



A survey on linear recurrent sequences associated with arithmetic triangles

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Abstract

The present survey investigates linear recurrences derived from sums of elements located on the transversals of some classical arithmetic triangles. We establish the corresponding linear recurrence relations, generating functions and deduce combinatorial identities. Finally, we provide the sums of the general term of the concerned linear recurrence sequences.

Keywords: Arithmetic triangle, recurrence relation, generating function, partial sums, triangle with pascal rule, biⁿomial coefficient triangle, Catalan triangle, triangle with Delannoy rule, quasi s -Delannoy triangle, Rhombus triangle

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1. Introduction


A generating function is a mathematical tool used in combinatorics, number theory, probability, and other areas of mathematics to transform problems about sequences into problems about formal series of functions (see [31, 45, 48]). It's a powerful way to manipulate sequences, coefficients, or terms of a sequence by representing them as coefficients of a formal power series or a polynomial. There are different types of generating functions, two primary types are (see [31, 48]):


- **Ordinary Generating Functions (OGF):** These represent a sequence of numbers by forming a power series where each term's coefficient corresponds one to one to a term in the sequence.
- **Exponential Generating Functions (EGF):** Similar to OGFs using factorials or exponential functions to encode sequences where combinatorial or counting problems are involved.

There is also other kinds of generating functions; we just need a way to extract the sequence from the coefficients of the given generating function.

Arithmetic triangles are a fundamental concept in discrete mathematics where numbers are arranged in a triangular pattern, following a specific rule to get the terms of the triangle. There are different types of arithmetic triangles, including Binomial Coefficient Triangle (BCT), Delannoy Triangle, and other variations. The BCT starts with 1 at

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the top and a sequence of zeros on the right and left sides of 1. Each number is the sum of the two numbers directly above it, forming a pattern of binomial coefficients $\binom{n}{k}$.

The triangle is generated using the rule, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, with $\binom{n}{0} = 1$ and the convention $\binom{n}{k} = 0$ for $k \notin \{1, \dots, n\}$, known as Pascal rule, as shown in Figure 1.

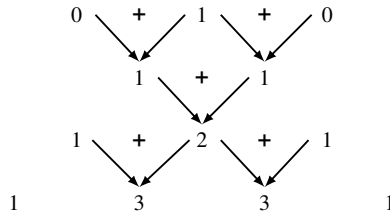


Figure 1: Construction of Binomial Coefficient Triangle

The Binomial Coefficient Triangle is shown in Figure 2.

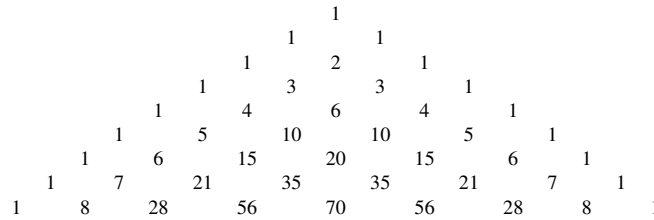


Figure 2: Binomial Coefficient Triangle (BCT)

BCT is a rich source for interesting integer sequences. For example, the Fibonacci sequence F_n is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, with initial conditions $F_0 = 0$ and $F_1 = 1$. This sequence and its generalizations possess many interesting combinatorial properties (see [35]). Moreover, it can be obtained as the sum of its diagonals, as shown in Figure 3, for $n \geq 0$, (see [40])

$$F_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}, \text{ with } F_0 = 0.$$

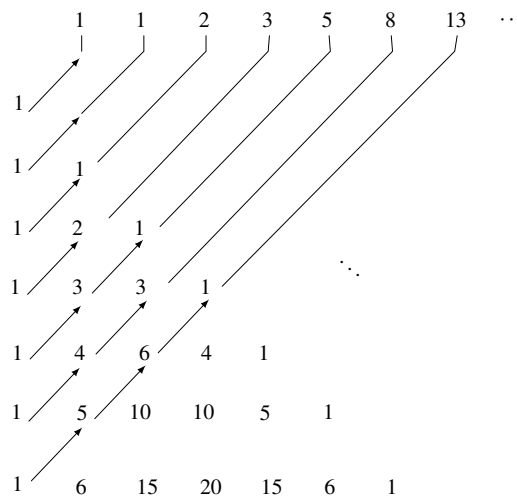


Figure 3: Fibonacci numbers in Binomial Coefficient Triangle

Lucas Triangle

Also, with the same rule, the concatenation of two consecutive BCT's yields the Lucas Triangle, generated by $\binom{n}{k}_L = \binom{n}{k} + \binom{n-1}{k-1}$, for $k \geq 1$. In other words, this approach suggests Lucas Triangle from two juxtaposed BCT's, see Figure 4.

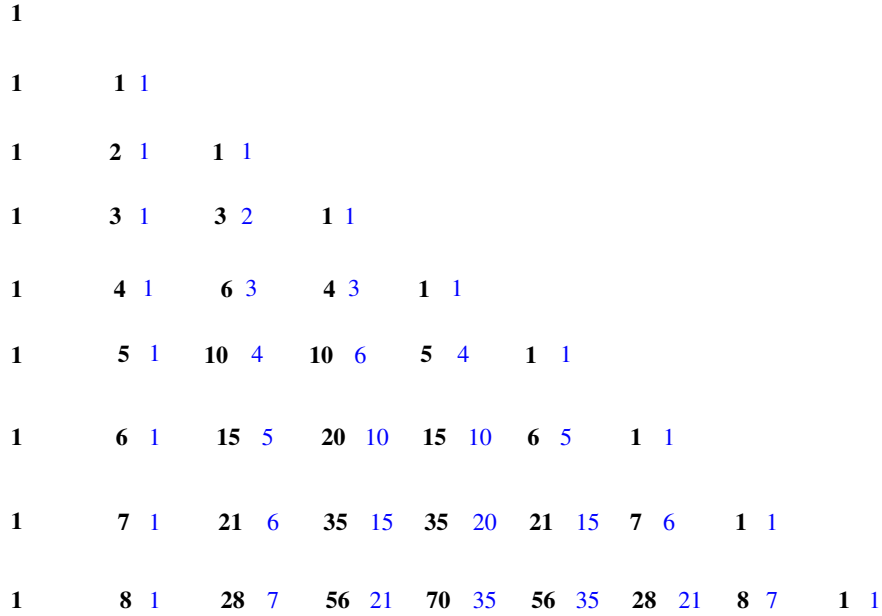


Figure 4: Juxtaposition of two BCT's gives Lucas Triangle

The Lucas Triangle is a weighted variant of the BCT, where each entry is defined by $\binom{n}{k}_L = \frac{n}{n-k} \binom{n}{k}$. The main difference between the two triangles is that the Lucas Triangle starts with 2 instead of 1 and has a leg of 2's. It is illustrated in Figure 5 as follows:

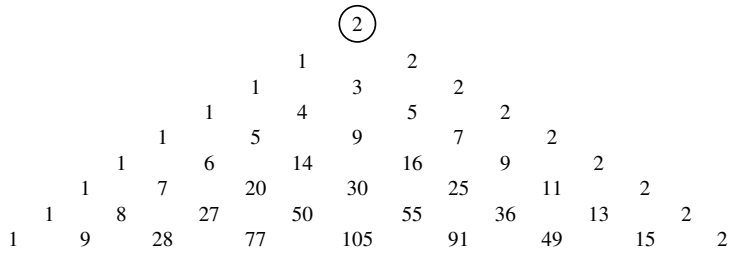


Figure 5: Lucas Triangle
The top value of the triangle is just an adjustment

The sum of the terms along the n th diagonal of the Lucas Triangle is equal to the n th term in the Lucas sequence (see Figure 6). This sequence, although similar to the Fibonacci sequence, has distinct initial conditions $L_0 = 2$, $L_1 = 1$ and follows the same recurrence relation $L_n = L_{n-1} + L_{n-2}$. These two sequences are intimately linked through the equation (for instance, see [47])

$$L_n = F_{n+1} + F_{n-1}.$$

The Lucas numbers is given explicitly by the following formula, for $n \geq 1$, (see [22])

