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Hardy spaces of monogenic and discrete monogenic functions revisited on octonionic half-space

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Abstract

In this short paper we revisit the definitions of Hardy spaces of monogenic and discrete monogenic functions on octonionic right half-space. Due to the lack of associativity generalizations of Hardy spaces are considered in the framework of the weaker notion of para-linear functionals. The definition of appropriate Hardy spaces often requires the implementation of an additional octonionic intrinsic weight factor of unit length to ensure the property of having a reproducing kernel. The latter compensates the non-associativity. In the particular setting of considering the octonionic half-space consisting of all octonions having a positive real part however these weight factors disappear canonically.

First we re-interprete the previously introduced Cauchy transform within the recently established concept of $\mathbb{O}$-para-linear operators. We revisit the dual Cauchy transform within the setting of generalized adjoint operators of $\mathbb{O}$-para-linear inner products. This leads to a meaningful definition of a generalized octonionic Kerzman-Stein operator within the theory of $\mathbb{O}$-para-linear function spaces. Alike the complex and Clifford analytic case, also this octonionic version of Kerzman Stein operator produces in the special half-space case to the classical eight-dimensional Hilbert-Riesz transform. After having explained these peculiarities for the continuous case we discuss a Weyl calculus-based approach to discretize the octonionic Cauchy-Riemann operator. We round off by re-discussing Hardy spaces of discrete monogenic functions on octonionic half-space together with appropriate peculiar definitions of a discretized version of the octonionic Cauchy transform.

Keywords: Octonionic monogenic functions, Cauchy transform, Kerzman-Stein kernel, Hilbert-Riesz transform, para-linearity

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1. Preliminaries

The number set of octonions $\mathbb{O}$ are hypercomplex numbers and form an eight-dimensional real vector space over $\mathbb{R}$. They arise by applying three-times the classical Cayley-Dickson doubling process to the reals.

Using the particular notation of [1] such an octonion can be expressed by

$$x = x_0 + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 + x_4\mathbf{e}_4 + x_5\mathbf{e}_5 + x_6\mathbf{e}_6 + x_7\mathbf{e}_7,$$

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where the elements e_i work like imaginary units, which means that we have $\mathbf{e}_i^2 = -1$ where $i \in \{1, 2, \ldots, 7\}$. We set $\Re(x) := x_0$. Like usual, we use the notation

$$x = x_0 - x_1\mathbf{e}_1 - x_2\mathbf{e}_2 - x_3\mathbf{e}_3 - x_4\mathbf{e}_4 - x_5\mathbf{e}_5 - x_6\mathbf{e}_6 - x_7\mathbf{e}_7,$$

for the conjugated octonion. The addition and scalar multiplication is inherited by the real vector space structure of $\mathbb{O} \cong \mathbb{R}^8$. Now, additionally to the vector space structure, octonions can also be multiplied with each other. In this paper we use the notation according to the multiplication table that has been given in [1].

Using the multiplication rules given in [1] one can draw the conclusion that the product if any two octonions always gives an octonion again. However, this multiplication is not associative. It is important to observe that $(\mathbf{e}_i\mathbf{e}_j)\mathbf{e}_k = -\mathbf{e}_i(\mathbf{e}_j\mathbf{e}_k)$ if i, j, k are pairwise distinct non-zero indices and provided $\mathbf{e}_i\mathbf{e}_j \neq \pm\mathbf{e}_k$. The other cases generate the associative part of the octonions. Despite of the loss of associativity in general, octonions still have some elegant and really particular calculation rules. Following for instance [1] one has the left alternative property stating that $x(xy) = (xx)y$ $(yx)x = y(xx)$, one has right alternativity and the flexibility condition $(xy)x = x(yx)$ which is true for all octonions $x, y \in \mathbb{O}$. From these properties, again following [1] one can deduce the famous Moufang rules, saying that $z(x(zy)) = ((zx)z)y$, $x(z(yz)) = ((xz)y)z$, $(zx)(yz) = (z(xy))z$ and $(zx)(yz) = z((xy)z)$, (*cf.* [1]). In particular, following [10] we still have the important rule that $(x\overline{x})y = (\overline{x}x)y = \overline{x}(xy)$. Later one we shall see that exactly this particular rule is responsible for ensuring that any of the eight real component octonionic monogenic function is a harmonic function in $\mathbb{R}^8$.

A further particular feature is that octonions still form a normed algebra. The Euclidean norm in $\mathbb{R}^8$ induces the standard norm in $\mathbb{O}$ and using the octonionic product the product octonion satisfies the relation $|xy| = |x||y|$ for any $x, y \in \mathbb{O}$.

This standard octonionic in turn induces the classical Euclidean real-valued inner product $\langle x, y\rangle = x_0y_0 + \sum_{i=1}^{7} x_iy_i = \Re(\overline{x}y)$. A simple calculation gives that $\langle xy, z\rangle = \langle y, \overline{x}z\rangle$ for any triple of octonions $x, y, z \in \mathbb{O}$. This relation plays a crucial role to meaningfully define an adjoint of a so-called para-linear operator, see [14, 15] where para-linearity replaces the strong linearity condition that is applied in the classical setting of real and complex Hilbert spaces

2. Octonionic monogenic Hardy spaces - continuous setting

In this section we revisit one possible way to extend the study of analytic Hardy spaces to the non-associative octonions.

Following for example [24, 25] a real differentiable octonion valued function f is called left octonionic monogenic if $\mathcal{D}f = 0$ where $\mathcal{D} := \frac{\partial}{\partial x_0} + \sum_{i=1}^{7} \frac{\partial}{\partial x_i}\mathbf{e}_i$ is the so-called octonionic generalized Cauchy-Riemann operator, see also [26] and others.

