Montes Taurus J. Pure Appl. Math. 6 (3), 375–386, 2024

Montes Taurus Journal of Pure and Applied Mathematics
https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

New Cusa-Huygens inequalities and approximations

Branko Malešević [a], Miloš Mićović [b], Dimitrije Jovanović [c]

[a] University of Belgrade, School of Electrical Engineering, Bulevar kralja Aleksandra 73, 11000 Belgrade, Serbia
[b] University of Belgrade, School of Electrical Engineering, Bulevar kralja Aleksandra 73, 11000 Belgrade, Serbia
[c] University of Belgrade, School of Electrical Engineering, Bulevar kralja Aleksandra 73, 11000 Belgrade, Serbia

Abstract

The subject of this paper is the Cusa-Huygens inequalities. We analyse the Cusa-Huygens inequality on $(-\infty, +\infty)$ and some one-parameter Cusa-Huygens-type inequalities for all real values of the parameter on the interval $(0, \pi)$.

Keywords: Cusa-Huygens inequality, MTP inequalities, stratified families of functions, a parametric method

2020 MSC: 41A44, 26D05

1. Introduction

We start with the trigonometric inequality of Cusa (Nicolaus de Cusa, 1401-1464):

$$\frac{3\sin x}{2+\cos x} < x, \tag{1.1}$$

which is valid for $x \in \left(0, \frac{\pi}{2}\right)$ (*cf.* [1, 29]). The first complete proof of this inequality was given by Christiaan Huygens (1629 -1695), as stated in [38]. Cusa's inequality is used for the purpose of the approximation:

$$\frac{3\sin x}{2+\cos x} \approx x, \tag{1.2}$$

for $x \in \left(0, \frac{\pi}{2}\right]$, see [37]. We call the previous approximation Cusa's approximation, which is stated in the book *K. T. Vahlen, Konstruktionen und Approximationen in systematischer Darstellung, Leipzig 1911.* (pp. 188–190), according to [37]. Extensions and generalisations of the Cusa-Huygens inequalities have been considered by many authors [3]-[7], [10]-[13], [16, 18, 30, 31, 34, 35, 39], [41]-[46]. In the papers [41, 42] the accuracy estimates of the Cusa Huygens approximation were considered.

Based on the papers [44] and [27] on the interval $(0, \pi)$, it holds

$$\frac{1}{180}x^5 < x - \frac{3\sin x}{2+\cos x} < \frac{1}{m_1}x^5 = \frac{1}{92.96406\ldots}x^5, \tag{1.3}$$

†*Article ID:* MTJPAM-D-24-00221

Email addresses: branko.malesevic@etf.bg.ac.rs (Branko Malešević), milos.micovic@etf.bg.ac.rs (Miloš Mićović), jd235018p@student.etf.bg.ac.rs (Dimitrije Jovanović)

Received:19 December 2024, Accepted:22 January 2025, Published:17 March 2025

⋆*Corresponding Author:* Branko Malešević

where $m_1 = 1/M_1$. For $M_1 = 0.010756\ldots$, the maximal value of the function $g(t) = \left(t - \dfrac{3\sin t}{2+\cos t}\right)\Big/t^5$ on $(0,\pi)$ is reached. We note that in the paper [19] it was proved that

$$\frac{1}{180}x^5 < x - \frac{3\sin x}{2+\cos x} < \frac{16(\pi-3)}{\pi^5}x^5 \tag{1.4}$$

for $x \in (0, \pi/2)$. The constants $\dfrac{1}{180} = 0.00\bar{5}$ and $\dfrac{16(\pi-3)}{\pi^5} = 0.00740306\ldots$ are the best possible.

We note that in the paper [19], the approximation

$$\frac{3\sin x}{2+\cos x} \approx x - 0.0072274\ldots x^5\,, \tag{1.5}$$

where $x \in (0, \pi/2)$, is derived. The previous approximation is an improvement of (1.2).

The goal of this paper is to prove that there exists a real constant p_0 such that the approximation

$$\frac{3\sin x}{2+\cos x} \approx x - p_0\, x^5 \tag{1.6}$$

holds for $x \in (0, \pi)$.

2. Preliminaries

We start with the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$, $\emptyset \neq \mathbb{P} \subseteq \mathbb{R}$, over the set $\mathbb{S}$, $\emptyset \neq \mathbb{S} \subseteq \mathbb{R}$. In this paper, we will use the parametric method for proving inequalities [22], which is based on the consideration of stratified families of functions. In order to consider inequalities in this paper, we divide preliminaries into three parts that are both definitional, as well as of an operational nature.

Stratified families of functions. According to [9, 19], [21]-[26], [28], the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$, is increasingly stratified over the set $\mathbb{S}$, if

$$(\forall x \in \mathbb{S})\,(\forall p_1, p_2 \in \mathbb{P})\;\; p_1 < p_2 \Longleftrightarrow \varphi_{p_1}(x) < \varphi_{p_2}(x)$$

and, conversely, the family $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is decreasingly stratified over the set $\mathbb{S}$ if

$$(\forall x \in \mathbb{S})\,(\forall p_1, p_2 \in \mathbb{P})\;\; p_1 < p_2 \Longleftrightarrow \varphi_{p_1}(x) > \varphi_{p_2}(x).$$

The parametric method [22] can be applied to those families $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ for which there exists a real function g such that the following equivalence holds:

$$\varphi_p(x) = 0 \iff p = g(x)\,.$$

Based on the previous equivalence, the function g determines those values of the parameter p for which functions from the family $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ have roots over the observed set $\mathbb{S}$. For all other parameter values, functions $\varphi_p(x)$ have a constant sign over the set $\mathbb{S}$. Based on these characteristics, in [22], cases when the function g is decreasing, increasing and when it has exactly one minimum were considered. The analogous statement can also be derived when the function g has exactly one maximum. For all considered cases, inequalities valid on $\mathbb{S}$ for each $p \in \mathbb{P}$ are listed (*cf.* [22, Theorems 5, 6 and 7]).