$$L_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k}.$$

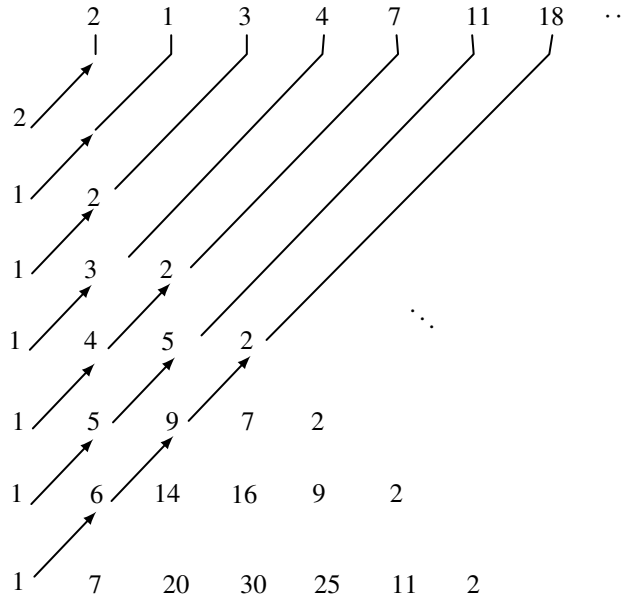


Figure 6: Lucas numbers in Lucas Triangle

Going to the BC weighted triangle

The variables x and y serve as weights; x represents the weight in the vertical or left direction, while y represents the weight in the horizontal or right direction, that are given for any positive integer $n, k \geq 1$ by the following relation:

$$\binom{n}{k}_{x,y} = x \binom{n-1}{k}_{x,y} + y \binom{n-1}{k-1}_{x,y} = \binom{n}{k} x^{n-k} y^k,$$

with $\binom{n}{0}_{x,y} = x^n, \binom{n}{n}_{x,y} = y^n$ with $\binom{n}{k}_{x,y} = 0$, for $k > n$ or $k < 0$.

The Bivariate Binomial Coefficient Triangle is illustrated by Figure 7.

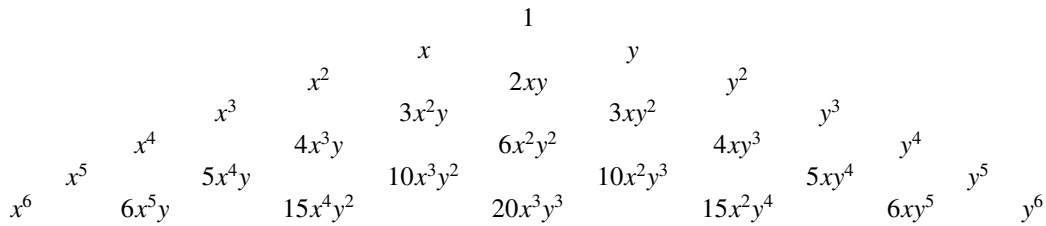


Figure 7: Bivariate Binomial Coefficients Triangle

For $x = y = 1$: $\binom{n}{k}_{1,1} = \binom{n}{k}$, we get the classical BCT.

Similarly, summing the elements along the rising diagonal lines in the weighted BCT (Bivariate BCT) yields the bivariate Fibonacci polynomials, as illustrated in Figure 8. For $n \geq 0$, we get

$$F_{n+1}(x, y) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}_{x,y} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^{n-2k} y^k.$$

These polynomials satisfy the recurrence relation (for more details, see [14])

$$F_n(x, y) = xF_{n-1}(x, y) + yF_{n-2}(x, y) \quad (n \geq 2), \text{ with } F_0(x, y) = 0 \text{ and } F_1(x, y) = 1.$$

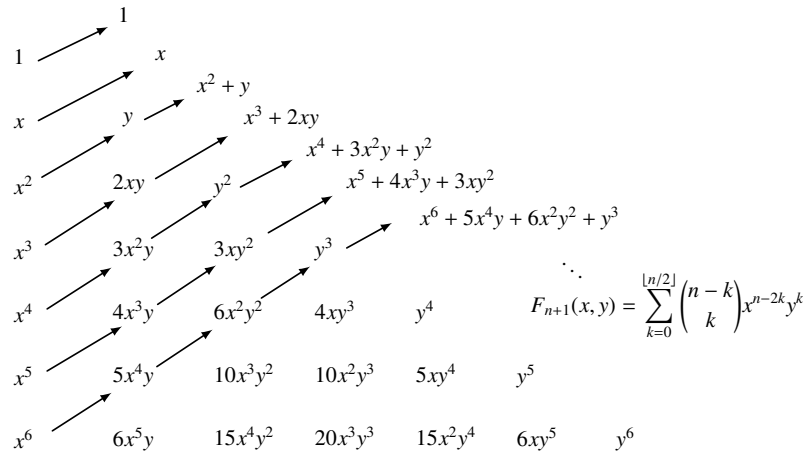


Figure 8: Bivariate Fibonacci polynomials in weighted BCT

In the following sections, we introduce the concept of directions in arithmetic triangles.

2. Directions in arithmetic triangles

Let $\langle \binom{n}{k} \rangle$ denote an Arithmetic Triangle, where $0 \leq k \leq n$.

Definition 2.1. For $n \in \mathbb{N} \cup \{0\}$, the sequence $\langle \binom{n-rk}{\beta+ak} \rangle$ defines the elements positioned along a specified direction (α, r, β) , with $r \in \mathbb{Z}$, $\alpha \in \mathbb{N}$, $0 \leq \beta < \alpha$, and $\alpha + r > 0$. The triple (α, r, β) represents α steps east and r steps north (for r positive) or $-r$ steps south (for r negative), and β indicates the position of the first element starting the line direction. The variables x and y serve as weights; x represents the weight in the vertical (or left) direction, while y represents the weight in the horizontal (or right) direction.

The condition $\alpha + r > 0$ suggests that the number of elements lying on the transversals is finite (see Figure 9, Figure 10); otherwise, it is infinite (see Figure 11).