In view of the above stated property $(\overline{x}x)y = \overline{x}(xy)$ we have the following important property: suppose that f is left octonionic monogenic, i.e. $\mathcal{D}f = 0$, then f (and hence each real component function of f) is harmonic, because $\Delta f = (\overline{\mathcal{D}}\mathcal{D})f = \overline{\mathcal{D}}(\mathcal{D}f) = 0$. It is important to mention, that in the Cayley-Dickson algebras beyond the octonions, such as the sedenions, this important property is not true any longer, see also [11]. However, octonions still admit this property. Although they are non-associative, they still are nice enough to form a so-called Moufang loop admitting this property. Sedenions and Cayley algebras beyond do not form Moufang loops any more.

Now, a substantial difference between octonionic and Clifford analysis is reflected in the fact that the set of left octonionic monogenic functions does not form a right $\mathbb{O}$-module, cf. for instance [16]. As the example $f(x) = x_1 - x_2\mathbf{e}_4$ from [16] shows we do not have that $\mathcal{D}(f\alpha) = (\mathcal{D}f)\alpha$ for arbitrary $\alpha \in \mathbb{O}$. This has several serious consequences.

Before going into details, let us fix to always consider an orientable domain $\Omega \subset \mathbb{O}$ that is simply connected and that is equipped with a strongly Lipschitz boundary $\partial\Omega$, where the exterior normal field $n(x)$ exists at almost every boundary point $x \in \partial\Omega$, except of countable many ones. Let us denote the scalar surface measure of $\partial\Omega$ by dS.

Now first of all there is no direct analogue of Stokes' formula, such as already proved in [24]. Instead, see [26], one has the following integral formula

$$\int_{\partial G} g(x)\ (n(x)f(x))dS(x) = \int_G \Big(g(x)(\mathcal{D}f(x)) + (g(x)\mathcal{D})f(x) - \sum_{j=0}^{7} [\mathbf{e}_j, \mathcal{D}g_j(x), f(x)] \Big) dV,$$

where $[a,b,c] := (ab)c - a(bc)$ is the first associator. This term vanishes in the case of associativity. We call this formula the first Stokes formula. Actually, this formula coincides with the usual Stokes formula, when the associator vanishes, which is always the case in an associative algebra, such as in Clifford analysis. Now, there is also a second Stokes' formula:

$$\int_{\partial\Omega} (g(x)\ n(x))f(x)dS(x) = \int_{\Omega} \Big(g(x)(\mathcal{D}f(x)) + (g(x)\mathcal{D})f(x) + \sum_{j=0}^{7} [g(x), \mathcal{D}f_j(x), \mathbf{e}_j] \Big) dV(x). \tag{2.1}$$

On the basis of these two generalizations of Stokes formula Nono [27], Xingmin-Li and Li-Zong Peng [25] were able to prove the following

Proposition 2.1 (*cf.* [25, 27], (Cauchy's integral formulae))**.** *Let $U \subseteq \mathbb{O}$ be open and $G \subseteq U$ be an* 8*-D compact oriented manifold with a strongly Lipschitz boundary ∂G.*

(i) *If $f : U \to \mathbb{O}$ is left octonionic monogenic, then for all $x \in G$*

$$f(x) = \frac{3}{\pi^4} \int_{\partial G} q_{\mathbf{0}}(y-x)\Big(n(y)f(y)dS(y)\Big),$$

where $q_{\mathbf{0}}(x) = \frac{\overline{x}}{|x|^8}$.

(ii) *Under the same geometric and analytic condition, one also has*

$$f(x) = \frac{3}{\pi^4} \int_{\partial G} \Big(q_{\mathbf{0}}(y-x)n(y)\Big)f(y)dS(y) - \int_G \sum_{i=0}^{7} \Big[q_{\mathbf{0}}(y-x), \mathcal{D}f_i(y), \mathbf{e}_i\Big] dy_0 \cdots dy_7.$$

Before we concretely talk about Hardy spaces of octonionic (left) monogenic functions it makes sense to recall the currently used definition of generalized octonionic Hilbert spaces given [14, 15] (based also on the classical definition given in [13]):

Definition 2.2 (*cf.* [23])**.** Let $\alpha \in \mathbb{O}$ and $r \in \mathbb{R}$. An octonionic Hilbert space H is a right $\mathbb{O}$-module with an octonion-valued inner product $(\cdot,\cdot) : H \times H \to \mathbb{O}$ such that $(H, \langle\cdot,\cdot\rangle_0)$ is a real Hilbert space, where $\langle\cdot,\cdot\rangle_0 := \Re(\cdot,\cdot)$ and where any $f, g, h \in H$ has to comply to the following six rules:

(i) $(f+g, h) = (f,h) + (g,h)$;

(ii) $(f,g) = \overline{(g,f)}$;

(iii) $(f,f) \in \mathbb{R}$ with $(f,f) \geq 0$ and $(f,f) = 0$ if and only if $f = 0$;

(iv) $(f, gr) = (f,g)r$;

(v) $(f, f\alpha) = (f,f)\alpha$;

(vi) $\langle f, g\alpha\rangle_0 = \Re\{(f, g\alpha)\} = \Re\{(f,g)\alpha\}$.

The property of Condition (vi) is called $\mathbb{O}$-para-linearity in the papers [14, 15].