Minimax approximant of a family of functions. We are particularly interested in cases when $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is a family of continuous functions and when $\mathbb{S}$ is some finite interval in $\mathbb{R}$ with the condition of eventually including the edge points, that is, when $\mathbb{P}$ is some connected set in $\mathbb{R}$. In the following, we will consider the families for which we assume that for each $p \in \mathbb{P}$, there exist values $\sup_{x\in\mathbb{S}} \left|\varphi_p(x)\right| \in \mathbb{R}$. For such families, for the function $\delta^{(p)} = \sup_{x\in\mathbb{S}} \left|\varphi_p(x)\right| : \mathbb{P} \to \mathbb{R}$, we aim to determine a value of the parameter $p = p_0 \in \mathbb{P}$ (if it exists) for which

$$d_0 = \sup_{x\in\mathbb{S}} \left|\varphi_{p_0}(x)\right| = \inf_{p\in\mathbb{P}} \sup_{x\in\mathbb{S}} |\varphi_p(x)|\,. \tag{2.1}$$

Especially, if we consider the family for the segments $\mathbb{S} = [a, b]$ and $\mathbb{P} = [c, d]$, then:

$$d_0 = \min_{p\in[c,d]} \max_{x\in[a,b]} \left|\varphi_p(x)\right| \tag{2.2}$$

and there exists a unique parameter $p_0 \in [c, d]$ for which (2.1) holds. Then, for the parameter p_0, the function $\varphi_{p_0}(x)$ is called the *minimax approximant* on $[a, b]$. Alongside the minimax approximant, we introduce the minimax approximation

$$\varphi_{p_0}(x) \approx 0\,, \tag{2.3}$$

over $[a, b]$, which has the bound of the error d_0 in relation to the norm $||.||_\infty$.

Mixed trigonometric polynomial inequalities. In this paper, while examining the monotonicity of the function g, we will use the method for proving mixed trigonometric polynomial (MTP) inequalities [9, 20].

According to [9, 20], an MTP inequality is an inequality of the form

$$\phi(x) > 0\,, \quad x\in\mathbb{I}\,, \tag{2.4}$$

where $\mathbb{I}$ is an open, semi-open, or closed interval, and

$$\phi(x) = \sum_{i=1}^{n} \alpha_i\, x^{p_i} \cos^{q_i}x\, \sin^{r_i}x\,, \tag{2.5}$$

for $\alpha_i\in\mathbb{R}\setminus\{0\}$, $p_i, q_i, r_i\in\mathbb{N}_0$ and $n\in\mathbb{N}$, is an MTP function.

The method for proving MTP inequalities from [9, 20] is based on determining a positive downward polynomial approximation $P(x)$ (if it exists) for the function $\phi(x)$. Polynomial approximation $P(x)$ is obtained using Taylor expansions for the trigonometric functions that appear in (2.5), according to the method described in [9, 20]. Therefore, proving the inequality (2.4) reduces to proving the polynomial inequality $P(x) > 0$ for $x\in\mathbb{I}$. To effectively prove the inequality $P(x) > 0$ for $x\in\mathbb{I}$, the Sturm's theorem is used (*cf.* [40], [15, Theorem 4.1 and 4.2]).

Let us note that in [22] methods for isolating zeros and extrema of MTP functions are also given based on the previously described method for proving MTP inequalities. It is obvious that we can apply those methods to determine zeros and extrema if the function is the quotient of two MTP functions. We will also use those methods in the next section of the paper.

3. Main results

In the following two subsections, we extend the Cusa-Huygens inequality and the Cusa-Huygens-type inequality from (*cf.* [19]).

3.1. Extension of the Cusa-Huygens inequality

In this part, we analyse the Cusa-Huygens inequality (1.1) on the interval $(-\infty, +\infty)$.

With that aim, we introduce the function

$$f(x) = x - \frac{3\sin x}{2+\cos x}$$

on the interval $(-\infty, +\infty)$. Let us examine the monotonicity of the function $f(x)$.

Since

$$f'(x) = \frac{(\cos x - 1)^2}{(\cos x + 2)^2} \geq 0$$

for each $x\in\mathbb{R}$, it follows that $f(x)$ is monotonically increasing on the interval $(-\infty, +\infty)$. Since $f(0) = 0$, it follows that $f(x) > 0$ for each $x > 0$, and $f(x) < 0$ for each $x < 0$.

Based on the previous analysis, the following theorem is valid.

Theorem 3.1. *It holds:*

(i)
$$\frac{3\sin x}{2+\cos x} > x\,, \quad \text{for each} \quad x\in(-\infty,0)\ .$$

(ii)
$$\frac{3\sin x}{2+\cos x} < x\,, \quad \text{for each} \quad x\in(0,+\infty)\ .$$

Remark 3.2. The error of the approximation of the function $\dfrac{3\sin x}{2+\cos x}$ from the function x on the interval $[-t,t]$, $t>0$, is equal to $f(t)$. In particular, for $t=\pi/2$, the approximation error is $f(\pi/2)=\dfrac{\pi-3}{2}$, while for $t=\pi$, the approximation error is $f(\pi)=\pi$.

3.2. Extension of the Cusa-Huygens-type inequalities

In [19], the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$, where

$$\varphi_p(x)=\begin{cases} x-\dfrac{3\sin x}{2+\cos x}-p\,x^5 & , \quad x\in(0,\pi/2] \\ 0 & , \quad x=0 \end{cases}$$

was considered for $\mathbb{P}=\mathbb{R}^+$. In this paper, we will analyse the same family of functions, but on the interval $[0,\pi]$ and for $\mathbb{P}=\mathbb{R}$. The following assertions hold:

Lemma 3.3. *The family of functions* $\{\varphi_p(x)\}_{p\in\mathbb{P}}$, $\mathbb{P}=\mathbb{R}$ *is decreasingly stratified on the interval* $(0,\pi]$.