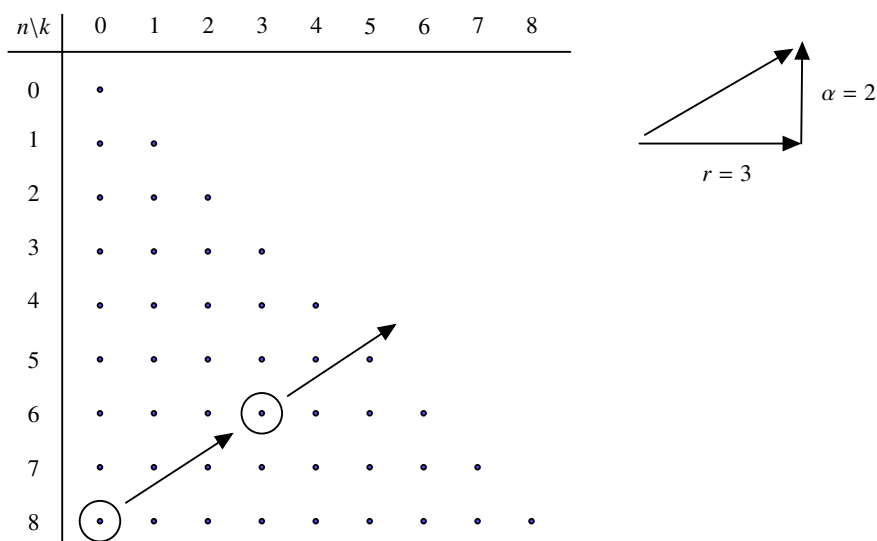


Figure 9: Direction (3, 2, 0) on an arithmetic triangle

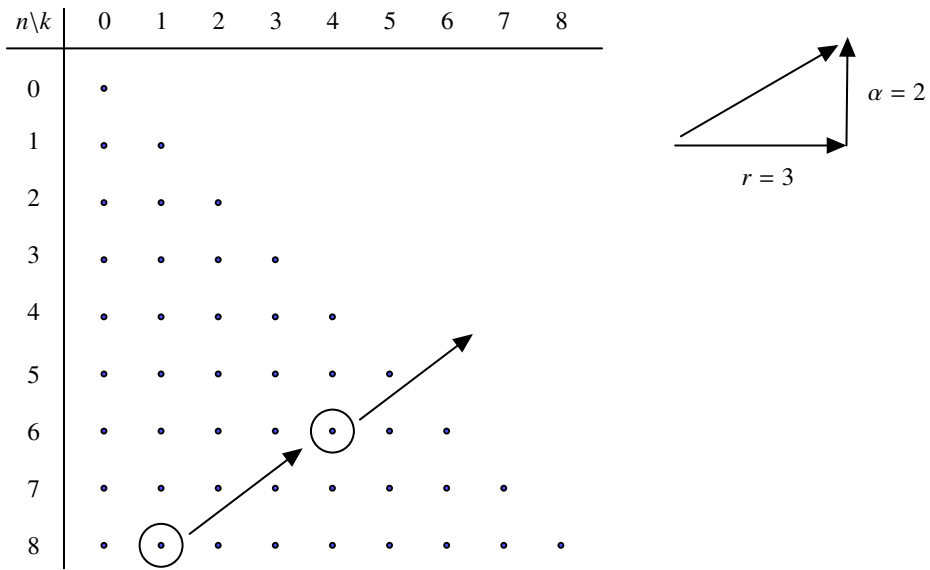


Figure 10: Direction (3, 2, 1) on an arithmetic triangle

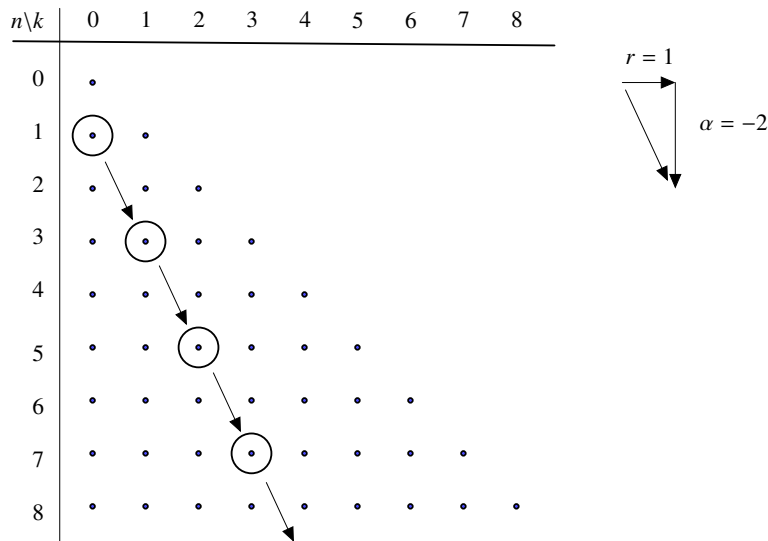


Figure 11: Direction (1, -2, 0) on an arithmetic triangle

In the following section, we explore various types of triangles that follow Pascal's Rule.

3. Triangles with Pascal's Rule

Triangles that follow Pascal's Rule are sequences of numbers arranged in a triangular pattern where each number is the sum of the two numbers directly above it. The most well-known example is BCT, which generates binomial coefficients. The parameters introduced in Section 2 deserves for the present section.

3.1. Directions in Binomial Coefficient Triangle (BCT)

The elements lying over transversals in the Binomial Coefficient Triangle can be expressed, for $0 \leq k \leq \lfloor \frac{n-\beta}{\alpha+r} \rfloor$, in the form

$$\binom{n-rk}{\beta+\alpha k} x^{n-\beta-(\alpha+r)k} y^{\beta+\alpha k}.$$

Let $u_n^{(\alpha,r,\beta)}$ be the general term of the sum of elements situated on all finite directions of the BCT. For brevity, we denote it by u_n . For $n \geq 0$, we define

$$u_{n+1} := \sum_{k=0}^{\lfloor (n-\beta)/(\alpha+r) \rfloor} \binom{n-rk}{\beta+\alpha k} x^{n-\beta-(\alpha+r)k} y^{\beta+\alpha k}, \text{ with } u_0 = 0. \tag{3.1}$$

In the following subsection, we study the case $\alpha = 1$ (thus $\beta = 0$) for fixed r .

3.1.1. Raab direction (1, r, 0)

Raab [40] introduced and explored the sums of elements lying along direction (1, r, 0), and provided the corresponding explicit relation,

$$u_{n+1}^{(1,r,0)} = F_{r,n+1}(x, y) = \sum_{k=0}^{\lfloor n/(r+1) \rfloor} \binom{n-rk}{k} x^{n-(r+1)k} y^k.$$

This satisfies the linear recurrence relation:

$$F_{r,n+1}(x, y) = xF_{r,n}(x, y) + yF_{r,n-r}(x, y), \quad (n \geq r),$$

with $F_{r,0}(x, y) = 0$ and $F_{r,i}(x, y) = x^{i-1}$ for $i = 1, \dots, r$. This sequence is known as the bivariate r -Fibonacci sequence.

The associated generating function is given by (see for instance [1, 20])

$$U(z) := \sum_{n \geq 0} F_{r,n+1}(x, y) z^n = \frac{1}{1 - xz - yz^{r+1}}.$$

In the following theorem, we provide the formula for the sum of the general term of the sequence $(F_{r,n}(x, y))_n$.

Theorem 3.1. *The sum of terms of the sequence $(F_{r,n}(x, y))_n$, is given by*

$$\sum_{j=0}^n F_{r,j}(x, y) = \frac{F_{r,n+r+1}(x, y) - F_{r,r}(x, y)}{x + y - 1} + \frac{1 - x}{x + y - 1} \left(\sum_{j=0}^{r-1} (F_{r,n+j+1}(x, y) - F_{r,j}(x, y)) \right),$$

with $x + y \neq 1$ and the convention that the empty sum is 0.

Proof. The proof follows by induction. □

For $r = 1$, we get the following result.

Corollary 3.2. *The sum of terms of the bivariate Fibonacci polynomials, $F_n(x, y)$, is given by*

$$\sum_{j=0}^n F_j(x, y) = \frac{F_{n+2}(x, y) + (1 - x)F_{n+1}(x, y) - 1}{x + y - 1}.$$

For $r = 2$, we get the following result.

Corollary 3.3. *The sum of terms of the sequence $(F_{2,n}(x, y))_n$, is given by*

$$\sum_{j=0}^n F_{2,j}(x, y) = \frac{F_{2,n+3}(x, y) + (1 - x)(F_{2,n+1}(x, y) + F_{2,n+2}(x, y)) - 1}{x + y - 1}.$$

Companion sequences

The family of companion sequences associated with the r -Fibonacci sequence is the r -Lucas sequences of type s , where $1 \leq s \leq r$. For more details, see [1].

Definition 3.4 (cf. [1]). For any integers n, r and s ($1 \leq s \leq r$), the companion sequence family of $(F_{r,n}(x, y))_n$ is defined by the following recurrence

$$\begin{cases} L_0^{(r,s)}(x, y) = s + 1, L_k^{(r,s)}(x, y) = x^k, & (1 \leq k \leq r), \\ L_{n+1}^{(r,s)}(x, y) = xL_n^{(r,s)}(x, y) + yL_{n-r}^{(r,s)}(x, y), & (n \geq r). \end{cases}$$

The terms of the sequence $(L_n^{(r,s)}(x, y))_n$ are called r -Lucas polynomials of type s .

Remark 3.5 (cf. [1]). Note that when $s = 0$, we get the shifted bivariate r -Fibonacci sequence.

The following theorem gives an explicit formula for $(L_n^{(r,s)}(x, y))$ in terms of s and $(F_{r,n}(x, y))$.

Theorem 3.6 ([1]). Let r and s be nonnegative integers such that $1 \leq s \leq r$. For $n \geq r$, we have the following

$$L_n^{(r,s)}(x, y) = F_{r,n+1}(x, y) + syF_{r,n-r}(x, y).$$

We also obtain the explicit form for $n \geq 1$, given by the following expression

Theorem 3.7 (cf. [1]). For $n \geq 1$, we have

$$L_n^{(r,s)}(x, y) = \sum_{k=0}^{\lfloor \frac{n}{r+1} \rfloor} \frac{n - (r - s)k}{n - rk} \binom{n - rk}{k} x^{n-(r+1)k} y^k.$$

In the following, we give the generating function of $(L_n^{(r,s)}(x, y))_n$.

Theorem 3.8 (cf. [1]). The generating function of the sequence $(L_n^{(r,s)}(x, y))_{n \geq 0}$ is given by

$$V(z) = \sum_{n \geq 0} L_n^{(r,s)}(x, y) z^n = \frac{(1 + s) - sxz}{1 - xz - yz^{r+1}}.$$

In the following theorem, we provide the formula for the sum of the general term of the sequence $(L_n^{(r,s)}(x, y))_n$.

Theorem 3.9. The sum of terms of the sequence $(L_n^{(r,s)}(x, y))_n$, is given by

$$\sum_{j=0}^n L_j^{(r,s)}(x, y) = \frac{L_{n+r+1}^{(r,s)}(x, y) - L_r^{(r,s)}(x, y)}{x + y - 1} + \frac{1 - x}{x + y - 1} \left(\sum_{j=0}^{r-1} (L_{n+j+1}^{(r,s)}(x, y) - L_j^{(r,s)}(x, y)) \right),$$

with $x + y \neq 1$ and the convention that the empty sum is 0.

Proof. The proof follows by induction. □

For $r = 1$, we get the following result.

Corollary 3.10. The sum of terms of the bivariate Lucas polynomials of type s , $L_n^{(1,s)}(x, y)$, is given by

$$\sum_{j=0}^n L_j^{(1,s)}(x, y) = \frac{L_{n+2}^{(1,s)}(x, y) + (1 - x)L_{n+1}^{(1,s)}(x, y) + (x - 1)s - 1}{x + y - 1}.$$

3.1.2. Direction (α, r, β)

Now, we deal with the general case, starting with $r \geq 0$. According to (3.1), we have the following result.