A naive idea would be to work with functionals defined by $\mathcal{T}_g : H \to \mathbb{O}, \mathcal{T}_g(f) := (f,g)$. Such a functional however would be octonion-valued in general. But the non-associativity prevents us from getting a direct analogue of the Cauchy-Schwartz inequality in the classical sense. Hence, we do not get a direct generalization of the Fischer-Riesz representation theorem, either. Besides some trivial cases one cannot even define meaningfully an adjoint operator using this strict sense of $\mathbb{O}$-linearity. Also in the recent work [14] the authors observed that strict $\mathbb{O}$-linearity, demanding that $\mathcal{T}_g(f\alpha) = \mathcal{T}_g(f)\alpha$ is a too strong requirement for the development of a powerful theory of octonionic Hilbert function spaces.

In [14, 15] the authors proposed instead that a functional should only be $\mathbb{O}$-para-linear in the sense of Definition 2.2 part (vi). Behind this theoretical background it makes completely sense to consider octonionic generalizations of Hardy spaces in the $\mathbb{O}$-para-linear sense. The consideration of the $\mathbb{O}$-linear setting also allows us to define an adjoint operator in a meaningful sense. One then has to interprete the dual space as the space of $\mathbb{O}$-para-linear functionals. Note that in contrast to the classical setting, the dual of the dual space will not be isomorphic to the original space again in general. This is a price that we have to pay for applying the weaker notion of $\mathbb{O}$-para-linearity.

As already proposed independently in the work by Constales and Kraußhar [8] one may define $H^2(\partial\Omega, \mathbb{O})$ as the closure of the set of $L^2(\partial\Omega)$-octonion-valued functions being left octonionic monogenic functions inside of Ω and having additionally a continuous extension to $\partial\Omega$.

To include the particular cases of the unit ball and the half-space considered in [31, 30] we suggested in [17, 8] to endow $H^2(\partial\Omega, \mathbb{O})$ attached to a general domain Ω with the following inner product

$$(f, g)_{\partial\Omega} := \frac{3}{\pi^4} \int_{\partial\Omega} \overline{(n(x)g(x))}\ (n(x)f(x))dS(x).$$

In the case of the unit ball we simply have that $n(x) = \frac{x}{|x|}$ and we recover the particular choice for the inner product considered in [30]. Now in the case of the octonionic right half-space $H^+(x) = \{x \in \mathbb{O} \mid \Re(x) > 0\}$ we get $n(x) = -1$, so we would recover the classical unweighted L^2 inner product. Note that $n(x)$ is an intrinsic octonionic weight factor of unit length. In the case of associativity these factors would disappear due to $\overline{n(x)}n(x) = 1$.

Endowing $(H^2(\partial\Omega), (\cdot,\cdot)_{\partial\Omega}$ with this particular choice of inner product makes it to a right $\mathbb{O}$-para-linear Hilbert in the sense of Definition 2.2. Applying Artin's theorem this inner product induces the classical L^2 norm on the boundary via $\|f\|^2_{\partial\Omega} = (f, f)_{\partial\Omega}$.

Next, quoting [10] Prop. 1.4 (e), each octonionic triple $a, b, c \in \mathbb{O}$ satisfies $\Re(a(bc)) = \Re((ab)c)$, (*cf.* [10] Prop. 1.4 (e)), we may infer that

$$\langle f\alpha, g\rangle_{\partial\Omega} = \Re(f\alpha, g)_{\partial\Omega} = \Re(f, g)_{\partial\Omega}\alpha = \langle f, g\rangle_{\partial\Omega}\alpha \text{ for all } \alpha \in \mathbb{O}.$$

Thus, this inner product is $\mathbb{O}$-para-linear.

On the level of the real-valued inner product $\langle\cdot,\cdot\rangle_{\partial\Omega}$ we have the Fischer-Riesz representation theorem and the $\mathbb{O}$-para-linear language allows us to translate this theorem into the language of octonionic Hilbert spaces.

In particular, if $\mathcal{T}$ is an $\mathbb{O}$-para-linear bounded operator induced by an integral kernel $k(x, y)$ in the explicit way

$$[\mathcal{T} f](x) = \langle f(\cdot), k(x, \cdot)\rangle_{\partial\Omega}$$

then we may express

$$\langle \mathcal{T} f, g\rangle_{\partial\Omega} = \langle f, \mathcal{T}^* g\rangle_{\partial\Omega}$$

with a uniquely defined $\mathbb{O}$-para-linear adjoint operator $\mathcal{T}^*$ acting on the correspondence dual space in the weaker $\mathbb{O}$-para-linear sense, see comments above. Remember here again, that H^{**} is not isomorphic to H.

In view of $\langle uv, w\rangle = \langle v, \overline{u}w\rangle$, the integral kernel of the adjoint operator is then simply given by the octonionic conjugate of the kernel $k(x, y)$ such as defined in the preliminary section.

We shall now revisit this new functional analytic concept on the particular example of the Cauchy transform.

Suppose first that x is a point of the interior of the domain Ω. Assume that f is an octonionic-valued function that is square integrable over the boundary $\partial\Omega$, i.e. $f \in L^2(\partial\Omega)$. In terms of the inner product $(\cdot,\cdot)_{\partial\Omega}$ the octonionic Cauchy projection previously defined in [26], Theorem 2.2 can then be expressed in the form

$$[Cf](x) = (f, g_x)_{\partial\Omega} = \frac{3}{\pi^4} \int_{\partial\Omega} \overline{(n(y)g_x(y))}\ (n(y)f(y))dS(y),$$

where we set
$$g_x(y) = \overline{n(y)}\frac{y-x}{|y-x|^8}.$$
Since $[Cf](x)$ is left octonionic monogenic at each $x \in \Omega$, Cf is a mapping from $L^2(\partial\Omega, \mathbb{O})$ into $H^2(\partial\Omega, \mathbb{O})$. In our notation the Cauchy integral formula is simply given in the way
$$(f, g_x)_{\partial\Omega} = f$$
under the particular condition that f is also left octonionic monogenic.