Proof. It holds that $\dfrac{d\,\varphi_p(x)}{d\,p}=-x^5<0$ for each $x\in(0,\pi]$. □

We now proceed with the analysis of the considered family. First, the equivalence

$$\varphi_p(x)=x-\frac{3\sin x}{2+\cos x}-p\,x^5=0 \iff p=g(x)=\frac{x\cos x-3\,\sin x+2\,x}{x^5\,(2+\cos x)}\,,$$

for $x\in(0,\pi]$, determines the function g over $(0,\pi]$. Then

$$\lim_{x\to0+} g(x)=A:=\frac{1}{180}=0.000\bar{5} \quad \text{and} \quad g(\pi)=B:=\frac{1}{\pi^4}=0.010265\ldots\ .$$

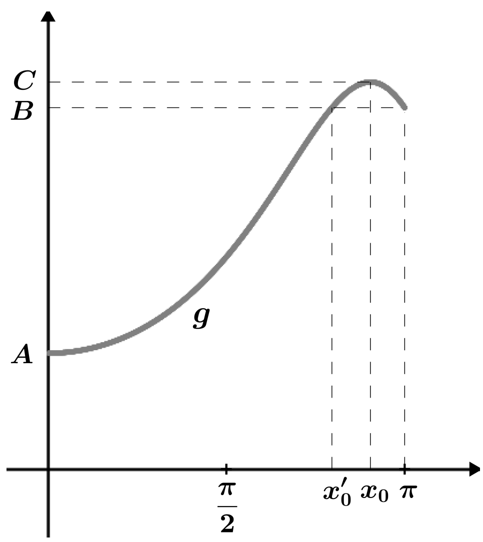

Figure 1. Graph of the function g

The function $g(x)$, as the quotient of two MTP functions, has the unique maximum

$$C = g(x_0) = 0.010756\ldots$$

at the point $x_0 = 2.83982\ldots \in (0,\pi)$. For details, see the Appendix based on [22].

Let us emphasise that all conclusions drawn in the next part of the analysis are based on [22]. *For each such conclusion, formal proof can be provided in the appropriate appendix.*

Let us notice that for the values of the parameter $p \in \mathbb{R}\backslash[A,C]$, the functions $\varphi_p(x)$ have a constant sign, and therefore, it is of particular interest to further consider the values of the parameter $p \in [A,C]$. The function $g(x)$ is monotonically increasing on $[0,x_0]$ and monotonically decreasing on $[x_0,\pi]$. Thus, for the parameter $p = g(\pi) = B$, there exists exactly one value $x'_0 = 2.50057\ldots \in (0,x_0)$ such that $g(x'_0) = g(\pi) = B$. For each $p \in g\big((0,x'_0)\big)$, there exists exactly one value $x^{(p)} \in (0,x'_0)$ such that $p = g(x^{(p)})$ and $\varphi_p(x^{(p)}) = 0$. For each $p \in g\big([x'_0,\pi]\big)\backslash\{C\}$, there exist exactly two values $x_1^{(p)}, x_2^{(p)} \in [x'_0,\pi]$, $0 < x_1^{(p)} < x'_0 < x_2^{(p)} < \pi$, such that $p = g(x_1^{(p)}) = g(x_2^{(p)})$ and $\varphi_p(x_1^{(p)}) = \varphi_p(x_2^{(p)}) = 0$. For $p = C$, there exists the unique value $x = x_1^{(C)}$ such that $\varphi_p(x_1^{(C)}) = 0$.

The first derivative of the function $\varphi_p(x)$ exists on $(0,\pi)$. The equivalence:

$$\frac{d\varphi_p(x)}{dx} = -\frac{\sin^2 x}{(2+\cos x)^2} + 5\,px^4 = 0 \iff p = g_1(x) = \frac{1}{5}\frac{(1-\cos x)^2}{x^4\,(2+\cos x)^2}$$

determines the function g_1 over $(0,\pi]$. It holds that

$$\lim_{x\to 0+} g_1(x) = A \quad \text{and} \quad g_1(\pi) = B_1 := \frac{4}{5\pi^4} = 0.0082127\ldots.$$

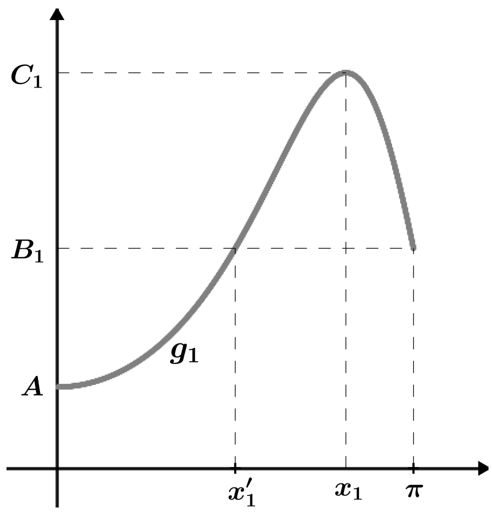

Figure 2. Graph of the function g_1

The function $g_1(x)$, as the quotient of two MTP functions, has the unique maximum

$$C_1 = g_1(x_1) = 0.011573\ldots$$

at the point $x_1 = 2.54595\ldots$.

The function $g_1(x)$ is monotonically increasing on $(0,x_1)$ and monotonically decreasing on (x_1,π). Thus, for the parameter $p = g_1(\pi) = B_1$, there exists exactly one value $x'_1 = \pi/2 \in (0,x_1)$ such that $g_1(x'_1) = g_1(\pi) = B_1$. For each $p \in g_1\big((0,x'_1)\big)$, there exists exactly one value $t^{(p)} \in (0,x'_1)$ such that $p = g_1(t^{(p)})$ and $d/d\,x(\varphi_p(x))\,|_{x=t^{(p)}} = 0$. For each $p \in g_1\big([x'_1,\pi]\big)\backslash\{C_1\}$, there exist exactly two values $t_1^{(p)}, t_2^{(p)} \in [x'_1,\pi]$, $0 < t_1^{(p)} < x'_1 < t_2^{(p)} < \pi$, such that

$p = g_1(t_1^{(p)}) = g_1(t_2^{(p)})$ and $d/d\,x(\varphi_p(x))\,|_{x=t_1^{(p)}} = d/d\,x(\varphi_p(x))\,|_{x=t_2^{(p)}} = 0$. For $p = C_1$, there exists the unique value $x = t^{(C_1)}$ such that $d/d\,x(\varphi_p(x))\,|_{x=t^{(C_1)}} = 0$.