Theorem 3.11 (cf. [20]). *The sequence $(u_n)_n$ satisfies the following linear recurrence relation, given by*

$$u_n - \binom{\alpha}{1} x u_{n-1} + \binom{\alpha}{2} x^2 u_{n-2} + \dots + (-1)^\alpha \binom{\alpha}{\alpha} x^\alpha u_{n-\alpha} = y^\alpha u_{n-r-\alpha}, \quad (n \geq \alpha + r),$$

with $u_0 = u_1 = \dots = u_\beta = 0$ and $u_{k+1} = \binom{k}{\beta} x^{k-\beta} y^\beta$ for $\beta \leq k \leq \beta + \alpha + r - 1$.

In the case where $r < 0$, the sequence $(u_n)_n$ is of order α for any r ($-\alpha < r \leq 0$), and the coefficient y^α of $u_{n-\alpha-r}$ is subtracted from one of the coefficients of the terms u_{n-1}, \dots, u_{n-r} . Otherwise, the terms of a sequence can overlap. This is what we call the **Morgan-Voyce phenomenon**, see [2].

The second recurrence sequence, satisfied when $r < 0$, is given by the following theorem.

Theorem 3.12 (cf. [2, 20]). *The sequence $(u_n)_n$ satisfies the following linear recurrence relation, given by*

$$u_n - \binom{\alpha}{1} x u_{n-1} + \dots + \left((-x)^{r+\alpha} \binom{\alpha}{\alpha+r} - y^\alpha \right) u_{n-\alpha-r} + \dots + (-1)^\alpha \binom{\alpha}{\alpha} x^\alpha u_{n-\alpha} = 0, \quad (n \geq \alpha),$$

with $u_0 = u_1 = \dots = u_\beta = 0$ and $u_{k+1} = \binom{k}{\beta} x^{k-\beta} y^\beta$ for $\beta \leq k \leq \beta + \alpha$.

Theorem 3.13 (cf. [20]). *The generating function associated to the sum of elements lying along a finite ray crossing the Binomial Coefficient Triangle is given by*

$$U(z) = \sum_{n \geq 0} u_{n+1} z^n = \frac{y^\beta z^\beta (1 - xz)^{\alpha-\beta-1}}{(1 - xz)^\alpha - y^\alpha z^{\alpha+r}}.$$

We provide another approach to prove this result since the original paper contains a misprint as the power of z is $\beta + 1$. We need the following lemma.

Lemma 3.14 (cf. [20]). *For $s \geq k$, we have*

$$\sum_{j=0}^k (-1)^j \binom{\beta+s}{k-j} \binom{\beta+j}{j} = \binom{s-1}{k}.$$

Here we propose a direct way to establish Theorem 3.13 as an alternative to the one given in [20].

Proof of Theorem 3.13. Let consider the generating function of $(u_n)_{n \geq 0}$ given by $U(z) = \sum_{n \geq 0} u_{n+1} z^n$. We know that the linear recurrence relation of u_n is of the form (see [13]):

$$u_n = \sum_{j=1}^{\alpha+r} (-1)^{j-1} a_j u_{n-j},$$

where

$$a_j = \begin{cases} x^j \binom{\alpha}{j}, & \text{for } j = 1, \dots, \alpha + r - 1, \\ y^\alpha (-1)^{j+1}, & \text{for } j = \alpha + r. \end{cases}$$

Then, we have

$$\begin{aligned} U(z) &= \sum_{n=0}^{\alpha+r-1} u_{n+1}z^n + \sum_{n \geq \alpha+r} u_{n+1}z^n \\ &= \sum_{n=0}^{\alpha+r-1} u_{n+1}z^n + \sum_{n \geq \alpha+r} \left(\sum_{j=1}^{\alpha+r} (-1)^{j-1} a_j u_{n+1-j} \right) z^n \\ &= \sum_{n=0}^{\alpha+r-1} u_{n+1}z^n + \sum_{j=1}^{\alpha+r} (-1)^{j-1} a_j z^j \sum_{n \geq \alpha+r} u_{n+1-j} z^{n-j} \\ &= \sum_{n=0}^{\alpha+r-1} u_{n+1}z^n + \sum_{j=1}^{\alpha+r} (-1)^{j-1} a_j z^j \left(U(z) - \sum_{n=0}^{\alpha+r-j-1} u_{n+1}z^n \right). \end{aligned}$$

Thus

$$\left(\sum_{j=0}^{\alpha+r} (-1)^j a_j z^j \right) U(z) = \sum_{j=0}^{\alpha+r} (-1)^j a_j z^j \left(\sum_{n=0}^{\alpha+r-j-1} u_{n+1}z^n \right) = \sum_{k=0}^{\alpha+r-1} \left(\sum_{j=0}^k (-1)^j a_j u_{k-j+1} \right) z^k.$$

Therefore

$$U(z) = \frac{\sum_{k=0}^{\alpha+r-1} \left(\sum_{j=0}^k (-1)^j a_j u_{k-j+1} \right) z^k}{\sum_{j=0}^{\alpha+r} (-1)^j a_j z^j} = \frac{A(z)}{B(z)}.$$

Let evaluate $A(z)$ and $B(z)$, we have

$$B(z) = \sum_{j=0}^{\alpha+r} (-1)^j a_j z^j = \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} x^j z^j + y^\alpha z^{\alpha+r} = (1 - xz)^\alpha - y^\alpha z^{\alpha+r}$$

and

$$\begin{aligned} A(z) &= \sum_{k=0}^{\alpha+r-1} \left(\sum_{j=0}^k (-1)^j a_j u_{k-j+1} \right) z^k \\ &= \sum_{k=0}^{\alpha+r-1} \left(\sum_{j=0}^k (-1)^j \binom{\alpha}{j} x^j \binom{k-j}{\beta} y^\beta x^{k-j-\beta} \right) z^k \\ &= y^\beta \sum_{k=0}^{\alpha+r-1} \left(\sum_{j=0}^k (-1)^j \binom{\alpha}{j} \binom{k-j}{\beta} \right) x^{k-\beta} z^k. \end{aligned}$$

We have

$$\sum_{j=0}^k (-1)^j \binom{\alpha}{j} \binom{k-j}{\beta} = (-1)^k \sum_{j=\beta}^k (-1)^j \binom{\alpha}{k-j} \binom{j-\beta+\beta}{\beta} = (-1)^{k-\beta} \sum_{j=0}^{k-\beta} (-1)^j \binom{\alpha}{k-\beta-j} \binom{j+\beta}{\beta}.$$

We apply Lemma 3.14, we obtain

$$A(z) = y^\beta \sum_{k=0}^{\alpha+r-1} (-x)^{k-\beta} z^k \binom{\alpha-\beta-1}{k-\beta} = (yz)^\beta \sum_{l=0}^{\alpha+r-\beta-1} (-xz)^l \binom{\alpha-\beta-1}{l} = (yz)^\beta (1 - xz)^{\alpha-\beta-1}.$$

The desired result is obtained. □

A bivariate generating function

Note that for $u = xz$ and $v = \frac{y^\alpha}{x^{\alpha+r}}$, we obtain the following result.

Corollary 3.15. The bivariate generating function of the sequence $\left\{\binom{n-rk}{\beta+\alpha k}\right\}_{n,k}$ is given by

$$\sum_{n,k} \binom{n-rk}{\beta+\alpha k} u^n v^k = \frac{u^\beta(1-u)^{\alpha-\beta-1}}{(1-u)^\alpha - vu^{\alpha+r}}.$$

Proof. Note that

$$\frac{u^\beta(1-u)^{\alpha-\beta-1}}{(1-u)^\alpha - vu^{\alpha+r}} = \frac{u^\beta}{(1-u)^{\beta+1}} \frac{1}{1 - \frac{u^{\alpha+r}}{(1-u)^\alpha} v}.$$

This gives that

$$\frac{u^\beta(1-u)^{\alpha-\beta-1}}{(1-u)^\alpha - vu^{\alpha+r}} = \frac{u^\beta}{(1-u)^{\beta+1}} \sum_{n \geq 0} \left(\frac{u^{\alpha+r}}{(1-u)^\alpha}\right)^n v^n = \sum_{n \geq 0} \left(\frac{u^{(\alpha+r)n+\beta}}{(1-u)^{\alpha n+\beta+1}}\right) v^n.$$

It is well known that

$$\sum_{n \geq 0} \binom{n-rk}{\beta+\alpha k} u^n = \frac{u^{(\alpha+r)k+\beta}}{(1-u)^{\alpha k+\beta+1}}.$$

Therefore

$$\sum_{n,k} \binom{n-rk}{\beta+\alpha k} u^n v^k = \frac{u^\beta(1-u)^{\alpha-\beta-1}}{(1-u)^\alpha - vu^{\alpha+r}}.$$

□

Remark 3.16. This result can be deduced also by using

$$\sum_{n \geq 0} \binom{n+\alpha}{\beta} u^n = \frac{u^{\beta-\alpha}}{(1-u)^{\beta+1}}.$$

Sum of terms of sequence (u_n)

The following theorem provides the formula for the partial sums of the terms of the sequence $(u_n)_n$.

