a generating function for the sum of products of the harmonic numbers and inverse binomial coefficients and use it to obtain identities and explicit expressions for infinite sums.

2. Identities for the harmonic numbers

In the paper, we use a method associated with finding the coefficients of the composition of generating functions [12]-[14]. The method for obtaining identities is defined as follows:

1. We write the composition of generating functions as follows:

$$G(x)=A(F(x)),$$

providing $F(0)=0$;

2. For coefficients of the generating function composition, we write the expression

$$g(n) = \sum_{k=0}^{n} F^{\Delta}(n,k)a(k),$$

where $F^{\Delta}(n,k) = [x^n]F(x)^k$, $a(n) = [x^n]A(x)$ and $G(n) = [x^n]G(x)$;

3. We define the coefficients

$$c(n) = [x^n]G(x);$$

4. Next, we set the equation $c(n) = g(n)$ and carry out the transformations.

Let us use this method to obtain desired identities.

Theorem 2.1. *For the harmonic numbers $H(k)$ and the central binomial numbers we have*

$$\sum_{k=1}^{n} \binom{k}{n-k}(-1)^{n-k}\binom{2k}{k}H(k) = 2\sum_{k=1}^{n}\binom{n}{k}H(k). \tag{2.1}$$

Proof. Let there be a generating function (1.1) as follows:

$$A(x) = \frac{2}{\sqrt{1-4x}}\log\left(\frac{\sqrt{1-4x}+1}{2\sqrt{1-4x}}\right).$$

Consider the composition of functions $G(x) = A(F(x))$, where $F(x) = x - x^2$. In accordance with the above method, we write the coefficients of the generating function $F(x)$ in power k:

$$F^{\Delta}(n,k) = [x^n]F(x)^k = \binom{k}{n-k}(-1)^{n-k}.$$

Therefore

$$g(n) = \sum_{k=0} F^{\Delta}(n,k)a(k) = \sum_{k=0}^{n}\binom{k}{n-k}(-1)^{n-k}\binom{2k}{k}H(k).$$

On the other hand,

$$\frac{2}{\sqrt{1-4(x-x^2)}} = \frac{2}{1-x}$$

and

$$\log\left(\frac{\sqrt{1-4x+4x^2}+1}{2\sqrt{1-4x+4x^2}}\right) = \log\left(\frac{1-x}{1-2x}\right).$$

Then

$$G(x) = \frac{2}{1-2x}\log\left(\frac{1-x}{1-2x}\right).$$

The generating function $x\,G(x)$ can also be considered as a composition of functions

$$x\,B(x) = 2\left(\frac{x}{1-x}\right)H\left(\frac{x}{1-x}\right),$$

where

$$H(x) = -\frac{\log(1-x)}{1-x}.$$

Coefficients of the generating function $\left(\frac{x}{1-x}\right)^k$ have the following expression:

$$[x^n]\left(\frac{x}{1-x}\right)^k = \binom{n-1}{n-k}.$$

Then, using the compositional formula, we obtain the expression for the coefficients $G(x)$

$$[x^n]x\,G(x) = 2\sum_{k=2}^{n}\binom{n-1}{n-k}H(k-1)$$

or

$$g(n) = 2\sum_{k=1}^{n}\binom{n}{k}H(k).$$

On the other hand, for the generating function $F(x) = x - x^2$ we get

$$F^{\Delta}(n,k) = [x^n](x-x^2)^k = \binom{k}{n-k}(-1)^{n-k}.$$

Then

$$[x^n]A(x-x^2) = \sum_{k=0}^{n}F^{\Delta}(n,k)\binom{2\,k}{k}H(k).$$

Substituting resulting expression

$$\sum_{k=0}^{n}\binom{k}{n-k}(-1)^{n-k}\binom{2\,k}{k}H(k),$$

we obtain the desired identity

$$\sum_{k=1}^{n}\binom{k}{n-k}(-1)^{n-k}\binom{2\,k}{k}H(k) = 2\sum_{k=1}^{n}\binom{n}{k}H(k).$$

□

The resulting identity defines the relationship between the central binomial numbers, the harmonic numbers and the binomial transform of the harmonic numbers.

Now let us obtain the second identity as follows.