Let us consider the function

$$h(x) = \varphi_{g_1(x)}(x) = \frac{1}{5}\frac{4\,x\cos^2 x + 22\,x\cos x - 15\cos x\sin x - 30\sin x + 19\,x}{(2+\cos^2 x)^2}$$

over $(0, \pi]$ in the following. The function $h(x)$ has the unique minimum

$$C_2 = h(x_1) = -0.12789\ldots,$$

at the point $x_1 = 2.54595\ldots$ and the unique root $x_0 = 2.83982\ldots$ (where the function $g(x)$ has the local maximum).

For the values of the parameter $p \in [A, C_2]$, the functions $\varphi_p(x)$ have exactly one minimum at the points $x = t_1^{(p)} \in (0, x_2)$ and exactly one maximum at the points $x = t_2^{(p)} \in (x_3, \pi)$. Let us consider the equation $h(x) = \varphi_C(x)$ which, on $(0, \pi)$, has the solution $x_2 = 2.18506\ldots$. Thus, for the values of the parameter $p \in [A, C]$, the functions $\varphi_p(x)$ have uniquely determined minima at the points $x = t_1^{(p)} \in (0, x_2)$ and uniquely determined maxima at the points $x = t_2^{(p)} \in (x_0, \pi)$.

The function $h_1(x) = h(x) : [0, x_2] \to [C, 0]$ is a continuous monotonically decreasing bijection whose values are the values of minima of $\varphi_p(x)$. Similarly, the function $h_2(x) = h(x) : [x_0, \pi] \to [0, \pi/5]$ is a continuous monotonically increasing bijection whose values are the values of maxima of $\varphi_p(x)$. Therefore, there exist exactly two values $\bar{t}_1^{(p)} \in (0, x_2)$ and $\bar{t}_2^{(p)} \in (0, \pi/5)$ such that

$$-h(\bar{t}_1^{(p)}) = h(\bar{t}_2^{(p)})\,.$$

Using the Maple function

```
fsolve({diff(f(u,p),u)=0, diff(f(v,p),v)=0, -f(u,p)=f(v,p)}, {u=0..2.5,v=2.5..Pi,p=A..C});
```

the numerical values

$$p_0 = \mathrm{p} := 0.010451\ldots$$
$$\bar{t}_1^{(p)} = \mathrm{u} := 2.10829\ldots$$
$$\bar{t}_2^{(p)} = \mathrm{v} := 2.88858\ldots$$

are obtained, as well as the value

$$\delta = 0.058888\ldots$$

for which

$$\varphi_{p_0}(\bar{t}_1^{(p)}) = -\delta \quad \text{and} \quad \varphi_{p_0}(\bar{t}_2^{(p)}) = \delta\,.$$

Let us notice that for other values of the parameter p, it holds that $|\varphi_p(t^{(p)})| > \delta$ for $x \in (0, \pi)$, since due to the stratification, a greater value is reached either through the minimum or the maximum.

It holds that

$$\max_{x\in[0,\pi]} |\varphi_p(x)| = \max_{x\in[0,\pi]} \{|h(x)|, \varphi_{g_1(x)}(\pi)|\}.$$

Regarding $p = g_1(x) \in [A, C]$, there are two possible cases:

1. $\varphi_p(\pi) \notin [-\delta, \delta]$, then

$$\max_{x\in[0,\pi]} |\varphi_p(x)| = \max_{x\in[0,\pi]} \{|h(x)|, |\varphi_{g_1(x)}(\pi)|\} \geq |\varphi_p(\pi)| > \delta\,.$$

2. $\varphi_p(\pi) \in [-\delta, \delta]$, then, we distinguish the following subcases:

2.1 If $p = p_0$, then

$$\varphi_{p_0}(\bar{t}_1^{(p_0)}) = h(\bar{t}_1^{(p_0)}) = -0.058888\ldots,$$
$$\varphi_{p_0}(\bar{t}_2^{(p_0)}) = h(\bar{t}_2^{(p_0)}) = +0.058888\ldots,$$
$$\varphi_{p_0}(\pi) = -0.056851\ldots.$$

Thus,

$$\max_{x\in[0,\pi]} |\varphi_p(x)| = \max_{x\in[0,\pi]} \{|h(x)|, |\varphi_{g_1(x)}(\pi)|\} = \delta\,.$$

2.2 If $p \neq p_0$ and $\varphi_p(\pi) \in [-\delta, \delta]$, then $\max\limits_{x\in[0,\pi]} \{|h(x)|\} > \delta$ (due to the stratification, a greater value is reached either through the minimum or the maximum); thus,

$$\max_{x\in[0,\pi]} |\varphi_p(x)| = \max_{x\in[0,\pi]} \{|h(x)|, \varphi_{g_1(x)}(\pi)|\} > \delta\,.$$

Finally,

$$d_0 = \min_{p\in\mathbb{R}} \max_{x\in[0,\pi]} \left|\varphi_p(x)\right| = \min_{p\in[A,C]} \max_{x\in[0,\pi]} \left|\varphi_p(x)\right| = \delta\,.$$

Based on the entire previous analysis, the following theorem holds.