Theorem 3.17. The sum of the terms of the sequence $(u_n)_n$ is given by

- For $r \geq 0$

$$\sum_{j=0}^n u_j = \frac{1}{y^\alpha - (1-x)^\alpha} \left(\sum_{j=0}^{\alpha+r-1} \left(\sum_{i=0}^{\alpha+r-1-j} (-1)^i \binom{\alpha}{i} x^i \right) (u_{n+j+1} - u_j) \right).$$

- For $r < 0$

$$\sum_{j=0}^n u_j = \frac{1}{y^\alpha - (1-x)^\alpha} \left(\sum_{j=0}^{\alpha-1} \left(\sum_{i=0}^{\alpha-1-j} \left((-1)^i \binom{\alpha}{i} x^i - \delta_{i,\alpha+r} y^\alpha \right) \right) (u_{n+j+1} - u_j) \right),$$

where $\delta_{i,j}$ is the Kronecker delta.

Proof. The proof is established through induction.

□

Refer to [17] for the linear recurrence sequences related to the rays of the negatively extended BCT.

3.1.3. Hyper (α, r) -Fibonacci polynomials

For a fixed non-negative integer k , Dil and Mező [26] introduced the hyper-Fibonacci numbers $F_n^{(k)}$, defined by the recurrence

$$F_n^{(k+1)} = \sum_{j=0}^n F_j^{(k)}, F_0^{(k)} = 0, F_n^{(0)} = F_n$$

and provided the generating function

$$\sum_{n \geq 0} F_n^{(k)} z^n = \frac{z}{(1 - z - z^2)(1 - z)^k}.$$

The hyper-Fibonacci numbers $F_n^{(k)}$ also satisfy the recurrence relation

$$F_n^{(k)} = F_{n-1}^{(k)} + F_n^{(k-1)}, n \geq 1, k \geq 1$$

and have the explicit formula

$$F_{n+1}^{(k)} = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n+k-j}{k+j}.$$

In [12], the authors introduced the bivariate hyper-Fibonacci polynomials $F_n^{(k)}(x, y)$, defined by the recurrence relation

$$\begin{cases} F_0^{(k)}(x, y) = 0, \\ F_n^{(0)}(x, y) = F_n(x, y), \\ F_n^{(k)}(x, y) = xF_{n-1}^{(k)}(x, y) + yF_n^{(k-1)}(x, y), \end{cases} \quad \begin{matrix} n \geq 0 \\ \\ k \geq 1, n \geq 1. \end{matrix}$$

Also, they provide the associated generating function

$$\sum_{n \geq 0} F_n^{(k)}(x, y) z^n = \frac{y^k z}{(1 - xz - yz^2)(1 - xz)^k},$$

and the closed-form expression

$$F_{n+1}^{(k)}(x, y) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n+k-j}{k+j} x^{n-2j} y^{k+j}.$$

In this subsection, we introduce the hyper (α, r) -Fibonacci polynomials.

Definition 3.18 (cf. [21]). Let $\alpha \in \mathbb{N}, r \in \mathbb{Z}$ such that $\alpha + r > 0$. Then for a fixed non negative integer k , the hyper (α, r) -Fibonacci polynomials, $G_n^{(k;\alpha,r)}$, are defined, for $n \geq 0$, as

$$G_{n+1}^{(k;\alpha,r)} = \sum_{j=0}^{\lfloor n/(\alpha+r) \rfloor} \binom{n+k-rj}{k+\alpha j} x^{n-(\alpha+r)j} y^{k+\alpha j},$$

with $G_0^{(k;\alpha,r)} = 0$.

According to Definition 3.18 and (3.1), for $n \geq 0$, we have

$$G_{n+1}^{(k;\alpha,r)} = u_{n+k+1}^{(\alpha,r,k)}, \text{ when } k < \alpha.$$

Note that for a fixed non negative integer k ,

$$G_{m+1}^{(k;\alpha,r)} = \binom{m+k}{k} x^m y^k, \text{ for } 0 \leq m \leq \alpha + r - 1.$$

Remark 3.19. The formula in Definition 3.18 represents a truncated sum of the elements lying along a finite ray crossing the Binomial Coefficient Triangle.

Theorem 3.20 (cf. [21]). Let $\alpha \in \mathbb{N}$, $r \in \mathbb{N}$. The terms of the sequence $(G_n^{(k;\alpha,r)})_n$ satisfy the recurrence relation

$$\begin{cases} G_n^{(k;\alpha,r)} - x \binom{\alpha}{1} G_{n-1}^{(k;\alpha,r)} + \dots + (-x)^\alpha \binom{\alpha}{\alpha} G_{n-\alpha}^{(k;\alpha,r)} = y^\alpha G_{n-\alpha-r}^{(k;\alpha,r)}, & \text{for } 0 \leq k < \alpha, \\ G_n^{(k;\alpha,r)} - x \binom{\alpha}{1} G_{n-1}^{(k;\alpha,r)} + \dots + (-x)^\alpha \binom{\alpha}{\alpha} G_{n-\alpha}^{(k;\alpha,r)} = y^\alpha G_n^{(k-\alpha;\alpha,r)}, & \text{for } k \geq \alpha. \end{cases}$$

Note that the second relation is a non-homogeneous linear recurrence relation.

As a consequence of Theorem 3.20, the following corollary provides a connection between $G_n^{(k;\alpha,r)}$ and $G_j^{(k-\alpha;\alpha,r)}$, $0 \leq j \leq \alpha$.

Corollary 3.21 (cf. [21]). For $k \geq \alpha$, $r \geq 0$ and $n \geq \alpha$, we have

$$G_n^{(k;\alpha,r)} = y^\alpha \sum_{j=0}^{n-1} x^{n-j-1} \binom{n+\alpha-2-j}{\alpha-1} G_{j+1}^{(k-\alpha;\alpha,r)}.$$

We now present a non-homogeneous recurrence relation of $(G_n^{(k;\alpha,r)})$.

Theorem 3.22 (cf. [21]). Let $\alpha \in \mathbb{N}$, $r \in \mathbb{Z}$ such that $\alpha + r > 0$. Then for a fixed non negative integer k , we have

$$\sum_{j=0}^{\alpha} (-x)^j \binom{\alpha}{j} G_{n-j}^{(k;\alpha,r)} - y^\alpha G_{n-\alpha-r}^{(k;\alpha,r)} = \begin{cases} \binom{n+k-1-\alpha}{k-\alpha} x^{n-1} y^k, & \text{for } k \geq \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Now, we give the generating function of $(G_n^{(k;\alpha,r)})$.

Theorem 3.23 (cf. [21]). For a fixed non negative integers k and r , the generating function of $(G_n^{(k;\alpha,r)})_n$ is given by

$$\sum_{n \geq 0} G_{n+1}^{(k;\alpha,r)} z^n = \frac{y^k (1-xz)^{\alpha-k-1}}{(1-xz)^\alpha - y^\alpha z^{\alpha+r}}.$$

3.1.4. Incomplete (α, r) -Fibonacci polynomials

The incomplete Fibonacci numbers were introduced by Flipponi [29] and are defined, for $n \geq 0$, by

$$F_{n+1}(l) = \sum_{j=0}^l \binom{n-j}{j}, \quad 0 \leq l \leq \lfloor n/2 \rfloor.$$

They satisfy, for $n \geq 1$, the recurrence relation

$$F_{n+1}(l) = F_n(l) + F_{n-1}(l-1).$$

Recent studies have proposed and explored interesting generalisations, as highlighted by [27, 37, 39, 41, 46]. For instance, the authors in [46] defined and examined the incomplete r -Fibonacci numbers, given by

$$F_{r,n+1}(l) = \sum_{j=0}^l \binom{n-rj}{j}, \quad 0 \leq l \leq \lfloor n/(r+1) \rfloor.$$

They satisfy for $n \geq r$, the recurrence relation

$$F_{r,n+1}(l) = F_{r,n}(l) + F_{r,n-r}(l-1).$$

In the following, we introduce the incomplete (α, r) -Fibonacci polynomials.