Remark. Only the first version of Proposition 2.1(i) *can be so easily expressed in terms of the use of the inner product. If we depart from the second version of Proposition* 2.1(ii) *instead, then we can not put the factor $n(x)$ inside the parenthesis to join the factor $f(x)$. To re-express the second version of the Cauchy formula in terms of this inner product is not that canonical.*

Under additional conditions, for instance when claiming that f is μ-höldercontinuous over the boundary of the domain with $0 < \mu < 1$ one can extend our representation for Cf, such as also mentioned in [26] Theorem 2.2, to the boundary of the domain. For boundary values $x \in \partial\Omega$ the octonionic Cauchy transform then has the form
$$[Cf](x) = \frac{1}{2}f(x) + p.v.\frac{3}{\pi^4}\int\limits_{\partial\Omega}\overline{(n(y)g_x(y))}\,(n(y)f(y))dS(y),$$
where $p.v.$ as usual stands for the Cauchy principal value such as defined earlier in the associative settings, see for example [2, 3, 8, 9]. The singular integral
$$[\mathcal{H}f](x) := 2p.v.\frac{3}{\pi^4}\int\limits_{\partial\Omega}\overline{(n(y)g_x(y))}\,(n(y)f(y))dS(y)$$
is nothing else than the octonionic Hilbert transform defined and investigated in our preceding work [8] where we already proved that $\mathcal{H}$ and C are L^2-bounded integral operators.

It is noteworthy that Cf actually is $\mathbb{O}$-para linear in the spirit of [15].

Considering the real-valued inner product induced by the real part of the octonionic inner product, namely $\langle\cdot,\cdot\rangle_{\partial\Omega}$, one may constate that there exists a unique adjoint transform C^* in the sense
$$\langle Cf, g\rangle_{\partial\Omega} = \langle f, C^*g\rangle_{\partial\Omega}.$$
In fact this is exactly the $\mathbb{O}$-para-linear adjoint in the spirit of the definition of [15].

More precisely,

Theorem 2.3 (*cf.* [8])**.** *The $\mathbb{O}$-para-linear adjoint octonionic Cauchy transform C^* with respect to $\langle\cdot,\cdot\rangle_{\partial\Omega}$ can be expressed by*
$$C^*\colon C^\mu(\partial\Omega) \to L^2(\partial\Omega)\colon \quad [C^*f](x) = \frac{1}{2}f(x) + p.v.\frac{3}{\pi^4}\int\limits_{\partial\Omega}\overline{(n(y)\overline{g_y(x)})}\,(n(y)f(y))dS(y) = (f, \overline{g_y})_{\partial\Omega}.$$

The integral kernel of C^* is exactly the octonionic conjugate of C.

Since C is $\mathbb{O}$-para-linear, continuous and bounded, the same is true for its $\mathbb{O}$-para-linear adjoint transform. The property $\|Cf\|_{L^2} = \|C^*f\|_{L^2}$ now follows from a classical density argument.

The following definition thus makes sense.

Definition 2.4 (*cf.* [8])**.** The octonionic monogenic Kerzman-Stein operator $\mathcal{A}\colon L^2(\partial\Omega) \to L^2(\partial\Omega)$ is well-defined in terms
$$[\mathcal{A}f](x) := (f, A_x)_{\partial\Omega} = \frac{3}{\pi^4}\int\limits_{\partial\Omega}\overline{(n(y)A(x,y))}\,(n(y)f(y))dS(y),$$
where explicitly
$$A(x,y) := g_x(y) - \overline{g_y(x)} = \overline{n(y)}\frac{y-x}{|y-x|^8} - \frac{\overline{x-y}}{|y-x|^8}n(x)$$
for all $x, y \in \partial\Omega \times \partial\Omega$ with $(x \neq y)$.

Note that the Cauchy transform is self-adjoint in the $\mathbb{O}$-para-linear setting if and only if Ω is the octonionic unit ball, because in this setting the Kerzman- Stein kernel is identically zero.

It is also noteworthy that the Kerzman-Stein operator $\mathcal{A}$ is not singular, so it is defined for all functions $f \in L^2(\partial\Omega)$.

Alike in the complex and Clifford analysis case $\mathcal{A}$ is a compact and a skew symmetric operator, i.e. $\mathcal{A}^* = -\mathcal{A}$. For details, see [8].

Following [8] also in the octonionic case the compact operator $\mathcal{A}$ can be expressed in terms of the Hilbert transform by

$$\mathcal{A} = \frac{1}{2}\mathcal{H} - \frac{1}{2}\mathcal{H}^*,$$

where

$$[\mathcal{H}^* f](x) := 2\frac{3}{\pi^4} p.v. \int_{\partial\Omega} \overline{(n(y)\overline{g_y(x)})}\,(n(y)f(y))dS(y), \quad f \in C^\mu(\partial\Omega)$$

is the $\mathbb{O}$-para-linear adjoint of the Hilbert transform introduced before. Here, we assume for instance that the functions f are additionally μ-höldercontinuous with $0 < \mu < 1$.

Using the real-valued inner product one has $\langle \mathcal{H}f, g\rangle_{\partial\Omega} = \langle f, \mathcal{H}^* g\rangle_{\partial\Omega}$ when working in the spaces of μ-Hölder continuous functions.