Theorem 3.4. *Let*:

$$A = \frac{1}{180} = 0.00\overline{5}\,, \quad B = \frac{1}{\pi^4} = 0.010265\ldots\,,$$

$$C = 0.010756\ldots \quad \text{and} \quad B_1 = \frac{4}{5\pi^4} = 0.0082127\ldots\,.$$

Then, it holds:

(i) *If $p \in (-\infty, A]$, then*

$$x \in (0,\pi) \implies x - \frac{3\sin x}{2+\cos x} > A\,x^5 \geq p\,x^5\,.$$

(ii) *If $p \in (A, B]$, then the equality*

$$\varphi_p(x) = x - \frac{3\sin x}{2+\cos x} - p\,x^5$$

has a unique solution $x_0^{(p)}$ on $(0,\pi)$, and it holds that

$$x \in (0, x_0^{(p)}) \implies x - \frac{3\sin x}{2+\cos x} < p\,x^5$$

and

$$x \in (x_0^{(p)}, \pi) \implies x - \frac{3\sin x}{2+\cos x} > p\,x^5\,.$$

Each function $\varphi_p(x)$ has exactly one minimum $t_1^{(p)} \in \left(0, x_0^{(p)}\right)$ for $p \in (A, B_1]$ on the interval $(0,\pi)$. For $p \in (B_1, B)$, each function $\varphi_p(x)$ has exactly one minimum $t_1^{(p)} \in \left(0, x_0^{(p)}\right)$ and exactly one maximum $t_2^{(p)} \in \left(x_0^{(p)}, \pi\right)$ on the interval $(0,\pi)$.

(iii) *If $p \in (B, C)$, then the equality*

$$\varphi_p(x) = x - \frac{3\sin x}{2+\cos x} - p\,x^5$$

has exactly two solutions $x_0^{(p)}$ and $x_1^{(p)}$ $\left(x_0^{(p)} < x_1^{(p)}\right)$ on $(0,\pi)$, and it holds that

$$x \in (0, x_0^{(p)}) \cup x \in (x_1^{(p)}, \pi) \implies x - \frac{3\sin x}{2+\cos x} < p\,x^5$$

and

$$x \in (x_0^{(p)}, x_1^{(p)}) \implies x - \frac{3\sin x}{2+\cos x} > p\,x^5\,.$$

Each function $\varphi_p(x)$ has exactly one minimum $t_1^{(p)} \in \left(0, x_0^{(p)}\right)$ and exactly one maximum $t_2^{(p)} \in \left(x_0^{(p)}, \pi\right)$ for $p \in (B, C)$ on the interval $(0,\pi)$.

(iv) *If $p \in [C, \infty)$, then*

$$x \in (0,\pi) \implies x - \frac{3\sin x}{2+\cos x} < C\,x^5 \leq p\,x^5\,.$$

(v) *There exists exactly one solution to the equation*

$$\varphi_p\left(t_1^{(p)}\right) = \left|\varphi_p\left(t_2^{(p)}\right)\right|$$

with respect to the parameter $p \in (B, C)$*, which is numerically determined as*

$$p_0 = 0.010451\ldots.$$

For the value

$$d_0 = \varphi_p\left(t_1^{(p)}\right) = 0.058888\ldots$$

it holds that

$$d_0 = \min_{p\in\mathbb{R}} \max_{x\in[0,\pi]} \left|\varphi_p(x)\right|.$$

(vi) *For the value* p_0*, the minimax approximant of the family of functions on the interval* $(0, \pi)$ *is*

$$\varphi_{p_0}(x) = x - \frac{3\sin x}{2+\cos x} - p_0\, x^5,$$

which determines the corresponding minimax approximation

$$x - \frac{3\sin x}{2+\cos x} \approx 0.010451\ldots\, x^5$$

on the interval $(0, \pi)$ *with the error* d_0.

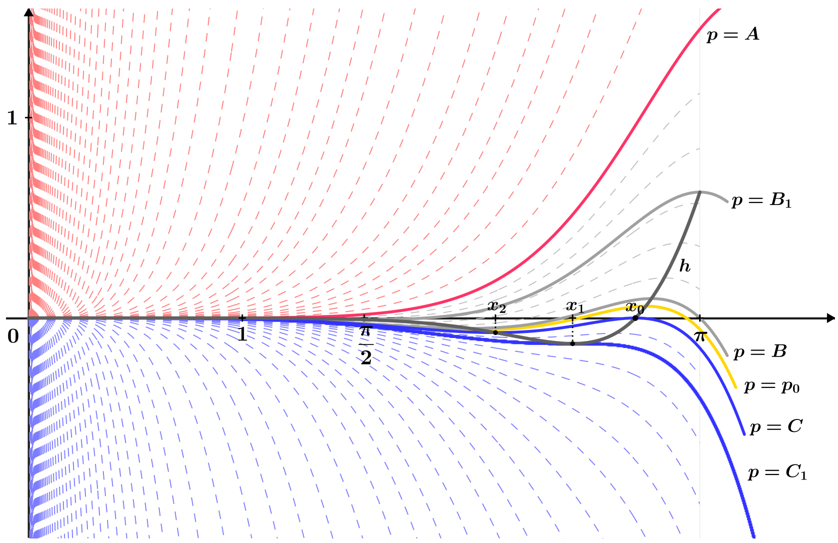

Figure 3. The stratified family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ and the function h

As a corollary of the previous analysis, by considering the functions g and g_1 on the interval $(0, \pi/2)$ in the same manner, it is easy to show that the following theorem holds, which extends the result from [19].

Theorem 3.5. *Let:*

$$A = \frac{1}{180} = 0.00\overline{5} \quad \text{and} \quad B = \frac{16\,(\pi-3)}{\pi^5} = 0.0074030\ldots.$$

Then, it holds:

(i) *If $p \in (-\infty, A]$, then*

$$x \in \left(0, \frac{\pi}{2}\right) \implies x - \frac{3\sin x}{2+\cos x} > A\,x^5 \geq p\,x^5 \,.$$

(ii) *If $p \in (A, B)$, then the equality*

$$\varphi_p(x) = x - \frac{3\sin x}{2+\cos x} - p\,x^5$$

has a unique solution $x_0^{(p)}$ on $(0, \pi/2)$, and it holds that

$$x \in \left(0, x_0^{(p)}\right) \implies x - \frac{3\sin x}{2+\cos x} < p\,x^5$$

and

$$x \in \left(x_0^{(p)}, \frac{\pi}{2}\right) \implies x - \frac{3\sin x}{2+\cos x} > p\,x^5 \,.$$

Each function $\varphi_p(x)$ has exactly one minimum $t_0^{(p)} \in \left(0, x_0^{(p)}\right)$ for $p \in (A, B)$.

