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Classification and solution methods for Schlömilch-type integral equations

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Abstract

Schlömilch's integral equations are extensively used in the fields of terrestrial physics and ionospheric research. In recent years, numerous studies have focused on developing solution methods and exploring applications for these equations. In this work, we define, investigate, and propose a solution method for Schlömilch-type integral equations. We start by classifying Schlömilch-type integral equations and developing a simple yet effective method for solving each class. Additionally, we explore an application problem that leads to the Rayleigh equation and provide its solution. To demonstrate the efficiency and practicality of our findings, we include illustrative examples from the literature for each type of equation discussed.

Keywords: Integral equations, orthogonal polynomials, regularization

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1. Introduction

Integral equations, in simple terms, are equations in which the unknown function appears under the integral sign. They occur extensively in various areas of mathematics, physics, and engineering. There are situations where they are more useful than differential equations. In particular, in the process of modeling a practical problem involving some restrictions such as boundary conditions or initial conditions, integral equations offer advantages over differential equations since these conditions can be condensed into a single integral equation rather than appearing as separate side conditions in a differential equation.

The Schlömilch's integral equation is a special type of integral equation which has numerous applications in mathematical physics and engineering. The standard form of the *Schlömilch's integral equation* is given by

$$f(x) = \frac{2}{\pi} \int_0^{\pi/2} \phi(x \sin\theta)\, d\theta, \quad -\pi \le x \le \pi,$$

where f is given and a continuous differential function on $[-\pi, \pi]$. Many works have been conducted on Schlömilch's integral equations and their solutions (*cf.* [8, 9, 17, 18]). Additionally, we list some recent works below.

Bougoffa et al. [5] introduced a convenient technique for solving Fredholm integral equations of the first kind. Considering the Schlömilch's integral equations as a special subclass of these, they extended this method to find solutions for Schlömilch's integral equations and obtained very accurate results. Wazwaz [20] combined the regularization

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method with the Adomian decomposition method (ADM) to solve the equation. Altürk [2] proposed an algorithm for finding solutions for linear and nonlinear Schlömilch's integral equations. Altürk and Arabacıoğlu [3] modified the homotopy perturbation method to find solutions for a variety of Schlömilch's integral equations. Parand and Delkhosh [11] used the generalized fractional order of the Chebyshev orthogonal functions to find accurate approximate solutions for both linear and nonlinear cases. A wide variety of additional materials is available for readers who need a more advanced treatment of Schlömilch integral equations (*cf.* [1, 7]).

In this work, we focus on the Schlömilch-type integral equations introduced recently by Wazwaz [20]. The standard *linear Schlömilch-type integral equation* is defined as

$$f(x) = \frac{2}{\pi}\int_0^{\pi/2} u(x\cos\theta)\,d\theta, \qquad f,u \in C'([-\pi,\pi]), \tag{1.1}$$

where f is a given function and u is the unknown function.

In addition to equation (1.1), we define and investigate the following pair of equations:

- The generalized Schlömilch-type integral equations.
- The nonlinear Schlömilch-type integral equations.

The rest of the article is organized as follows:
In section 2, we study the Schlömilch-type integral equations including the generalized and nonlinear Schlömilch-type integral equations and obtain some significant algorithms that produce a solution for a given Schlömilch-type equation. Section 3 includes examples mainly from the literature to demonstrate the applicability of the proposed methods and enable comparisons with existing results. Section 4 considers an application problem that leads to Rayleigh equation describes the generalized Schlömilch-type integral equations and provides an algorithm for their solutions. The final section concludes the article.

2. Main section

2.1. *The Schlömilch-type integral equation*

In this section, we consider the standard *linear Schlömilch-type integral equation* that admits the following form:

$$p(x) = \frac{2}{\pi}\int_0^{\pi/2} u(x\cos\theta)\,d\theta, \qquad p,u \in C'([-\pi,\pi]), \tag{2.1}$$

where p is a given function and u is the unknown function. In addition, Abel equation is defined as

$$f(x) = \int_0^t \frac{u(t)}{\sqrt{x-t}}\,dt, \tag{2.2}$$

where its solution is given by

$$u(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)}{\sqrt{x-t}}\,dt \tag{2.3}$$