Theorem 2.2. *For the harmonic numbers $H(k)$ and the Catalan numbers C_k we have*

$$\frac{n\,(n+1)}{2}\sum_{k=1}^{n}\binom{k+1}{n-k}(-1)^{n-k}C_k\,H(k) = 2^n - 1. \tag{2.2}$$

Proof. Let us consider the following integral

$$D(x) = \int A(x)dx = -\log\left(\frac{\sqrt{1-4\,x}+1}{2\,\sqrt{1-4\,x}}\right)\,\sqrt{1-4\,x} - \log\left(\sqrt{1-4\,x}+1\right) + c.$$

Now let's write its expansion in the Taylor series

$$-\log 2 + x^2 + 3\,x^3 + \frac{55\,x^4}{6} + \frac{175\,x^5}{6} + \frac{959\,x^6}{10} + \frac{1617\,x^7}{5} + \cdots.$$

Next, we write the function $D(x)$ without the free term.

$$D(x) = -\log\left(\frac{\sqrt{1-4\,x}+1}{2\,\sqrt{1-4\,x}}\right)\,\sqrt{1-4\,x} - \log\left(\sqrt{1-4\,x}+1\right) + \log(2)$$

$$= -\log\left(\frac{\sqrt{1-4\,x}+1}{2\,\sqrt{1-4\,x}}\right)\,\sqrt{1-4\,x} - \log\left(\frac{\sqrt{1-4\,x}+1}{2}\right).$$

The condition now holds

$$[x^n]D(x) = \frac{1}{n}\binom{2\,n-2}{n-1}H(n-1) = C_{n-1}H(n-1),$$

where C_n are the Catalan numbers.

Consider the compositions of functions $D(x - x^2)$, after substitution we get

$$\log\left(\frac{1-x}{1-2\,x}\right)(2\,x-1) - \log(1-x) = \log(1-2\,x)\,(1-2\,x) - 2\,\log(1-x)\,(1-x)\,.$$

After transformations, this yields

$$[x^n]\log(1-2\,x)\,(1-2\,x) - 2\,\log(1-x)\,(1-x) = \frac{2^n-2}{n^2-n}.$$

On the other hand, for the composition $D(x - x^2)$ one can write

$$[x^n]D(x-x^2) = \sum_{k=2}^{n}\binom{k}{n-k}(-1)^{n-k}\frac{1}{k}\binom{2\,k-2}{k-1}H(k-1).$$

Whence we obtain the identity

$$\sum_{k=2}^{n}\binom{k}{n-k}(-1)^{n-k}C_{n-1}\,H(k-1) = \frac{2^n-2}{n^2-n}.$$

After simple transformations, we obtain the desired identity

$$\frac{n\,(n+1)}{2}\sum_{k=1}^{n}\binom{k+1}{n-k}(-1)^{n-k}C_k\,H(k) = 2^n-1.$$

□

The resulting identity is related to the triangular numbers, the Catalan numbers, the harmonic numbers and the Mersenne numbers.

Theorem 2.3. *For the harmonic numbers $H(k)$ and the Catalan numbers C_k we have*

$$n\,(n+1)\sum_{k=1}^{n}\binom{k+1}{n-k}(-1)^{n-k}C_k\,H(2\,k) = 2^{n+1}-1. \tag{2.3}$$

Proof. Let there be a generating function

$$E(x) = \frac{1}{\sqrt{1-4\,x}}\log\left(\frac{\sqrt{1-4\,x}+1}{2\,(1-4\,x)}\right).$$

It is known [2] that the coefficients of this generating function have the following expression:

$$[x^n]E(x) = \binom{2\,n}{n}H(2\,n).$$

Now we consider the following integral:

$$\int E(x)dx = \frac{2\left(-\frac{\sqrt{1-4\,x}}{2} - \frac{\log\left(\sqrt{1-4\,x}+1\right)}{2}\right) - \log\left(\frac{\sqrt{1-4\,x}+1}{2\,(1-4\,x)}\right)\sqrt{1-4\,x}}{2} + c.$$

Next, we perform the same actions as for the identity (2.1). We obtain the desired identity

$$n\,(n+1)\sum_{k=1}^{n}\binom{k+1}{n-k}(-1)^{n-k}C_k\,H(2\,k) = 2^{n+1}-1.$$

□

Now we use the identities (2.2) and (2.3) and obtain the following identity:

$$\sum_{k=1}^{n} \frac{(-1)^{n-k}}{k+1} \binom{2\,k}{k} \binom{k+1}{n-k} (H(2\,k) - H(k)) = \frac{1}{n\,(n+1)}.$$

Then,

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{(-1)^{n-k}}{k+1} \binom{2\,k}{k} \binom{k+1}{n-k} (H(2\,k) - H(k)) = 1.$$

3. A generating function for the harmonic numbers and inverse binomial coefficients

The harmonic numbers and inverse binomial coefficients are classic mathematical objects of high importance for various branches of mathematics and related fields.