(iii) *If $p \in [B, \infty)$, then*

$$x \in \left(0, \frac{\pi}{2}\right) \implies x - \frac{3\sin x}{2+\cos x} < B\,x^5 \leq p\,x^5 \,.$$

(iv) *There exists exactly one solution to the equation*

$$\left|\varphi_p\left(t_0^{(p)}\right)\right| = \varphi_p\left(\frac{\pi}{2}-\right)$$

with respect to the parameter $p \in (A, B)$, which is numerically determined as

$$p_0 = 0.0072274\ldots\,.$$

For the value

$$d_0 = \varphi_{p_0}\left(\frac{\pi}{2}-\right) = 0.0016797\ldots$$

it holds that

$$d_0 = \min_{p \in [0,+\infty)} \max_{x \in [0,\pi/2]} \left|\varphi_p(x)\right|.$$

(v) *For the value $p_0 = 0.0072274\ldots$, the minimax approximant of the family of functions on the interval $(0, \pi/2)$ is*

$$\varphi_{p_0}(x) = x - \frac{3\sin x}{2+\cos x} - p_0\,x^5 \,,$$

which determines the corresponding minimax approximation

$$x - \frac{3\sin x}{2+\cos x} \approx 0.0072274\ldots\, x^5$$

on the interval $(0, \pi/2)$ with the error d_0.

4. Conclusion

In this paper, the Cusa-Huygens inequalities are analysed. The extension of the Cusa-Huygens inequality to the interval $(-\infty, +\infty)$ was obtained. Additionally, the one-parameter Cusa-Huygens-type inequalities [19] were extended to all real values of the parameter and from the interval $(0, \pi/2)$ to the interval $(0, \pi)$. As a result, new Cusa-Huygens inequalities and approximations were obtained.

In a similar manner, many inequalities from papers and monographs [1, 2, 8, 14, 17, 29, 32, 33, 36] could be improved.

Acknowledgments

This paper is dedicated to Professor Yilmaz Simsek on the occasion of his 60th anniversary.

This research was conducted at the Center for Applied Mathematics, the Palace of Science; therefore, the authors wish to express their gratitude to the Miodrag Kostić Endowment, Belgrade.

Author Contributions: The authors contributed equally to this paper.

Conflict of Interest: The authors declare no conflicts of interest.

Funding (Financial Disclosure): This work was financially supported by the Ministry of Science, Technological Development and Innovation of the Republic of Serbia under contract numbers: 451-03-65/2024-03/200103 (for the first author) and 451-03-66/2024-03/200103 (for the second author).

References

[1] G. D. Anderson, M. Vuorinen, X. Zhang, *Topics in special functions III*, In: Analytic number theory, approximation theory and special functions (Ed. by G. V. Milovanović and M. Th. Rassias), Springer, New York, 297–345, 2014.

[2] G. A. Anastassiou, *Intelligent comparisons: Analytic inequalities*, Springer, Cham, Switzerland, 2016.

[3] Y. J. Bagul, B. Banjac, C. Chesneau, M. Kostić and B. Malešević, *New refinements of Cusa-Huygens inequality*, Results Math. **76 (107)**, 1–16, 2021.

[4] Y. J. Bagul and C. Chesneau, *Refined forms of Oppenheim and Cusa-Huygens type inequalities*, Acta Comment. Univ. Tartu. Math. **24 (2)**, 183–194, 2020.

[5] Y. J. Bagul and C. Chesneau, *Sharp extensions of a Cusa-Huygens type inequality*, Armen. J. Math. **16 (14)**, 1–12, 2024.

[6] Y. J. Bagul, C. Chesneau and M. Kostić, *The Cusa-Huygens inequality revisited*, Novi Sad J. Math. **50 (2)**, 149–159, 2020.

[7] Y. J. Bagul, C. Chesneau and M. Kostić, *On the Cusa-Huygens inequality*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **115 (29)**, 1–12, 2021.

[8] B. Banjac, *System for automatic proving of some classes of analytic inequalities*, Ph. D. Thesis, School of Electrical Engineering, University of Belgrade, Belgrade, Serbia, 2019. (In Serbian)

[9] B. Banjac, B. Malešević, M. Mićović, B. Mihailović and M. Savatović, *The best possible constants approach for Wilker-Cusa-Huygens inequalities via stratification*, Appl. Anal. Discrete Math. **18 (1)**, 244–288, 2024.

[10] G. Bercu, *Fourier series method related to Wilker-Cusa-Huygens inequalities*, Math. Inequal. Appl. **22 (4)**, 1091–1098, 2019.

[11] C.-P. Chen and C. Mortici, *The relationship between Huygens' and Wilker's inequalities and further remarks*, Appl. Anal. Discrete Math. **17 (1)**, 92–100, 2023.

[12] A. R. Chouikha, *Other approaches related to Huygens trigonometric inequalities*, 1–15, 2022; HAL (Preprint) ID: hal03769843.

[13] A. R. Chouikha, C. Chesneau and Y. J. Bagul, *Some refinements of well-known inequalities involving trigonometric functions*, J. Ramanujan Math. Soc. **36 (3)**, 193–202, 2021.

[14] M. J. Cloud, B. C. Drachman and L. P. Lebedev, *Inequalities with applications to engineering*, Springer, Cham, Switzerland, 2014.

[15] N. Cutland, *Computability: An introduction to recursive function theory*, Cambridge University Press, 1980.

[16] R. M. Dhaigude, C. Chesneau and Y. J. Bagul, *About trigonometric-polynomial bounds of sinc function*, Math. Sci. Appl. E-Notes **8 (1)**, 100–104, 2020.

[17] N. D. Kazarinoff, *Analytic inequalities*, Holt, Rinehart and Winston: New York, NY, USA, 1961.