Definition 3.24 (cf. [21]). The incomplete (α, r) -Fibonacci polynomials defined, for $\alpha \in \mathbb{N}$, $r \in \mathbb{Z}$, $\beta < \alpha$, $\alpha + r > 0$, and $n \geq 0$ by

$$u_{n+1,\beta}(l) = \sum_{j=0}^l \binom{n-rj}{\beta+\alpha j} x^{n-\beta-(\alpha+r)j} y^{\beta+\alpha j}, \quad 0 \leq l \leq \lfloor (n-\beta)/(\alpha+r) \rfloor,$$

with $u_{0,\beta}(l) = 0$.

Note that

$$u_{n+1,\beta}(l) = \binom{n}{\beta} x^{n-\beta} y^{\beta} \text{ for } \beta \leq n < \beta + \alpha + r.$$

As long as specifying the value of β is not required, we will denote the sequence $(u_{n,\beta}(l))_n$ simply as $(u_n(l))_n$.

We now present the recurrence relation and some properties of $(u_n(l))_n$.

Theorem 3.25 (cf. [21]). For $n \geq \max(\alpha, \alpha + r)$,

$$\sum_{j=0}^{\alpha} (-x)^j \binom{\alpha}{j} u_{n-j}(l+1) = y^{\alpha} u_{n-\alpha-r}(l).$$

This recurrence can be rewritten as a non-homogeneous recurrence, which is necessary to establish the generating function of $(u_n(l))_n$.

Theorem 3.26 (cf. [21]). For $n \geq \max(\alpha, \alpha + r)$,

$$\sum_{j=0}^{\alpha} (-x)^j \binom{\alpha}{j} u_{n-j}(l) = y^{\alpha} u_{n-\alpha-r}(l) - \binom{n-1-r(l+1)-\alpha}{\beta+\alpha l} x^{n-\beta-1-(\alpha+r)(l+1)} y^{\beta+\alpha(l+1)}.$$

The generating function of $(u_n(l))_n$ is given by the following theorem.

Theorem 3.27 (cf. [21]). For $\alpha \in \mathbb{N}$, $r \in \mathbb{Z}^+$ and $\beta < \alpha$, we have

$$\sum_{n \geq 0} u_n(l) z^n = \frac{-z^{\beta+1+\alpha(l-1)+r(l+1)} y^{\beta+\alpha(l+1)}}{x^{2\alpha} [(1-xz)^{\alpha} - y^{\alpha} z^{\alpha+r}] (1-xz)^{\beta+\alpha l+1}}.$$

For a fixed k and α , we establish a connection between the sequences $(u_n(l))_n$ and $(G_n^{(k;\alpha,r)})_n$.

Theorem 3.28 (cf. [21]). Let k , α and s be non negative integers such that $k \equiv s[\alpha]$. Then for $n \geq 0$,

$$u_{n+k+r\lfloor k/\alpha \rfloor}^{(\alpha,r,s)} = G_n^{(k;\alpha,r)} + u_{n+k+r\lfloor k/\alpha \rfloor, s} ([k/\alpha] - 1).$$

Corollary 3.29 (cf. [21]). Let k , α and s be non negative integers such that $k \equiv s[r]$. Then for $n \geq 0$,

$$G_n^{(k;\alpha,r)} = u_{n+k+r\lfloor k/\alpha \rfloor}^{(\alpha,r,s)} - \sum_{j=0}^{\lfloor k/\alpha \rfloor - 1} \binom{n+k+r\lfloor k/\alpha \rfloor - 1 - rj}{s + \alpha j} x^{n+k-1-s+r\lfloor k/\alpha \rfloor - (\alpha+r)j} y^{s+\alpha j}.$$

3.2. Direction in Lucas Triangle

The directions in the Lucas Triangle are expressible in the form

$$\frac{n + \beta + (\alpha - r)k}{n - rk} \binom{n - rk}{\beta + \alpha k} x^{n-\beta-(\alpha+r)k} y^{\beta+\alpha k}.$$

For a fixed direction (α, r, β) and a fixed value of β , let us consider the sequence $(v_n)_{n \geq 1}$ defined by

$$v_n := \sum_{k=0}^{\lfloor (n-\beta)/(\alpha+r) \rfloor} \frac{n + \beta + (\alpha - r)k}{n - rk} \binom{n - rk}{\beta + \alpha k} x^{n-\beta-(\alpha+r)k} y^{\beta+\alpha k}$$

with $v_j = 0$ for $j = 0, \dots, \beta - 1$, and $v_n = \frac{n+\beta}{n} \binom{n}{\beta} x^{n-\beta} y^\beta$ for $\beta \leq n < \beta + \alpha + r$.

Theorem 3.30. *The terms of the sequence $(v_n)_n$ satisfy the recurrence relation*

$$v_n = \binom{\alpha}{1} x v_{n-1} - \binom{\alpha}{2} x^2 v_{n-2} + \dots + (-1)^{\alpha+1} \binom{\alpha}{\alpha} x^\alpha v_{n-\alpha} + y^\alpha v_{n-r-\alpha} \quad (n \geq \alpha + r), \quad \text{for } r \geq 0$$

and

$$v_n = \binom{\alpha}{1} x v_{n-1} + \dots + \left(y^\alpha - (-x)^{r+\alpha} \binom{\alpha}{\alpha+r} \right) v_{n-\alpha-r} + \dots + (-1)^{\alpha+1} \binom{\alpha}{\alpha} x^\alpha v_{n-\alpha} \quad (n \geq \alpha), \quad \text{for } r < 0.$$

Theorem 3.31. *The generating function of the sequence $(v_n)_{n \geq 0}$, is given by*

$$V(z) = \sum_{n \geq 0} v_n z^n = \frac{y^\beta z^\beta (2 - xz)(1 - xz)^{\alpha-\beta-1}}{(1 - xz)^\alpha - y^\alpha z^{\alpha+r}}.$$

Remark 3.32. The relation between the generating functions of the sequences $(u_n)_n$ and $(v_n)_n$ is given by

$$V(z) = (2 - xz)U(z).$$

Note that, for $u = xz$ and $v = \frac{y^\alpha}{x^{\alpha+r}}$, we obtain the following result.

Corollary 3.33. *The bivariate generating function of the sequence $\left\{ \frac{n+\beta+(\alpha-r)k}{n-rk} \binom{n-rk}{\beta+\alpha k} \right\}_{n,k}$ is given by*

$$\sum_{n,k} \frac{n + \beta + (\alpha - r)k}{n - rk} \binom{n - rk}{\beta + \alpha k} u^n v^k = \frac{u^\beta (2 - u)(1 - u)^{\alpha-\beta-1}}{(1 - u)^\alpha - u^{\alpha+r} v}.$$

Theorem 3.34. *The sum of the terms of the elements of sequence $(v_n)_n$, is given by*

- for $r \geq 0$

$$\sum_{j=0}^n v_j = \frac{1}{y^\alpha - (1 - x)^\alpha} \left(\sum_{j=0}^{\alpha+r-1} \left(\sum_{i=0}^{\alpha+r-1-j} (-1)^i \binom{\alpha}{i} x^i \right) (v_{n+j+1} - v_j) \right),$$

- for $r < 0$

$$\sum_{j=0}^n v_j = \frac{1}{y^\alpha - (1 - x)^\alpha} \left(\sum_{j=0}^{\alpha-1} \left(\sum_{i=0}^{\alpha-1-j} \left((-1)^i \binom{\alpha}{i} x^i - \delta_{i,\alpha+r} \right) \right) (v_{n+j+1} - v_j) \right).$$

Proof. The proof is established through induction. □

3.3. Direction in Bi-periodic Binomial Coefficient Triangle

We begin with the definition of the bi-periodic Binomial Coefficient Triangle, which generalizes the Binomial Coefficient Triangle.

Definition 3.35 (cf. [10]). Let $\binom{n}{k}^{(a,b)}$ be the k -th element of the n -th row of the bi-periodic BCT, this triangle is given by

$$\binom{n}{k}^{(a,b)} = \begin{cases} a\binom{n-1}{k}^{(a,b)} + \binom{n-1}{k-1}^{(a,b)}, & \text{for } n+k \text{ odd,} \\ b\binom{n-1}{k}^{(a,b)} + \binom{n-1}{k-1}^{(a,b)}, & \text{for } n+k \text{ even,} \end{cases}$$

with $\binom{n}{0}^{(a,b)} = a^{\xi(n)}(ab)^{\lfloor n/2 \rfloor}$, $\binom{n}{n}^{(a,b)} = 1$, and $\binom{n}{k}^{(a,b)} = 0$ for $k > n$ or $k < 0$.

This is illustrated by Figure 12.