For further study, let us consider the following identity for the dilogarithm [11]:

$$\mathrm{Li}_2(-x) - \mathrm{Li}_2(1-x) + \frac{1}{2}\mathrm{Li}_2(1-x^2) = -\frac{\pi^2}{12} - \log(x)\log(1+x), \tag{3.1}$$

where

$$\mathrm{Li}_2(x) = \sum_{n>0} \frac{x^n}{n^2}.$$

In the following theorem we give a generating function for the sum of products of the harmonic numbers and inverse binomial coefficients.

Theorem 3.1. *The generating function for the sum of products of the harmonic numbers and inverse binomial coefficients is*

$$\begin{aligned} A(x) = & \frac{-\mathrm{Li}_2((2-x)\;x) + 2\,\mathrm{Li}_2(x) - 2\;\log(1-x)\;(x-1)}{(x-2)^2} + \\ & + \frac{\log(1-x)\;\left(4\,x^2 + \log(1-x)\;x - 14\,x - \log(1-x) + 12\right)}{2\;(x-2)^2\;(x-1)}. \end{aligned} \tag{3.2}$$

Proof. We use the identity obtained in [1] :

$$\sum_{k=1}^{n} \frac{H(k)}{\binom{n}{k}} = \frac{1+(n+1)H(n)}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k} - \frac{n+1}{2^{n+2}} \sum_{k=1}^{n+1} \frac{H(k)2^k}{k} - \frac{n+1}{2^{n+2}} \sum_{k=1}^{n+1} \frac{2^k}{k^2}. \tag{3.3}$$

Now we find the generating functions for the right-hand side terms of the identity. Let us decompose this expression into its components

$$\frac{1+(n+1)H(n)}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k} = \frac{1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k} + \frac{(n+1)H(n)}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k}.$$

Let us find a generating function for the expression

$$\frac{1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k}.$$

For that purpose, we write

$$\frac{2^k}{k} = [x^k]\,(-\log(1-2x))\,.$$

Then we have the following sum

$$\sum_{k=1}^{n} \frac{2^k}{k} = [x^k]\frac{-\log(1-2x)}{(1-x)}. \tag{3.4}$$

Therefore

$$\frac{1}{2^{n+1}}\sum_{k=1}^{n+1}\frac{2^k}{k} = [x^k]\frac{-\log(1-x)}{(1-\frac{x}{2})x}.$$

Now let us find the generating function for the expression

$$\frac{(n+1)H(n)}{2^{n+1}}\sum_{k=1}^{n+1}\frac{2^k}{k}.$$

Consider the expression

$$H(n)\sum_{k=1}^{n+1}\frac{2^k}{k} = \left(H(n-1)+\frac{1}{n}\right)\left(\sum_{k=1}^{n}\frac{2^k}{k}+\frac{2^{n+1}}{n+1}\right) = H(n-1)\sum_{k=1}^{n}\frac{2^k}{k} + H(n-1)\frac{2^{n+1}}{n+1} + \frac{1}{n}\sum_{k=1}^{n}\frac{2^k}{k} + \frac{2^n}{n^2}. \tag{3.5}$$