[18] N. Lord, 106.19 *Some observations on inequalities related to Huygens' inequality*, Math. Gaz. **106 (566)**, 316–318, 2022.

[19] B. Malešević and D. Jovanović, *Frame's types of inequalities and stratification*, Cubo **26 (1)**, 1–19, 2024.

[20] B. Malešević and M. Makragić, *A method for proving some inequalities on mixed trigonometric polynomial functions*, J. Math. Inequal. **10 (3)**, 849–876, 2016.

[21] B. Malešević and M. Mićović, *Exponential polynomials and stratification in the theory of analytic inequalities*, J. Sci. Arts. **23 (3)**, 659–670, 2023.

[22] B. Malešević, M. Mićović and B. Mihailović, *A parametric method for proving some analytic inequalities*, Axioms **13 (8)**, 1–23, 2024; Article ID: 520.

[23] B. Malešević, M. Mićović, B. Mihailović and T. Lutovac, *A new approach to selecting constants for some analytic inequalities*, 1–23, 2024; HAL (Preprint) ID: hal-04676658.

[24] B. Malešević, M. Mićović, B. Mihailović, T. Lutovac and V. Šešum Čavić, *The best constants in some new sharp bounds for sinc function*, ResearchGate (Preprint), 1–16, 2024.

[25] B. Malešević and B. Mihailović, *A minimax approximant in the theory of analytic inequalities*, Appl. Anal. Discrete Math. **15 (2)**, 486–509, 2021.

[26] B. Malešević, B. Mihailović, M. Nenezić Jović, M. Mićović and L. Milinković, *Some generalisations and minimax approximants of D'Aurizio trigonometric inequalities*, 1–11, 2024; HAL (Preprint) ID: hal-03550277v2.

[27] B. Malešević, M. Nenezić, L. Zhu, B. Banjac and M. Petrović, *Some new estimates of precision of Cusa-Huygens and Huygens approximations*, Appl. Anal. Discrete Math. **15 (1)**, 243–259, 2021.

[28] M. Mićović and B. Malešević, *Jordan-type Inequalities and Stratification*, Axioms **13 (4)**, 262, 1–25, 2024.

[29] D. S. Mitrinović, *Analytic inequalities*, Springer-Verlag, New York, 1970.

[30] C. Mortici, *The natural approach of Wilker-Cusa-Huygens inequalities*, Math. Inequal. Appl. **14 (3)**, 535–541, 2011.
[31] R.B. Nelsen, *Elementary proofs of the trigonometric inequalities of Huygens, Cusa, and Wilker*, Math. Mag. **93 (4)**, 276–283, 2020.
[32] M. Nenezić Jović, *Stratified families of functions in the theory of analytical inequalities with applications*, Ph. D. Thesis, School of Electrical Engineering, University of Belgrade, Belgrade, Serbia, 2023. (In Serbian)
[33] B. G. Pachpatte, *Analytic inequalities: Recent advances*, Atlantis Press, Paris, France, 2012.
[34] I. Pinelis, *Huygens' final unproved inequality* (Mathoverflow), Avaliable at https://mathoverflow.net/questions/475597/huygens-final-unproved-inequality (Accession date: 24 Dec. 2024).
[35] I. Pinelis, *On a trigonometric inequality by Huygens* (Mathoverflow), Avaliable at https://mathoverflow.net/questions/475372/huygens-trigonometric-inequality (Accession date: 24 Dec. 2024).
[36] Y. Qin, *Analytic inequalities and their applications in PDEs*, Birkhäuser, Cham, Switzerland, 2017.
[37] *Queries - Replies*, Mathematics of computation **3**, 561–563, 1949.
[38] J. Sándor and M. Bencze, *On Huygen's trigonometric inequality*, RGMIA Res. Rep. Collection **8 (3)**, 1–4, 2005.
[39] J. Sándor and R. Oláh-Gál, *On Cusa-Huygens type trigonometric and hyperbolic inequalities*, Acta Univ. Sapientiae Math. **4 (2)**, 145–153, 2012.
[40] J.C.F. Sturm, *Mémoire sur la résolution des équations numériques*, Bulletin des Sciences de Ferussac **11**, 419–425, 1829.
[41] M. Szyszkowicz, *Approximations of pi and squaring the circle*, J. Multidiscip. Eng. Sci. Technol. **2 (1)**, 330–332, 2015.
[42] M. B. Villarino, *Huygens and π*, 2024; ArXiv:2412.10880.
[43] Y. Wu and G. Bercu, *New refinements of Becker-Stark and Cusa-Huygens inequalities via trigonometric polynomials method*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **115 (87)**, 1–12, 2021.
[44] L. Zhu, *On Frame's inequalities*, J. Inequal. Appl. **2018**, 2018; Article ID: 94.
[45] L. Zhu, *An unity of Mitrinovic-Adamovic and Cusa-Huygens inequalities and the analogue for hyperbolic functions*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **113**, 3399–3412, 2019.
[46] L. Zhu, *New inequalities of Cusa-Huygens type*, Mathematics **9 (17)**, 2021; Article ID: 2101.

How to cite this article: B. Malešević, M. Mićović and D. Jovanović, *New Cusa-Huygens inequalities and approximations*, Montes Taurus J. Pure Appl. Math. **6 (3)**, 375–386, 2024; Article ID: MTJPAM-D-24-00221.

Appendix

For the function

$$g(x) = \frac{x\cos x - 3\,\sin x + 2\,x}{x^5\,(2+\cos x)}\,,$$

there exists a derivative function

$$g'(x) = \frac{-4\,x\cos^2 x + 15\,\sin x\cos x - 22\,x\cos x + 30\,\sin x - 19\,x}{x^6\,(2+\cos x)^2}$$

over $(0,\pi)$. The numerator of the derivative function is an MTP function

$$f(x) = -4\,x\cos^2 x + 15\,\sin x\cos x - 22x\cos x + 30\,\sin x - 19\,x$$

over $(0,\pi)$. We prove that the considered MTP function has the unique root $x_0 = 2.83982\ldots \in (0,\pi)$. We use Theorem for isolating zeros of an MTP function [22] for $a_0 = 2.7$ and $b = 3.0$.