If Ω is octonionic right half space $H^+(\mathbb{O}) := \{x \in \mathbb{O} \mid x_0 > 0\}$, then the associated octonionic Kerzman-Stein operator induces nothing else than the classical Hilbert-Riesz transform in the x_0-direction of the eight dimensional Euclidean space $\mathbb{R}^8$, that means we have

$$[\mathcal{A}f](x) = 2p.v. \int_{\mathbb{R}^7} \frac{y_0 - x_0}{|y - x|^8} f(y)dy_1 \cdots dy_7.$$

For the detailed proof we refer the reader to [8].

Next, as shown in [17], the Hardy space of left octonionic monogenic functions over $H^+(\mathbb{O})$ possesses a reproducing kernel $S(x, y)$ which is called the octonionic Szegö kernel. As shown in [17] on the octonionic right half-space the latter has the explicit form

$$S(x, y) = \frac{\overline{x} + y}{|\overline{x} + y|^8}$$

and satisfies

$$[\mathcal{S}f](x) := (S(x, \cdot), f)_{\mathbb{R}^7} = f$$

for all f satisfying $\mathcal{D}f = 0$ on $H^+(\mathbb{O})$. Note that in case of the right octonionic half-space, the exterior normal field is $n(x) \equiv -1$ for all $x \in \partial H^+(\mathbb{O}) = \mathbb{R}^7$, so it is canceled out in the general intrinsic definition of $(\cdot, \cdot)$.

As shown in [8] this kernel obviously satisfies $S(x, y) = \overline{S(y, x)}$, so the associated Szegö projection $[\mathcal{S}f] := (S(x, \cdot), f)_{\mathbb{R}^7}$ defined for arbitrary functions $f \in L_2(H^+\mathbb{O})$ is self-adjoint concerning the real part of the inner product, i.e. we have $\langle \mathcal{S}f, g\rangle_0 = \langle f, \mathcal{S}g\rangle_0$.

Finally, we want to remark that we always have the property that $\mathcal{S}\mathcal{C} = \mathcal{C}$, that means for every strongly Lipschitz domain any octonion-valued $L^2(\Omega)$ functions satisfies the operator relations

$$\mathcal{S}\mathcal{C}[f](x) := \mathcal{S}[\mathcal{C}f](x) = \mathcal{C}[f](x),$$

such as already proposed in our previous paper [8].

3. Hardy spaces of discrete octonionic monogenic functions on the half-space

To introduce a discrete counterpart of Hardy spaces of $\mathbb{O}$-valued functions, we establish over $\mathbb{R}^8$ the lattice $h\mathbb{Z}^8$ with $h > 0$ as follows:

$$h\mathbb{Z}^8 := \left\{\mathbf{x} \in \mathbb{R}^8 \mid \mathbf{x} = (m_0 h, m_1 h, \ldots, m_7 h), m_j \in \mathbb{Z}, j = 0, 1, \ldots, 7\right\}.$$

Further, we define also the upper half-spaces, while the lower half-space can be defined analogously, as follows

$$h\mathbb{Z}^8_+ := \left\{(h\underline{m}, hm_7) \colon \underline{m} \in \mathbb{Z}^7, m_7 \in \mathbb{Z}_+\right\}.$$

In recent years, two general approaches to the discretization of octonionic analysis have been proposed: a direct discretisation of partial derivatives in the Cauchy-Riemann operator by finite differences, see [20, 19, 22, 29], and the ideas grounded in the classical Weyl calculus [18, 21]. In this paper, we follow the latter approach, which consists in splitting of $\mathbf{e}_k$, $k = 0, 1, \ldots, 7$, into two directions: positive and negative, namely $\mathbf{e}_k^+$ and $\mathbf{e}_k^-$, $k = 0, 1, \ldots, 7$, i.e., $\mathbf{e}_k = \mathbf{e}_k^+ + \mathbf{e}_k^-$, see also [4, 12]. Additionally, the following conditions must be satisfied:

$$\begin{cases} \mathbf{e}_j^-\mathbf{e}_k^- + \mathbf{e}_k^-\mathbf{e}_j^- &= 0, \\ \mathbf{e}_j^+\mathbf{e}_k^+ + \mathbf{e}_k^+\mathbf{e}_j^+ &= 0, \\ \mathbf{e}_j^+\mathbf{e}_k^- + \mathbf{e}_k^-\mathbf{e}_j^+ &= -\delta_{jk}, \end{cases} \tag{3.1}$$

with δ_{jk} being the Kronecker delta. Additionally, non-associativity relations have to be respected:

$$\begin{array}{rclrcl} \left(\mathbf{e}_i^+\mathbf{e}_j^+\right)\mathbf{e}_k^+ &=& -\mathbf{e}_i^+\left(\mathbf{e}_j^+\mathbf{e}_k^+\right), & \left(\mathbf{e}_i^+\mathbf{e}_j^+\right)\mathbf{e}_k^- &=& -\mathbf{e}_i^+\left(\mathbf{e}_j^+\mathbf{e}_k^-\right), \\ \left(\mathbf{e}_i^-\mathbf{e}_j^+\right)\mathbf{e}_k^+ &=& -\mathbf{e}_i^-\left(\mathbf{e}_j^+\mathbf{e}_k^+\right), & \left(\mathbf{e}_i^-\mathbf{e}_j^+\right)\mathbf{e}_k^- &=& -\mathbf{e}_i^-\left(\mathbf{e}_j^+\mathbf{e}_k^-\right), \\ \left(\mathbf{e}_i^+\mathbf{e}_j^-\right)\mathbf{e}_k^+ &=& -\mathbf{e}_i^+\left(\mathbf{e}_j^-\mathbf{e}_k^+\right), & \left(\mathbf{e}_i^+\mathbf{e}_j^-\right)\mathbf{e}_k^- &=& -\mathbf{e}_i^+\left(\mathbf{e}_j^-\mathbf{e}_k^-\right), \\ \left(\mathbf{e}_i^-\mathbf{e}_j^-\right)\mathbf{e}_k^+ &=& -\mathbf{e}_i^-\left(\mathbf{e}_j^-\mathbf{e}_k^+\right), & \left(\mathbf{e}_i^-\mathbf{e}_j^-\right)\mathbf{e}_k^- &=& -\mathbf{e}_i^-\left(\mathbf{e}_j^-\mathbf{e}_k^-\right), \end{array} \tag{3.2}$$