f is a given function and u is the unknown function to be determined (*cf.* [19]). In [9], the author showed that Abel and Schlömilch integral equations are closely related and obtained a method for the solution of Schlömilch's integral equation by making the connection between these two equations apparent. We now show here that that connection is preserved for standard linear Schlömilch-type integral equation.
Let $x\cos(\theta) = \sqrt{\mu}$ and $x^2 = \eta$. These substitutions transform the equation (2.1) into

$$p(\sqrt{\eta}) = \frac{1}{\pi}\int_0^{\eta} \frac{u(\sqrt{\mu})}{\sqrt{\mu}\sqrt{\eta-\mu}}\,d\mu,$$

or, equivalently,

$$p(\sqrt{\eta}) = \int_0^{\eta} \frac{U(\mu)}{\sqrt{\eta-\mu}}\,d\mu, \tag{2.4}$$

where $U(\mu) = \frac{u(\sqrt{\mu})}{\pi\sqrt{\mu}}$.
Since the equation (2.4) is an Abel equation, it's solution is given by

$$U(\eta) = \frac{1}{\pi}\frac{d}{d\eta}\int_0^\eta \frac{p(\sqrt{\mu})}{\sqrt{\eta-\mu}}\,d\mu$$

or, equivalently

$$u(\sqrt{\eta}) = \sqrt{\eta}\frac{d}{d\eta}\int_0^\eta \frac{p(\sqrt{\mu})}{\sqrt{\eta-\mu}}\,d\mu. \tag{2.5}$$

Reintroducing the original variable amounts to

$$\begin{aligned}
u(x) &= x\frac{d}{d\eta}\int_0^{\pi/2} p(x\cos\theta)2x\cos\theta\,d\theta \\
&= x\frac{dx}{d\eta}\frac{d}{dx}\int_0^{\pi/2} p(x\cos\theta)2x\cos\theta\,d\theta \\
&= \frac{d}{dx}\int_0^{\pi/2} p(x\cos\theta)x\cos\theta\,d\theta \\
&= \int_0^{\pi/2}\left[x\cos^2\theta p'(x\cos\theta) + \cos\theta p(x\cos\theta)\right]d\theta \\
&= p(0) + x\int_0^{\pi/2} p'(x\cos\theta)\,d\theta.
\end{aligned} \tag{2.6}$$

The last step is the result of an application of integration by parts. Thus, a solution is given by

$$u(x) = p(0) + x\int_0^{\pi/2} p'(\varepsilon)\,d\theta, \qquad \varepsilon = x\cos\theta,$$

where $p'(\varepsilon)$ represents a derivative. Following similar approach to that used in [9] for Schlömilch integral equations, we show that Abel integral equation and Schlömilch-type integral equation are closely related. This relationship provides a method for solving Schlömilch-type integral equations.

We further investigate Schlömilch-type integral equations, recognizing the critical role of polynomials in mathematical modeling and approximation theory. Significant integral equations are usually constructed or appeared in the solutions of application problems [14, 15]. Consequently, in the following, we first examine the case where p is given as a polynomial function.

We then assert that if p is a polynomial function with a degree n, then there is a polynomial with degree n that satisfies equation (2.1).

Suppose that $p(x) = \sum_{j=0}^{n} p_j x^j$ and let

$$\begin{aligned}
q_0 &= p_0, \\
q_1 &= \frac{\pi p_1}{2}, \\
q_k &= \begin{cases} \frac{k}{k-1}\times\frac{k-2}{k-3}\times\cdots\times\frac{2}{1}\times p_k, & k \text{ even}, \\ \frac{k}{k-1}\times\frac{k-2}{k-3}\times\cdots\times\frac{3}{2}\times\frac{\pi}{2}\times p_k, & k \text{ odd} \end{cases}
\end{aligned} \tag{2.7}$$

for $k \geq 2$ and that $p_0, p_1, \ldots, p_n$ are constants.

Define $u(x) = \sum_{k=0}^{n} q_k x^k$. An application of mathematical induction on the degree of the polynomial p proves that u is the solution of equation (2.1).

Notice that it is easy to verify that $p_0 = q_0$.