It holds that

$$f(x) = -22\,x\cos x - 2\,x\cos 2x - 21\,x + 30\sin x + \frac{15}{2}\sin 2x\,.$$

1. We prove that

$$f(x) > 0$$

on the interval $(0, 2.7]$. If we approximate the cosine functions with the Maclaurin polynomials of degree 15, and the sine functions with the Maclaurin polynomial of degree 16, then the function $f(x)$ has the downward polynomial approximation

$$\begin{aligned} P(x) &= -22\,x\,T_{16}^{\cos,0}(x) - 2\,x\,T_{16}^{\cos,0}(2\,x) - 21\,x + 30\,T_{15}^{\sin,0}(x) + \tfrac{15}{2}T_{15}^{\sin,0}(2\,x) \\ &= -\tfrac{7283}{1162377216000}x^{17} + \tfrac{1367}{7264857600}x^{15} - \tfrac{59}{8108100}x^{13} + \tfrac{41}{221760}x^{11} - \tfrac{13}{5040}x^9 + \tfrac{1}{84}x^7 \\ &= x^7\left(-\tfrac{7283}{1162377216000}x^{10} + \tfrac{1367}{7264857600}x^8 - \tfrac{59}{8108100}x^6 + \tfrac{41}{221760}x^4 - \tfrac{1}{5040}3x^2 + \tfrac{1}{84}\right) \end{aligned}$$

on the interval $(0, 2.7]$. By applying the Sturm's theorem to the polynomial

$$-\frac{7283}{1162377216000}x^{10}+\frac{1367}{7264857600}x^{8}-\frac{59}{8108100}x^{6}+\frac{41}{221760}x^{4}-\frac{1}{5040}3x^{2}+\frac{1}{84}$$

on the segment $[0, 2.7]$, it can be concluded that this polynomial has no zeros on the segment $[0, 2.7]$. Hence, the polynomial $P(x)$ has no zeros on the interval $(0, 2.7]$. Considering that $P\left(\frac{27}{10}\right) = 0.53349\ldots > 0$, it holds that $P(x) > 0$ on the interval $(0, 2.7]$. Thus, on the interval $(0, 2.7]$, it holds that

$$f(x) > 0\,.$$

2. We prove that

$$f(x) < 0$$

on the interval $[3, \pi)$. If we approximate the sine functions with the Maclaurin polynomials of degree 17, and the cosine functions with the Maclaurin polynomial of degree 14, then the function $f(x)$ has the upward polynomial approximation

$$\begin{aligned}P(x) &= -22\,x\,T_{14}^{\cos,0}(x) - 2\,x\,T_{14}^{\cos,0}(2\,x) - 21\,x + 30\,T_{17}^{\sin,0}(x) + \tfrac{15}{2}T_{17}^{\sin,0}(2\,x)\\ &= \tfrac{331}{119760076800}x^{17} + \tfrac{1367}{7264857600}x^{15} - \tfrac{59}{8108100}x^{13} + \tfrac{41}{221760}x^{11} - \tfrac{13}{5040}x^{9} + \tfrac{1}{84}x^{7}\end{aligned}$$

on the interval $[3, \pi)$. By applying the Sturm's theorem to the polynomial $P(x)$ on the segment $[3, \pi]$, it can be concluded that this polynomial has no zeros on the segment $[3, \pi]$. Considering that $P(\pi) = -1.49377\ldots < 0$, it holds that $P(x) < 0$ on the interval $[3, \pi)$. Thus, on the interval $[3, \pi)$, it holds that

$$f(x) < 0\,.$$

3. We prove that

$$f'(x) = 26\,\cos^2 x + 8\,x \sin x \cos x + 8 \cos x + 22\,x \sin x - 34 < 0$$

on the interval $[2.7, 3]$. It holds that

$$f'(x) = 22\,x \sin x + 4\,x \sin 2x + 8 \cos x + 13 \cos 2x - 21\,.$$

If we approximate the sine functions with the Maclaurin polynomials of degree 13, and the cosine function with the Maclaurin polynomial of degree 16, then the function $f'(x)$ has the upward polynomial approximation

$$\begin{aligned}P(x) &= 22\,x\,T_{13}^{\sin,0}(x) + 4\,x\,T_{13}^{\sin,0}(2x) + 8\,T_{16}^{\cos,0}(x) + 13\,T_{16}^{\cos,0}(2\,x) - 21\\ &= \tfrac{11833}{290594304000}x^{16} + \tfrac{1367}{484323840}x^{14} - \tfrac{59}{623700}x^{12} + \tfrac{41}{20160}x^{10} - \tfrac{13}{560}x^{8} + \tfrac{1}{12}x^{6}\end{aligned}$$

on the interval $[2.7, 3]$. By applying the Sturm's theorem to the polynomial $P(x)$ on the segment $[2.7, 3]$, it can be concluded that this polynomial has no zeros on the segment $[2.7, 3]$. Considering that $P(3) = -6.48904\ldots < 0$, it holds that $P(x) < 0$ on the interval $[2.7, 3]$. Thus, on the interval $[2.7, 3]$, it holds that

$$f'(x) < 0\,.$$

Hence, based on Theorem for isolating zeros of an MTP function [22], the function $f(x)$ has exactly one zero x_0 on the interval $(0, \pi)$.

Let us notice that for $x \in (0, x_0)$, it holds that $f(x) > 0$, and that for $x \in (x_0, \pi)$, it holds that $f(x) < 0$. Therefore, for $x \in (0, x_0)$, the function $g(x)$ is increasing, and for $x \in (x_0, \pi)$, the function $g(x)$ is decreasing; thus, the function $g(x)$ has the unique maximum

$$C = g(x_0) = 0.010756\ldots$$

at the point $x_0 = 2.83982\ldots$.