which held for mutually distinct and non-zero indices i, j, k satisfying additionally $\mathbf{e}_i\mathbf{e}_j \neq \pm\mathbf{e}_k$, otherwise, the multiplication is associative $(\mathbf{e}_i\mathbf{e}_j)\mathbf{e}_k = \mathbf{e}_i(\mathbf{e}_j\mathbf{e}_k)$.

Now, we can introduce the pair of discrete Cauchy-Riemann operators $D_h^{+-} \colon l^p(\Omega_h, \mathbb{O}) \to l^p(\Omega_h, \mathbb{O})$ and $D_h^{-+} \colon l^p(\Omega_h, \mathbb{O}) \to l^p(\Omega_h, \mathbb{O})$ for $\Omega_h \subset h\mathbb{Z}^8$ in the classical way

$$D_h^{+-} := \sum_{j=0}^{7} \mathbf{e}_j^+\partial_h^{+j} + \mathbf{e}_j^-\partial_h^{-j}, \quad D_h^{-+} := \sum_{j=0}^{7} \mathbf{e}_j^+\partial_h^{-j} + \mathbf{e}_j^-\partial_h^{+j},$$

where $\partial_h^{\pm j}$ are classical finite differences

$$\begin{array}{rcl} \partial_h^{+j} f(mh) &:=& h^{-1}(f(mh + \mathbf{e}_j h) - f(mh)), \\ \partial_h^{-j} f(mh) &:=& h^{-1}(f(mh) - f(mh - \mathbf{e}_j h)). \end{array} \tag{3.3}$$

Octonionic monogenicity in the discrete setting is then defined as

Definition 3.1. A function $f \in l^p(\Omega_h, \mathbb{O})$ is called *discrete left monogenic* if $D_h^{+-} f = 0$ in Ω_h. Respectively, a function $f \in l^p(\Omega_h, \mathbb{O})$ is called *discrete left anti-monogenic* if $D_h^{-+} f = 0$ in Ω_h.

Similar to the continuous function theories, discrete theories utilise a discrete fundamental solution (DFS) of the discrete Cauchy-Riemann (or Dirac) operator. Moreover, considering that we have two operators D_h^{+-} and D_h^{-+}, we need also to distinguish their fundamental solutions:

Definition 3.2. The function $E_h^{+-} \colon h\mathbb{Z}^8 \to \mathbb{O}$ is called a *discrete fundamental solution* of D_h^{+-} if it satisfies

$$D_h^{+-} E_h^{+-} = \delta_h = \begin{cases} h^{-8}, & \text{for } mh = 0, \\ 0, & \text{for } mh \neq 0, \end{cases}$$

with $(mh) \in h\mathbb{Z}^8$. Analogously, the function $E_h^{-+} \colon h\mathbb{Z}^8 \to \mathbb{O}$ is called a *discrete fundamental solution* of D_h^{-+} if it satisfies

$$D_h^{-+} E_h^{-+} = \delta_h = \begin{cases} h^{-8}, & \text{for } mh = 0, \\ 0, & \text{for } mh \neq 0, \end{cases}$$

with $(mh) \in h\mathbb{Z}^8$.

In practice, a DFS is computed by using the (classical) discrete Fourier transform, which consequently leads to an integral representation of the DFS. We omit these technical details here and refer to [19, 21] for the octonionic case.

Before discussing the discrete octonionic Hardy spaces, we present the following theorem [21]:

Theorem 3.3. *Let E_h^{-+} be the discrete fundamental solution of the discrete Cauchy-Riemann operator D_h^{-+}. Then the discrete octonionic Borel-Pompeiu formula for the upper half-lattice $h\mathbb{Z}_+^8$ is given by*