For $n = 1$,

$$u(x) = q_0 + q_1 x.$$

Substitute this into the the equation (2.1), we verify that

$$\begin{aligned}&\frac{2}{\pi}\int_0^{\pi/2} q_0 + q_1 x\cos\theta\, d\theta,\\ &= q_0 + \frac{2}{\pi}q_1 x = p_0 + p_1 x.\end{aligned}$$

For $n = k$, assume

$$u(x) = q_0 + q_1 x + \cdots + q_k x^k = p_0 + p_1 x + \cdots + p_k x^k.$$

For $n = k + 1$,

$$u(x) = q_0 + q_1 x + \cdots + q_k x^k + q_{k+1} x^{k+1}.$$

Substitute this into the the equation (2.1), we only need to show that the last term in both sides of the equation (2.1) is identical. After evaluating the integral, we obtain

$$q_{k+1} = \frac{\sqrt{\pi}\Gamma\left(\frac{k+3}{2}\right)}{\Gamma\left(\frac{k}{2}+1\right)} p_{k+1},$$

where Γ represents the gamma function (*cf.* [4]). We want to emphasize that as long as p is a polynomial function, no question arises about the existence of the corresponding q_k's. The above discussion is concluded in the following theorem.

Theorem 2.1. *$deg(p) = n$ if and only if the solution of (2.1) is a polynomial function of the same degree.*

On the other hand, Schlömilch-type integral equation may also appear as

$$p(x) = \frac{2}{\pi}\int_0^{\pi/2} u\Big(x\cos(m\theta)\Big)\, d\theta, \quad m \in \mathbb{Z}^+ - \{1\}, \qquad -\pi \le x \le \pi. \tag{2.8}$$

We now investigate this case both for m being even and odd separately. One of the reason for considering as two separate cases is that the formulations of the solutions become easier to understand and to interpret. We handle both cases with the following theorems.

Theorem 2.2. *If p is a polynomial function of degree n, m is odd, and let*

$$\begin{aligned} q_0 &= p_0,\\ q_1 &= \begin{cases} \frac{-m\pi p_1}{2}, & m = 4n_0 - 1,\\ \frac{m\pi p_1}{2}, & m = 4n_0 + 1,\end{cases}\\ q_k &= \begin{cases} \frac{k}{k-1}\times\frac{k-2}{k-3}\times\cdots\times\frac{2}{1}\times p_k, & k \text{ is even},\\ \frac{k}{k-1}\times\frac{k-2}{k-3}\times\cdots\times\frac{3}{2}\times\frac{m\pi}{2\alpha}\times p_k, & k \text{ is odd},\end{cases}\end{aligned} \tag{2.9}$$

for $n_0 \in \mathbb{N}$, $k \ge 2$ and where $\alpha = \int_0^{m\pi/2}\cos(t)\, dt$, then $u(x) = \sum_{k=0}^{n} q_k x^k$ is a solution to (2.8).

Proof. For $n = 1$, $p(x) = p_0 + p_1 x$. We first assume that $m = 4n_0 - 1$ for some $n_0 \in \mathbb{N}$. We need to show that $u(x) = q_0 + q_1 x$ is a solution to (2.8).

$$\begin{aligned} p(x) &= \frac{2}{\pi}\int_0^{\pi/2}\left[p_0 - \frac{m\pi p_1}{2}x\cos(m\theta)\right] d\theta,\\ &= p_0 + p_1 x.\end{aligned}$$

For $n = k$, assume that $u(x) = q_0 + q_1 x + \ldots + q_k x^k$ is a solution to (2.8) where q_k's are given in (2.9). For $n = k + 1$, we only need to show that

$$p_{k+1} = \frac{2}{\pi}\int_0^{\pi/2} q_{k+1}\cos^{k+1}(m\theta)d\theta,$$

where q_{k+1} is given in (2.9).
If k is odd,

$$\begin{aligned}
I_{k+1} &= \frac{2}{\pi}\int_0^{\pi/2}\left[\frac{k+1}{k}\times\frac{k-1}{k-2}\times\cdots\times\frac{2}{1}\times p_{k+1}\right]\cos^{k+1}(m\theta)d\theta\\
&= \frac{2}{\pi}\left[\frac{k+1}{k}\times\frac{k-1}{k-2}\times\cdots\times\frac{2}{1}\times p_{k+1}\right]\int_0^{\pi/2}\cos^{k+1}(\theta)d\theta\\
&= \frac{1}{\sqrt{\pi}}\left[\frac{k+1}{k}\times\frac{k-1}{k-2}\times\cdots\times\frac{2}{1}\times p_{k+1}\right]\frac{\Gamma\left(\frac{k}{2}+1\right)}{\Gamma\left(\frac{k+1}{2}+1\right)}\\
&= p_{k+1}.
\end{aligned}$$