The desired generating function is as follows:

$$B(x) = \sum_n H(n)\sum_{k=1}^{n+1}\frac{2^k}{k}x^n.$$

Now we write the generating functions for all terms of the expression (3.5) and compose the functional equation

$$B(x) = xB(x) + B_1(x) + B_2(x) + B_3(x),$$

where

$$B_1(x) = \sum_n H(n-1)\frac{2^n}{n}x^n,$$

$$B_2(x) = \sum_n \frac{1}{n}\sum_{k=1}^{n}\frac{2^k}{k}x^n,$$

$$B_3(x) = \sum_n \frac{2^n}{n(n+1)}x^n.$$

Then

$$B(x) = \frac{1}{1-x}\left(B_1(x)+B_2(x)+B_3(x)\right).$$

Now we find expressions for $B_1(x)$ and $B_2(x)$. By using integral representation, we obtain

$$\frac{H(n-1)2^{n+1}}{n+1} = [x^n]\int\frac{-4x\,\log(1-2\,x)}{(1-2x)}dx.$$

After integration, the generating function has the following form:

$$B_1(x) = \frac{(4\,\log(1-2\,x)-4)\;x+\log^2(1-2\,x)-2\,\log(1-2\,x)}{2}.$$

Now, we expand it in a Taylor series

$$\frac{8\,x^3}{3}+6\,x^4+\frac{176\,x^5}{15}+\frac{200\,x^6}{9}+\frac{4384\,x^7}{105}+\cdots.$$

Note that the expansion coefficients appear in the third term, so the series needs to be shifted to the left by dividing the function by x. Then

$$B_1(x) = \frac{(4\,\log(1-2\,x)-4)\;x+\log^2(1-2\,x)-2\,\log(1-2\,x)}{2\,x}.$$

Now let us find the generating function $B_2(x)$. For that purpose, we write

$$\frac{1}{n}\sum_{k=1}^{n}\frac{2^k}{k}=\frac{1}{n}\sum_{k=1}^{n-1}\frac{2^k}{k}+\frac{2^n}{n^2},$$

$$\sum_{k=1}^{n}\frac{2^k}{k}=[x^n]\frac{-\log(1-2x)}{(1-x)}.$$

Then

$$\frac{1}{n}\sum_{k=1}^{n-1}\frac{2^k}{k}=[x^n]\int\frac{-\log(1-2x)}{(1-x)}dx=\mathrm{Li}_2(2\,x-1)+\log\left(1-2\,x\right)\,\log\left(2-2\,x\right).$$

For the second sum we have

$$\frac{2^n}{n^2}=[x^n]\mathrm{Li}_2(2x).$$

Therefore

$$\frac{1}{n}\sum_{k=1}^{n}\frac{2^k}{k}=[x^n]\mathrm{Li}_2(2\,x-1)+\log\left(1-2\,x\right)\,\log\left(2-2\,x\right)+\mathrm{Li}_2(2x).$$

Now we expand the resulting function in a Taylor series, the first terms are shown below

$$-\frac{\pi^2}{12}+2\,x+2\,x^2+\frac{20\,x^3}{9}+\frac{8\,x^4}{3}+\frac{256\,x^5}{75}+\frac{208\,x^6}{45}+\frac{4832\,x^7}{735}+\cdots.$$

Note that the resulting series differs from the desired one by the presence of the constant term $-\frac{\pi^2}{12}$. From where the generating function has the following form:

$$B_2(x)=\frac{\pi^2}{12}+\mathrm{Li}_2(2\,x-1)+\log\left(1-2\,x\right)\,\log\left(2-2\,x\right)+\mathrm{Li}_2(2x).$$

We use the identity for the dilogarithm (3.1), we substitute the variable $z=(1-x)$ and obtain the generating function $B_2(x)$:

$$B_2(x)=2\,\mathrm{Li}_2(2\,x)-\frac{\mathrm{Li}_2(4\,x-4\,x^2)}{2}.$$

Now we find $B_3(x)$, for which purpose we write

$$\frac{2^{n+1}}{n(n+1)}=\frac{2^{n+1}}{n}-\frac{2^{n+1}}{n+1}.$$

Whence, with the start value equal to zero, follows the expression for the desired function

$$B_3(x)=\frac{\log\left(1-2\,x\right)}{x}-2\,\log\left(1-2\,x\right)+2.$$

Then

$$B(x)=\frac{B_1(x)+B_2(x)+B_3(x)}{1-x}.$$

After substitution and simplification, we obtain the following expression of the function:

$$B(x)=\frac{x\,\mathrm{Li}_2(4\,x-4\,x^2)-4\,x\,\mathrm{Li}_2(2\,x)-\log^2\left(1-2\,x\right)}{2\,x^2-2\,x}.$$