$n \setminus k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	a	1								
2	ab	$2a$	1							
3	a^2b	$3ab$	$3a$	1						
4	a^2b^2	$4a^2b$	$6ab$	$4a$	1					
5	a^3b^2	$5a^2b^2$	$10a^2b$	$10ab$	$5a$	1				
6	a^3b^3	$6a^3b^2$	$15a^2b^2$	$20a^2b$	$15ab$	$6a$	1			
7	a^4b^3	$7a^3b^3$	$21a^3b^2$	$35a^2b^2$	$35a^2b$	$21ab$	$7a$	1		
8	a^4b^4	$8a^4b^3$	$28a^3b^3$	$56a^3b^2$	$70a^2b^2$	$56a^2b$	$28ab$	$8a$	1	
9	a^5b^4	$9a^4b^4$	$36a^4b^3$	$84a^3b^3$	$126a^3b^2$	$126a^2b^2$	$84a^2b$	$36ab$	$9a$	1

Figure 12: Bi-periodic Binomial coefficient triangle

Note that for $a = b = 1$, we obtain the classical BCT.

The bi-periodic Fibonacci sequence $(q_n)_n$ generalizes Fibonacci sequence and is defined for any non-zero real or complex numbers a and b by the following relation, [28],

$$q_0 = 0, q_1 = 1, \text{ and } q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{for } n \text{ even,} \\ bq_{n-1} + q_{n-2}, & \text{for } n \text{ odd,} \end{cases} \quad (n \geq 2).$$

It also satisfies the bi-quadratic linear recurrence relation [49]

$$q_n = (ab + 2)q_{n-2} - q_{n-4} \quad (n \geq 4).$$

In addition, the authors in [28] established the generating function for the bi-periodic Fibonacci sequence as

$$\sum_{n \geq 0} q_n z^n = \frac{z(1 + az - z^2)}{1 - (ab + 2)z^2 + z^4}.$$

The sums of the diagonals in bi-periodic BCT gives the bi-periodic Fibonacci sequence (see Figure 13),

$$q_{n+1} = a^{\xi(n)} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} (ab)^{\lfloor n/2 \rfloor - k},$$

where $\xi(n) = n - 2\lfloor n/2 \rfloor$ is the parity function, i.e., $\xi(n) = 0$ when n is even and $\xi(n) = 1$ when n is odd.

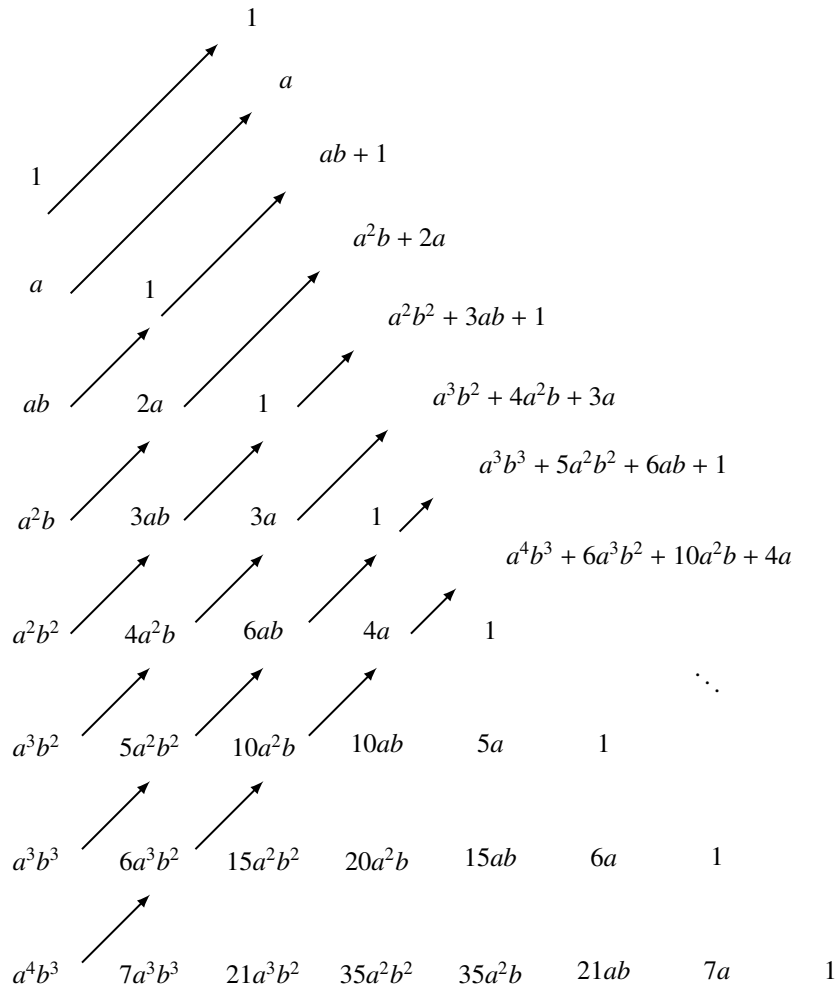


Figure 13: Bi-periodic Fibonacci sequence in bi-periodic BCT

The bivariate bi-periodic Fibonacci sequence is given by

$$u_n^{(a,b)} = \begin{cases} axu_{n-1}^{(a,b)} + yu_{n-2}^{(a,b)}, & \text{for } n \text{ even,} \\ bxu_{n-1}^{(a,b)} + yu_{n-2}^{(a,b)}, & \text{for } n \text{ odd.} \end{cases}$$

The relationship between the bi-periodic BCT and the bivariate BCT is given by

$$\binom{n}{k}^{(a,b)} x^{n-k} y^k = \left(\sqrt{\frac{a}{b}} \right)^{\xi(n-k)} \binom{n}{k}_{x,y} = a^{\xi(n-k)} \binom{n}{k} (ab)^{\lfloor (n-k)/2 \rfloor} x^{n-k} y^k. \tag{3.2}$$

From Definition 3.35 and the above relation, we obtain that the directions in the Binomial Coefficient Triangle are expressible in the form

$$\binom{n-rk}{\beta+\alpha k}^{(a,b)} x^{n-\beta-(\alpha+r)k} y^{\beta+\alpha k} = a^{\xi(n-\beta-(\alpha+r)k)} \binom{n-rk}{\beta+\alpha k} (ab)^{\lfloor (n-\beta-(\alpha+r)k)/2 \rfloor} x^{n-\beta-(\alpha+r)k} y^{\beta+\alpha k},$$

where x and y are variables.

Let $u_{n+1}^{(a,b)}$ be the sum of its elements, i.e., for $n \geq \beta$

$$u_{n+1}^{(a,b)} = \sum_{k=0}^{\lfloor (n-\beta)/(\alpha+r) \rfloor} a^{\xi(n-\beta-(\alpha+r)k)} \binom{n-rk}{\beta+\alpha k} (ab)^{\lfloor (n-\beta-(\alpha+r)k)/2 \rfloor} x^{n-\beta-(\alpha+r)k} y^{\beta+\alpha k}, \tag{3.3}$$

with some initial values $u_0^{(a,b)} = u_1^{(a,b)} = \dots = u_{\beta}^{(a,b)} = 0$.

3.3.1. Direction (1, r, 0)

The sequence associated to the direction (1, r, 0), with $r > 0$, is given by

$$u_{r,n+1}^{(a,b)} = \sum_{k=0}^{\lfloor n/(r+1) \rfloor} \binom{n-rk}{k}^{(a,b)} x^{n-(r+1)k} y^k = \sum_{k=0}^{\lfloor n/(r+1) \rfloor} a^{\xi(n-(r+1)k)} \binom{n-rk}{k} (ab)^{\lfloor (n-(r+1)k)/2 \rfloor} x^{n-(r+1)k} y^k,$$

with $u_{r,0}^{(a,b)} = 0$.

Theorem 3.36 (cf. [10]). *The terms of the sequence $(u_{r,n}^{(a,b)})_n$ satisfy, for $n \geq 2(r+1)$, the linear recurrence relation*

$$u_{r,n}^{(a,b)} = abx^2 u_{r,n-2}^{(a,b)} + 2y u_{r,n-r-1}^{(a,b)} - y^2 u_{r,n-2r-2}^{(a,b)}.$$

with $u_{r,0}^{(a,b)} = 0, u_{r,j}^{(a,b)} = a^{\xi(j-1)} (ab)^{\lfloor (j-1)/2 \rfloor} x^{j-1}$ for $j = 1, \dots, r$.

In addition, for odd r , the sequence $(u_{r,n})_n$ satisfies, for $n \geq r+1$, the shorter couple of recurrence relation

$$u_{r,n}^{(a,b)} = \begin{cases} axu_{r,n-1}^{(a,b)} + yu_{r,n-r-1}^{(a,b)}, & \text{for } n \text{ even;} \\ bxu_{r,n-1}^{(a,b)} + yu_{r,n-r-1}^{(a,b)}, & \text{for } n \text{ odd.} \end{cases}$$

The theorem below establishes the generating function for the sequence $(u_{r,n}^{(a,b)})_n$.