Here we use the fact that

$$\int_0^{\pi/2}\cos^{k+1}(m\theta)d\theta = \int_0^{\pi/2}\cos^{k+1}(\theta)d\theta.$$

If k is even,

$$\begin{aligned}
I_{k+1} &= \frac{2}{\pi}\int_0^{\pi/2}\left[\frac{k+1}{k}\times\frac{k-1}{k-2}\times\cdots\times\frac{3}{2}\times\frac{m\pi}{2\alpha}\times p_{k+1}\right]\cos^{k+1}(m\theta)d\theta,\\
&= -\frac{2}{m\pi}\left[\frac{k+1}{k}\times\frac{k-1}{k-2}\times\cdots\times\frac{3}{2}\times\frac{m\pi}{2\alpha}\times p_{k+1}\right]\int_0^{\pi/2}\cos^{k+1}(\theta)d\theta,\\
&= -\frac{1}{m\sqrt{\pi}}\left[\frac{k+1}{k}\times\frac{k-1}{k-2}\times\cdots\times\frac{3}{2}\times\frac{m\pi}{2\alpha}\times p_{k+1}\right]\frac{\Gamma\left(\frac{k}{2}+1\right)}{\Gamma\left(\frac{k+1}{2}+1\right)}\\
&= p_{k+1}.
\end{aligned}$$

Here we use the fact that

$$\int_0^{\pi/2}\cos^{k+1}(m\theta)d\theta = -\frac{1}{m}\int_0^{\pi/2}\cos^{k+1}(\theta)d\theta.$$

The case $m = 4n_0 + 1$ can be shown analogously. □

Theorem 2.3. *If p is an even polynomial function, m is even, and let*

$$\begin{aligned}
q_0 &= p_0,\\
q_k &= \begin{cases}\frac{k}{k-1}\times\frac{k-2}{k-3}\times\cdots\times\frac{2}{1}\times p_k, & k \text{ is even},\\ 0, & k \text{ is odd},\end{cases}
\end{aligned}$$

for $k \geq 1$, then $u(x) = \sum_{k=0}^{n} q_k x^k$ is a solution to (2.8).

The idea for the proof of this theorem is similar to the proof of the above theorems. These two theorems have important corollaries. Before we state a pair of them here, we assume that the equation that we consider is in the form of equation (2.8) and there exits a polynomial solution.

Corollary 2.4. *If p is an even polynomial function, then u is an even function.*

Corollary 2.5. *If p is an odd polynomial function, then u is an odd function.*

2.2. The generalized Schlömilch-type integral equation

We define the generalized Schlömilch integral equation as follows:

$$p(x) = \frac{2}{\pi}\int_0^{\pi/2} u(x\cos^r\theta)\,d\theta, \quad -\pi \le x \le \pi \quad \text{and} \quad r > 1. \tag{2.10}$$

The algorithm for the solution is similar to the case considered above. The power r will be a multiplier for the indices when evaluating corresponding q_k's. More precisely, we have the following theorem.

Theorem 2.6. *If p is a polynomial function of degree n, then*

$$u(x) = \sum_{j=0}^{n} q_j x^j$$

is a solution of (2.10)*, where*

$$\begin{aligned} q_0 &= p_0, \\ q_k &= s_{k\times r} \times p_k, \quad \textit{and} \\ s_k &= \begin{cases} \frac{k}{k-1} \times \frac{k-2}{k-3} \times \cdots \times \frac{2}{1}, & k \textit{ even}, \\ \frac{k}{k-1} \times \frac{k-2}{k-3} \times \cdots \times \frac{3}{2} \times \frac{\pi}{2}, & k \textit{ odd} \end{cases} \end{aligned}$$

for $k \ge 1$ and for $r \in \mathbb{Z}^+ - \{1\}$.

2.3. The nonlinear Schlömilch-type integral equation

We consider the nonlinear Schlömilch integral equation which has the following form:

$$p(x) = \frac{2}{\pi}\int_0^{\pi/2} F(u(x\cos\theta))\,d\theta, \quad -\pi \le x \le \pi, \tag{2.11}$$

where $F(u(x\cos\theta))$ is a nonlinear function of $u(x\cos\theta)$.
We assume that F is invertible so that letting that $F(u(x\cos\theta)) = v(x\cos\theta)$ will imply that $u(x\cos\theta) = F^{-1}(v(x\cos\theta))$. Thus, with this transformation, (2.11) becomes

$$p(x) = \frac{2}{\pi}\int_0^{\pi/2} v(x\cos\theta)\,d\theta, \tag{2.12}$$

which is equivalent to (1.1). We solve this equation for v and then use the inverse transform F^{-1} to get u.