Therefore

$$\frac{(n+1)H(n)}{2^{n+1}}\sum_{k=1}^{n+1}\frac{2^k}{k}=[x^n]\frac{\frac{d}{d\,x}B\left(\frac{x}{2}\right)\,x}{2}+\frac{B\left(\frac{x}{2}\right)}{2}.$$

The generating function has the expression

$$\frac{-\mathrm{Li}_2((2-x)x)+4\mathrm{Li}_2(x)-2\log(1-x)(x-1)+\log^2(1-x)}{(x-2)^2}-\frac{2\log(1-x)}{1-x}. \tag{3.6}$$

Now the desired generating function is equal to the sum of the generating functions (3.4) and (3.6)

$$B(x)=\frac{-\mathrm{Li}_2((2-x)x)+4\mathrm{Li}_2(x)-2\log(1-x)(x-1)+\log^2(1-x)}{(x-2)^2}-\frac{2\log(1-x)}{1-x}-\frac{2\,\log(1-x)}{(2-x)x}. \tag{3.7}$$

Consider the right-hand side term of the identity (3.3). Let us find the generating function for the expression under the summation sign

$$\frac{H(n)2^n}{n}=[x^n]-\int\frac{\log(1-2\,x)}{(1-2\,x)\;x}\,dx.$$

The integral is equal to

$$\int\frac{\log(1-2\,x)}{(1-2\,x)\;x}\,dx=\frac{2\,\mathrm{Li}_2(2\,x)+\log^2(1-2\,x)}{2}.$$

Therefore the sum has the following generating function

$$\sum_{k=1}^{n}\frac{H(k)2^k}{k}=[x^n]\frac{2\,\mathrm{Li}_2(2\,x)+\log^2(1-2\,x)}{2\,(1-x)}.$$

The resulting function is defined as follows:

$$C(x)=\frac{2\,\mathrm{Li}_2(2\,x)+\log^2(1-2\,x)}{2\,(1-x)}.$$

Now let us write the generating function for the following expression:

$$\frac{(n+1)}{2^{n+2}}\sum_{k=1}^{n}\frac{H(k)2^k}{k}=[x^n]\frac{1}{2}\frac{d}{d\,x}C\left(\frac{x}{2}\right)=[x^n]\frac{2\,\mathrm{Li}_2(x)+\log^2(1-x)}{2\,(2-x)^2}-\frac{\log(1-x)}{(x-2)\,(x-1)\,x}. \tag{3.8}$$

For the third term of the right-hand side of the identity (3.3) we write

$$\frac{2^n}{k^2}=[x^n]\mathrm{Li}_2(2x).$$

Next, we make the summation and differentiation for this expression, and obtain the following generating function

$$\frac{n+1}{2^{n+2}}\sum_{k=1}^{n+1}\frac{2^k}{k^2}=[x^n]\frac{\mathrm{Li}_2(x)}{(2-x)^2}-\frac{\log(1-x)}{(2-x)\;x}. \tag{3.9}$$

All generating functions for the right-hand side of the identity (3.3) have now been found. We sum the resulting functions (3.7), (3.8) and (3.9):

$$\begin{aligned}A(x)\quad=\quad&\frac{-\mathrm{Li}_2((2-x)\;x)+2\,\mathrm{Li}_2(x)-2\,\log(1-x)\;(x-1)}{(x-2)^2}\\&+\frac{\log(1-x)\left(4\,x^2+\log(1-x)\;x-14\,x-\log(1-x)+12\right)}{2\;(x-2)^2\;(x-1)}.\end{aligned}$$

Thus, we obtain the desired generating function

$$\sum_{k=1}^{n}\frac{H(k)}{\binom{n}{k}}=[x^n]A(x).$$

□

We write its integral representation as follows:

$$\frac{1}{n+1}\sum_{k=1}^{n}\frac{H(k)}{\binom{n}{k}} = [x^n]\int A(x)\,dx.$$

$$\int A(x)\,dx(x) = \frac{2\operatorname{Li}_2(2x-x^2) - 4\operatorname{Li}_2(x) - \log^2(1-x)}{x\,(2x-4)}.$$

Therefore

$$\frac{1}{n+1}\sum_{k=1}^{n}\frac{H(k)}{\binom{n}{k}} = [x^n]\frac{2\operatorname{Li}_2(2x-x^2) - 4\operatorname{Li}_2(x) - \log^2(1-x)}{x\,(2x-4)}. \tag{3.10}$$

This generating function is denoted by $I(x)$.