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}_+^8} E_h^{-+}(nh-mh)\left[D_h^{+-}f(nh)\right]h^8 - 2\sum_{m\in\mathbb{Z}_+^8}\sum_{s=1}^{7}\sum_{i\in I_s}\sum_{\substack{j\in I_s\\ j\neq i}}^{7}\sum_{\substack{k=1\\ k\notin I_s}}^{7}\Big[E_{h,i}^{-+}(nh-mh)(mh)\mathbf{e}_i\left(\partial_h^{+j}\mathbf{e}_j^{+}f_k(nh)\mathbf{e}_k\right)\\
&\qquad +E_{h,i}^{-+}(nh-mh)\mathbf{e}_i\left(\partial_h^{-j}\mathbf{e}_j^{-}f_k(nh)\mathbf{e}_k\right)\Big]h^8\\
&-2\sum_{\underline{m}\in\mathbb{Z}^7}\left[\sum_{i=1,6}\sum_{\substack{k=1\\ k\neq i}}^{6}\left(E_{h,i}^{-+}(\underline{n}h-\underline{m}h,0)\mathbf{e}_i\left(\mathbf{e}_7^{+}f_k(\underline{n}h,h)\mathbf{e}_k\right) + E_{h,i}^{-+}(\underline{n}h-\underline{m}h,h)\mathbf{e}_i\left(\mathbf{e}_7^{-}f_k(\underline{n}h,0)\mathbf{e}_k\right)\right)\right.\\
&\qquad \sum_{i=2,5}\sum_{\substack{k=1\\ k\neq i}}^{6}\left(E_{h,i}^{-+}(\underline{n}h-\underline{m}h,0)\mathbf{e}_i\left(\mathbf{e}_7^{+}f_k(\underline{n}h,h)\mathbf{e}_k\right) + E_{h,i}^{-+}(\underline{n}h-\underline{m}h,h)\mathbf{e}_i\left(\mathbf{e}_7^{-}f_k(\underline{n}h,0)\mathbf{e}_k\right)\right)\\
&\qquad \left.\sum_{i=3,4}\sum_{\substack{k=1\\ k\neq i}}^{6}\left(E_{h,i}^{-+}(\underline{n}h-\underline{m}h,0)\mathbf{e}_i\left(\mathbf{e}_7^{+}f_k(\underline{m}h,h)\mathbf{e}_k\right) + E_{h,i}^{-+}(\underline{n}h-\underline{m}h,h)\mathbf{e}_i\left(\mathbf{e}_7^{-}f_k(\underline{m}h,0)\mathbf{e}_k\right)\right)\right]h^7\\
&= \begin{cases} 0, & m\notin\mathbb{Z}_+^8,\\ -f(mh), & m\in\mathbb{Z}_+^8,\end{cases}
\end{aligned}$$

for any discrete function f such that the series converge, and index sets I_s, $s = 1,\ldots,7$ are given by

$$\begin{aligned}
&I_1 := \{1,2,4\}, \quad I_2 := \{1,3,5\}, \quad I_3 := \{1,6,7\}, \quad I_4 := \{2,3,6\},\\
&I_5 := \{2,5,7\}, \quad I_6 := \{3,4,7\}, \quad I_7 := \{4,5,6\}.
\end{aligned}$$

In the case when f is a discrete octonionic left monogenic function with respect to the operator D_h^{+-}, we obtain the discrete octonionic Cauchy formula for the upper half-lattice $h\mathbb{Z}_+^8$ in the form

$$\begin{aligned}
&2\sum_{m\in\mathbb{Z}_+^8}\sum_{s=1}^{7}\sum_{i\in I_s}\sum_{\substack{j\in I_s\\ j\neq i}}^{7}\sum_{\substack{k=1\\ k\notin I_s}}^{7}\left[E_{h,i}^{-+}(nh-mh)(mh)\mathbf{e}_i\left(\partial_h^{+j}\mathbf{e}_j^{+}f_k(nh)\mathbf{e}_k\right) + E_{h,i}^{-+}(nh-mh)\mathbf{e}_i\left(\partial_h^{-j}\mathbf{e}_j^{-}f_k(nh)\mathbf{e}_k\right)\right]h^8\\
&2\sum_{\underline{m}\in\mathbb{Z}^7}\left[\sum_{i=1,6}\sum_{\substack{k=1\\ k\neq i}}^{6}\left(E_{h,i}^{-+}(\underline{n}h-\underline{m}h,0)\mathbf{e}_i\left(\mathbf{e}_7^{+}f_k(\underline{n}h,h)\mathbf{e}_k\right) + E_{h,i}^{-+}(\underline{n}h-\underline{m}h,h)\mathbf{e}_i\left(\mathbf{e}_7^{-}f_k(\underline{n}h,0)\mathbf{e}_k\right)\right)\right.\\
&\qquad \sum_{i=2,5}\sum_{\substack{k=1\\ k\neq i}}^{6}\left(E_{h,i}^{-+}(\underline{n}h-\underline{m}h,0)\mathbf{e}_i\left(\mathbf{e}_7^{+}f_k(\underline{n}h,h)\mathbf{e}_k\right) + E_{h,i}^{-+}(\underline{n}h-\underline{m}h,h)\mathbf{e}_i\left(\mathbf{e}_7^{-}f_k(\underline{n}h,0)\mathbf{e}_k\right)\right)\\
&\qquad \left.\sum_{i=3,4}\sum_{\substack{k=1\\ k\neq i}}^{6}\left(E_{h,i}^{-+}(\underline{n}h-\underline{m}h,0)\mathbf{e}_i\left(\mathbf{e}_7^{+}f_k(\underline{m}h,h)\mathbf{e}_k\right) + E_{h,i}^{-+}(\underline{n}h-\underline{m}h,h)\mathbf{e}_i\left(\mathbf{e}_7^{-}f_k(\underline{m}h,0)\mathbf{e}_k\right)\right)\right]h^7\\
&= \begin{cases} 0, & m\notin\mathbb{Z}_+^8,\\ f(mh), & m\in\mathbb{Z}_+^8.\end{cases}
\end{aligned}$$