3. Examples

In this section, we provide some numerical examples that enables us to test the solution algorithms that we obtained in the previous sections. In order to be able to compare the algorithms and solutions with the existing ones, some of the examples are taken from the literature.

Example 3.1. Consider the following Schlömilch-type integral equation (*cf.* [1, 20]):

$$1 + 2x = \frac{2}{\pi}\int_0^{\pi/2} u(x\cos\theta)\,d\theta, \quad -\pi \le x \le \pi. \tag{3.1}$$

The data function $p(x) = 1 + 2x$. The equation belongs to the standard Schlömilch-type integral equation. We then use the algorithm given in equation (2.7) and obtain that

$$q_0 = p_0 = 1 \ \text{ and } \quad q_1 = \frac{\pi}{2}p_1 = \pi.$$

Thus, $u(x) = q_0 + q_1 x = 1 + \pi x$.
It can easily be verified that

$$u(x) = 1 + \pi x$$

is a solution of (3.1).
This is exactly the same solution obtained in [1, 20].

Example 3.2. Consider the following Schlömilch-type integral equation:

$$x - x^3 = \frac{2}{\pi}\int_0^{\pi/2} u(x\cos\theta)\, d\theta, \quad -\pi \le x \le \pi. \tag{3.2}$$

The data function $p(x) = x - x^3$. The equation is of the form of the standard Schlömilch-type integral equation. We then use the algorithm given in equation (2.7) and obtain that

$$\begin{aligned} q_0 &= p_0 = 0, \\ q_1 &= \frac{\pi}{2} p_1 = \frac{\pi}{2}, \\ q_2 &= 2p_2 = 0, \\ q_3 &= \frac{3\pi}{4} p_3 = -\frac{3\pi}{4}. \end{aligned}$$

Thus,

$$\begin{aligned} u(x) &= q_0 + q_1 x + q_2 x^2 + q_3 x^3 \\ &= \frac{\pi}{2}x - \frac{3\pi}{4}x^3. \end{aligned}$$

One can easily verify that

$$u(x) = \frac{\pi}{2}x - \frac{3\pi}{4}x^3$$

is a solution of (3.2).

Example 3.3. Consider the following Schlömilch-type integral equation:

$$\frac{1}{2}x^2 + \frac{1}{3}x^3 = \frac{2}{\pi}\int_0^{\pi/2} u(x\cos(3\theta))\, d\theta, \quad -\pi \le x \le \pi. \tag{3.3}$$

This equation is in the form of standard Schlömilch-type integral equation with that $m = 3$ and $p_0 = p_1 = 0, p_2 = \frac{1}{2}$ and $p_3 = \frac{1}{3}$. The hypothesis of theorem (2.2) are satisfied. It is easy to see that

$$q_0 = q_1 = 0, \;\; q_2 = 1 \quad \text{and} \quad q_3 = \frac{-3\pi}{4}$$

from the theorem (2.2).
This in turn implies that

$$u(x) = x^2 - \frac{3\pi}{4}x^3.$$

By a simple substitution, it can be easily shown that

$$u(x) = x^2 - \frac{3\pi}{4}x^3$$

is a solution to (3.3).

Example 3.4. Consider the following Schlömilch-type integral equation (*cf.* [20]):

$$\frac{1}{2}x^2 = \frac{2}{\pi}\int_0^{\pi/2} u(x\cos(2\theta))\,d\theta, \quad -\pi \le x \le \pi. \tag{3.4}$$

This equation belongs to standard Schlömilch-type integral equation with $m = 2$ and $p_0 = p_1 = 0$ and $p_2 = \frac{1}{2}$. If we use theorem (2.3), we obtain

$$q_0 = q_1 = 0 \quad \text{and} \quad q_2 = 1.$$

This in turn implies that

$$u(x) = x^2$$

is a solution to (3.4). It is also very easy to verify that $u(x) = x^2$ is a solution to (3.4).