For application of (3.2) and (3.10), we consider the special case with $x < 1$. Then, setting $x = \frac{1}{2}$ obtain the following convergent infinite sums:

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{H(k)}{\binom{n}{k}}\frac{1}{2^n} = -\frac{6\,\log^2 2 - 60\,\log 2 + 12\operatorname{Li}_2(\frac{3}{4}) - 2\pi^2}{27}$$

and

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{H(k)}{\binom{n}{k}}\frac{1}{(n+1)2^n} = -\frac{6\,\log^2 2 + 12\operatorname{Li}_2(\frac{3}{4}) - 2\pi^2}{9}.$$

Let us consider the application of the resulting functions based on composition methods [12]-[14]. Suppose we have a generating function

$$F(x) = 1 - \sqrt{1-x} = 2xC\left(\frac{x}{4}\right),$$

where $C(x)$ is the generating function for the Catalan numbers.

In order to find expressions of the composition, it is necessary to know the expressions of degree coefficients of the generating function. We have

$$\frac{k}{n}\binom{2n-k-1}{n-1} = [x](xC(x))^k.$$

Since

$$F^{\Delta}(n,k) = [x^n]1 - \sqrt{1-x} = \frac{k\,2^k}{n\,4^n}\binom{2\,n-k-1}{n-k},$$

the composition $I(F(x))$ has the following expression:

$$[x^n]I(F(x)) = \sum_{i=1}^{n}F^{\Delta}(n,i)\sum_{k=1}^{i}H(k)\binom{k}{i}^{-1}.$$

Then the composition coefficients has the expression

$$\frac{1}{n\,4^n}\sum_{i=1}^{n}i\,2^i\binom{2\,n-i-1}{n-i}\sum_{k=1}^{i}H(k)\binom{k}{i}^{-1}.$$

On the other hand,

$$I(F(x)) = \frac{-8\operatorname{Li}_2(x) + \log^2(1-x) + 16\operatorname{Li}_2(1-\sqrt{1-x})}{8\,x}.$$

Now let us find the expressions for coefficients for the components of this function

$$[x^n]\log(1-x)^2 = \sum_{k=1}^{n-1}\frac{1}{(n-k)k} = \frac{1}{n}\sum_{k=1}^{n-1}\frac{1}{(n-k)} + \frac{1}{k} = \frac{2H(n-1)}{n}.$$

$$[x^n]\mathrm{Li}_2\left(1-\sqrt{1-x}\right)=\sum_{i=1}^{n}F^{\Delta}(n,k)\frac{1}{k^2}=\sum_{k=1}^{n}\frac{k\,2^k}{n\,4^n}\binom{2\,n-k-1}{n-k}\frac{1}{k^2}.$$

After simple transformations, we obtain

$$-\frac{1}{(n+1)^2}+\frac{H(n)}{4(n+1)}+\frac{2}{(n+1)\,4^{n+1}}\sum_{k=1}^{n+1}\frac{2^k}{k}\binom{2\,n-k-1}{n-k}.$$

By equating the expressions of the decomposition coefficients to the composition of the generating function, we obtain the following identity:

$$\sum_{i=1}^{n}\frac{i\,2^i}{n\,4^n}\binom{2\,n-i-1}{n-i}\sum_{k=1}^{i}\frac{H(k)}{\binom{k}{i}}=\frac{H(n)}{4(n+1)}+\frac{1}{(n+1)\,4^n}\sum_{k=1}^{n+1}\frac{2^{k-1}}{k}\binom{2\,n-k-1}{n-k}-\frac{1}{(n+1)^2}.$$

4. Conclusion

In this article, we give some properties of the harmonic numbers. We apply a method of generating functions and a composition operation for getting new identities that are related to the harmonic, central binomial, triangular, Catalan and Mersenne numbers. Also we construct a new generating function for the sum of products of the harmonic numbers and inverse binomial coefficients. As an application we consider special cases for $x < 1$ in generating functions and give examples of calculating infinite sums and obtaining identities. Further studies related to getting relations of obtained results with different special numbers or polynomials will also have novelty and importance.

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