The goal now is to discuss construction of octonionic Hardy spaces in the discrete case. Consequently, the boundary behaviour of $\mathbb{O}$-valued discrete monogenic functions must be discussed. Additionally, it is interesting to observe, that the ideas from the discrete Clifford analysis [5, 6] can be directly adapted to the octonionic case, because the non-associativity does not influence the construction. This is due to the fact, that only the Fourier symbols of E_h^{-+} on the boundary layers are essential for the construction. The integral representation of E_h^{-+} is given by

$$E_h^{-+}(mh) = \frac{1}{(2\pi)^8}\int_{\xi\in[-\frac{\pi}{h},\frac{\pi}{h}]^8}\frac{\tilde{\xi}_-}{d^2}e^{-i\langle mh,\xi\rangle}d\xi, \quad m\in\mathbb{Z}^8$$

with $\tilde{\xi}_- = \sum_{j=0}^{7}\left(\mathbf{e}_j^+\xi_h^{-j} + \mathbf{e}_j^-\xi_h^{+j}\right)$ and $\xi = \sum_{j=0}^{7}\left(\mathbf{e}_j^+ + \mathbf{e}_j^-\right)\xi_j$. For the upper half-space, it is possible to compute the Fourier symbols of the DFS E_h^{-+} on boundary layers, i.e. for $m_7 = 0$ and $m_7 = 1$, explicitly [5]:

$$\begin{aligned}
\mathcal{F}_h^{(7)} E_h^{-+}(\underline{\xi}, 0) &= \frac{\widetilde{\underline{\xi}_-}}{\underline{d}\sqrt{4 + h^2\underline{d}^2}} + \left(\mathbf{e}_7^+ - \mathbf{e}_7^-\right)\left(\frac{1}{2} - \frac{h\underline{d}}{2\sqrt{4 + h^2\underline{d}^2}}\right), \\
\mathcal{F}_h^{(7)} E_h^{-+}(\underline{\xi}, h) &= \frac{\widetilde{\underline{\xi}_-}}{\underline{d}}\left(\frac{2 + h^2\underline{d}^2}{2\sqrt{4 + h^2\underline{d}^2}} - \frac{h\underline{d}}{2}\right) + \mathbf{e}_7^+\left(\frac{h\underline{d}}{2\sqrt{4 + h^2\underline{d}^2}} - \frac{1}{2}\right) - \mathbf{e}_7^-\left(-\frac{3h\underline{d} + h^3\underline{d}^3}{2\sqrt{4 + h^2\underline{d}^2}} + \frac{h^2\underline{d}^2 + 1}{2}\right),
\end{aligned}$$

with $\mathcal{F}_h^{(7)}$ denoting the 7-dimensional discrete Fourier transform, and $\underline{d}^2 = \frac{4}{h^2}\sum_{j=0}^{6}\sin^2\left(\frac{\xi_j h}{2}\right)$ is the symbol of the classical discrete Laplace operator.

Utilising the Fourier symbols introduced above, we can now define the operator

$$H_+ f := \mathcal{F}_h^{-1}\left[\frac{\widetilde{\underline{\xi}_-}}{\underline{d}}\left(\mathbf{e}_7^+\frac{h\underline{d} - \sqrt{4 + h^2\underline{d}^2}}{2} + \mathbf{e}_7^-\frac{2}{h\underline{d} - \sqrt{4 + h^2\underline{d}^2}}\right)\right],$$

which satisfy $(H_+)^2 = I$. Hence, the condition for a function f to be a boundary value of a discrete $\mathbb{O}$-valued monogenic function in $h\mathbb{Z}_+^8$ can now be equivalently written as

$$f(mh) = H_+ f(mh), \text{ for } \gamma_h^+.$$

Thus, finally the Hardy space of $\mathbb{O}$-valued discrete functions for the upper half-space $h\mathbb{Z}_+^8$ can be formally introduced as follows:

Definition 3.4. The space of discrete functions $f \in l^p(h\mathbb{Z}_+^8, \mathbb{O})$ whose discrete 7D-Fourier transform fulfils $f = H_+ f$ on $m_7 = 1$ is called the *upper discrete octonionic Hardy space*.

Moreover, the operator H_+ can be seen as a discrete Hilbert transform, and, hence, the Cauchy transform in the discrete case for $h\mathbb{Z}_+^8$ can be formally defined as

$$C_h^+ := \frac{1}{2}(I + H_+).$$

We would like to remark, that the construction of discrete octonionic Hardy space for $h\mathbb{Z}_-^8$ (lower half-space) is analogous, see [19, 21] for the details.

4. Conclusion

To introduce appropriate analogues of Hardy spaces in octonions serious obstacles in the development of the functional-analytic and function-theoretic background have to be surpassed. The non-associativity does not admit a direct analogue of the Cauchy-Schwartz inequality and hence we do not get a Fischer Riesz representation theorem when we consider the strong concept of linear functionals that are derived from octonion-valued inner products. The strong linearity concept has to be weakened in the sense of requesting some kind of linearity condition just on the level of the real components. This leads to a weaker notion of octonionic para-linearity. In this framework it is possible to consider adjoint operators and one gets reproducing kernel spaces, in particular a generalisation of the Hardy space. This paper shows how the concept of para-linearity allows to define a meaningful analogue of a dual Cauchy transform from which we can deduce an octonionic Kerzman-Stein operator. This is a compact operator which allows us to approximate the octonionic Szegö projection. It also gives rise to an octonionic Hilbert transform. In the particular case of the right octonionic half-space we show how we can recover the classical Hilbert-Riesz transform over $\mathbb{R}^7$ from it. Finally, we successfully find a way how to discretise the Cauchy and the Hilbert transform which will enable us to study boundary value problems in the octonionic setting. This paper thus successfully links the different puzzle pieces from previous works [8] on Hardy spaces on the one hand, the concept of $\mathbb{O}$-para-linear functionals on the other hand [14, 15], as well as on the different approaches of discretisation [18]-[21], [29] and revises this topic from the current state of knowledge which has grown significantly within the last three years. Also the different approaches for discretisations are joined together and reinterpreted from one common viewpoint.

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