Example 3.5. Consider the following integral equation.

$$1 + x + x^2 = \frac{2}{\pi}\int_0^{\pi/2} u(x\cos^2\theta)\,d\theta, \quad -\pi \le x \le \pi. \tag{3.5}$$

This equation is of the form of generalized Schlömilch-type integral equation and $p_0 = 1, p_1 = 1$, and $p_2 = 1$.
By using theorem (2.6), we obtain

$$\begin{aligned} q_0 &= p_0 = 1, \\ q_1 &= s_2 \times p_1 = 2, \\ q_2 &= s_4 \times p_2 = \frac{8}{3}. \end{aligned} \tag{3.6}$$

Thus, $u(x) = 1 + 2x + \frac{8}{3}x^2$ is a solution of (3.5).

Example 3.6. Consider the following nonlinear Schlömilch-type integral equation:

$$2x^2 = \frac{2}{\pi}\int_0^{\pi/2} u^2(x\cos\theta)\,d\theta, \quad -\pi \le x \le \pi. \tag{3.7}$$

Since it is a nonlinear equation, we replace u^2 with v to transform the equation into

$$2x^2 = \frac{2}{\pi}\int_0^{\pi/2} v(x\cos\theta)\,d\theta, \quad -\pi \le x \le \pi, \tag{3.8}$$

which becomes a linear equation. It turns out that the equation (3.8) belongs to the family of standard linear Schlömilch-type integral equation. We use the algorithm given in equation (2.7) and obtain that

$$\begin{aligned} q_0 &= p_0 = 0, \\ q_1 &= \frac{\pi}{2}p_1 = 0, \\ q_2 &= 2p_2 = 4. \end{aligned} \tag{3.9}$$

Thus, $v(x) = q_0 + q_1 x + q_2 x^2 = 4x^2$ is a solution to (3.8). Reintroducing the original variable u yields that

$$u(x) = \pm 2x$$

are solutions to (3.7).

4. An application problem

Consider the following second order nonlinear ordinary differential equation (*cf.* [1, 16])

$$\phi'' + F(\phi') + \phi = 0. \tag{4.1}$$

A polar form of this equation admits the form:

$$J(x) = \frac{2}{\pi} \int_0^{\pi/2} \cos\theta\, u(x\cos\theta)\, d\theta. \tag{4.2}$$

When $F(x) = x - \frac{x^3}{3}$ in equation (4.1) or $u(x) = x - \frac{x^3}{3}$ gives the Rayleigh equation

$$J(r) = \frac{1}{2\pi} \int_0^{2\pi} r\cos\theta - \frac{r^3}{8} cos^3\theta\, d\theta. \tag{4.3}$$

The Rayleigh equation is considered as one of the most important equation in fluid dynamics. It covers a broad range of applications from different branches of science and engineering. These involve an analytic method expressing liquid compositions in Rayleigh fractionation for ternary systems (*cf.* [21]). The extension of Rayleigh equation is used to improve quantification of biodegradation (*cf.* [6]). See for more details on this equation [10, 12, 13].

We can use the idea and methodology introduced in this paper for solving equation (4.2). We start with the equation

$$J(x) = \frac{2}{\pi} \int_0^{\pi/2} \cos\theta\, u(x\cos\theta)\, d\theta,$$

where $J(x) = \frac{x}{8}(4 - x^2)$ is used for the Rayleigh equation (*cf.* [16]).

Since J is a polynomial function, we expect that a solution of a polynomial form exists. In addition, we also expect that the solution is an odd function.

Alternatively, one can use one of the above theorems by modifying the algorithms because of the fact that appearance of $\cos\theta$ in the integrand will shift the coefficients. Direct calculation yields that

$$u(x) = q_1 x + q_3 x^3.$$

Evaluating the numerical values, we end up with

$$q_1 = 2 \times p_1 = 1$$
$$q_3 = \frac{8}{3} \times p_3 = -\frac{1}{3}.$$

As a result, the solution becomes $u(x) = q_1 x + q_3 x^3 = x - \frac{x^3}{3}$. It is easy to verify that u is a solution to equation (4.2).

5. Conclusion

In this work, the Schlömilch-type integral equations have been classified and investigated. For each type, the algorithm for its solution is given and supported by numerical examples. We also show that by using the methodology and idea in this work one can solve an application problem that leads to Rayleigh equation.

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