Theorem 3.37 (cf. [10]). *The generating function for the sequence $(u_{r,n}^{(a,b)})_n$ is given by*

$$\sum_{n \geq 0} u_{r,n+1}^{(a,b)} z^n = \frac{1 + axz - yz^{1+r}}{1 - abx^2 z^2 - 2yz^{r+1} + y^2 z^{2(1+r)}}.$$

The following theorem provides the formula for the partial sums of the terms of the sequence $(u_{r,n}^{(a,b)})_n$.

Theorem 3.38 (cf. [10]). *For $n \geq 0$, we have*

$$\sum_{j=0}^n u_{r,j}^{(a,b)} = \frac{abx^2 - 1 + 2y}{1 - abx^2 - 2y + y^2} \sum_{j=0}^r (u_{r,n+j+1}^{(a,b)} - u_{r,j}^{(a,b)}) + \frac{abx^2 - 1}{1 - abx^2 - 2y + y^2} \sum_{j=0}^{r-2} (u_{r,n+r+j+2}^{(a,b)} - u_{r,r+1+j}^{(a,b)}) - \frac{u_{r,n+2r+1}^{(a,b)} + u_{r,n+2r+2}^{(a,b)} - u_{r,2r+1}^{(a,b)} - u_{r,2r}^{(a,b)}}{1 - abx^2 - 2y + y^2}.$$

3.3.2. Direction (α, r, β)

The following theorem presents a linear recurrence relation for the terms of the sequence $(u_n^{(a,b)})_n$ when $\alpha + r > 0$.

Theorem 3.39 (cf. [10]). *The terms of the sequence $(u_n^{(a,b)})_n$, given by (3.3), satisfy, for $n \geq 2 \max(\alpha, \alpha + r)$, the following linear recurrence relation*

$$\sum_{i=0}^{\alpha} (-1)^i \binom{\alpha}{i} (abx^2)^i u_{n-2i}^{(a,b)} = 2y^{\alpha} \sum_{i=0}^{\lfloor \alpha/2 \rfloor} \binom{\alpha}{2i} (abx^2)^i u_{n-(\alpha+r+2i)}^{(a,b)} - y^{2\alpha} u_{n-2(\alpha+r)}^{(a,b)}. \tag{3.4}$$

In addition, for $(\alpha + r)$ even, the sequence $(u_n^{(a,b)})_n$ satisfies, for $n \geq \max(\alpha, \alpha + r)$, the shorter couple of recurrence relations:

$$y^\alpha u_{n-\alpha-r}^{(a,b)} = \begin{cases} \sum_{i=0}^\alpha (-1)^i a^{\xi(i)} \binom{\alpha}{i} (ab)^{\lfloor i/2 \rfloor} x^i u_{n-i}^{(a,b)}, & \text{for } (n + \beta) \text{ even,} \\ \sum_{i=0}^\alpha (-1)^i b^{\xi(i)} \binom{\alpha}{i} (ab)^{\lfloor i/2 \rfloor} x^i u_{n-i}^{(a,b)}, & \text{for } (n + \beta) \text{ odd.} \end{cases} \tag{3.5}$$

From the relationship between the bivariate bi-periodic BCT and the bivariate BCT given in (3.2), we establish the generating function of the sequence $(u_n^{(a,b)})_n$.

Theorem 3.40 (cf. [10]). *The generating function for the sequence $(u_n^{(a,b)})_n$ is given by*

$$\sum_{n \geq 0} u_{n+1}^{(a,b)} z^n = \frac{y^\beta z^\beta \tau (1 - \theta xz)^{\alpha-\beta-1} ((1 + \theta xz)^\alpha - y^\alpha z^{\alpha+r}) + \bar{\tau} (1 + \theta xz)^{\alpha-\beta-1} ((1 - \theta xz)^\alpha - y^\alpha z^{\alpha+r})}{2 ((1 + \theta xz)^\alpha - y^\alpha z^{\alpha+r}) ((1 - \theta xz)^\alpha - y^\alpha z^{\alpha+r})},$$

where $\tau = 1 + \sqrt{\frac{a}{b}}$, $\bar{\tau} = 1 - \sqrt{\frac{a}{b}}$, and $\theta = \sqrt{ab}$.

The sum of the terms of the sequence $(u_n^{(a,b)})_n$ is given by the following theorem.

Theorem 3.41 (cf. [10]). *For $n \geq 0$ and $d = \max(\alpha, \alpha + r)$, we have*

$$\sum_{j=0}^n u_j^{(a,b)} = \frac{\sum_{j=0}^{2d-1} C_j(x, y) (u_{n+j+1}^{(a,b)} - u_j^{(a,b)})}{((1 + \sqrt{ab}x)^\alpha - y^\alpha)((1 - \sqrt{ab}x)^\alpha - y^\alpha)},$$

where

$$C_j(x, y) = \sum_{i=0}^{\lfloor (2d-j-1)/2 \rfloor} \left((-1)^{i+1} \binom{\alpha}{i} (abx^2)^i - \delta_{i,\alpha+r} y^{2\alpha} \right) + 2y^\alpha \sum_{i=0}^{\lfloor (2d-\alpha-r-j-1)/2 \rfloor} \binom{\alpha}{2i} (abx^2)^i.$$

with $\delta_{i,j}$ is the Kronecker delta.

3.4. Direction in Generalized Arithmetic Triangle

The Generalized Arithmetic Triangle (GAT) is a arithmetic triangle that extends the concept of BCT. It preserves the weighted Pascal’s rule and replaces its legs with two arbitrary sequences, denoted by (a_n) and (b_n) , where $a_0 = b_0 = \Omega$ (constant). In the GAT the first element of each row corresponds to (a_n) , and the last element corresponds to (b_n) , while the interior elements are generated by the sum of the two elements directly above them Pascal’s rule.

Definition 3.42. Let $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ be two real or complex sequences, x and y two real or complex numbers. The Generalized Arithmetic Triangle, denoted by GAT, contains elements $\binom{n}{k}$ in the n th row and k th column defined, for $n \geq 2$, as follows:

$$\binom{n}{k}_{(x,y,a_n,b_n)} = x \binom{n-1}{k}_{(x,y,a_n,b_n)} + y \binom{n-1}{k-1}_{(x,y,a_n,b_n)}, \quad \text{for } 1 \leq k \leq n-1,$$

with $\binom{n}{0}_{(x,y,a_n,b_n)} = a_n$, $\binom{n}{n}_{(x,y,a_n,b_n)} = b_n$, $\binom{0}{0}_{(x,y,a_n,b_n)} = a_0 = b_0 = \Omega$, and the convention $\binom{n}{k} = 0$ whenever $k < 0$ or $k > n$.

The first few rows of the GAT are shown in Figure 14.

Ω				
a_1	b_1			
a_2	$ya_1 + xb_1$	b_2		
a_3	$y(xa_1 + a_2) + x^2b_1$	$y^2a_1 + x(yb_1 + b_2)$	b_3	
a_4	$y(x^2a_1 + xa_2 + a_3) + x^3b_1$	$y^2(2xa_1 + a_2) + x^2(2yb_1 + b_2)$	$y^3a_1 + x(y^2b_1 + yb_2 + b_3)$	b_4

Figure 14: First rows of the GAT

The authors in [18] provide the explicit formula for the GAT, as stated in the following theorem.

Theorem 3.43 (cf. [18]). Let $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ be two sequences and x and y two variables. The corresponding GAT has, for $(n, k) \neq (0, 0)$, the expression

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{(x,y,a_n,b_n)} = x^{n-k} y^k \left(\sum_{i=1}^{n-k} \binom{n-1-i}{k-1} \frac{a_i}{x^i} + \sum_{i=1}^k \binom{n-1-i}{k-i} \frac{b_i}{y^i} \right) + \left\lfloor \frac{n-k}{n} \right\rfloor a_n + \left\lfloor \frac{k}{n} \right\rfloor b_n.$$

Note that the GAT does not have symmetry. It is concerned by a quasi-symmetry, i.e.,

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{(x,y,a_n,b_n)} = \left\langle \begin{matrix} n \\ n-k \end{matrix} \right\rangle_{(y,x,b_n,a_n)}.$$

For a fixed direction (α, r, β) and a fixed value of β , we denote the sequence $(G_n^{(\alpha,r,\beta)})_n$ by

$$\mathcal{G}_{n+1}^{(\alpha,r,\beta)} := \sum_{k=0}^{\lfloor (n-\beta)/(\alpha+r) \rfloor} \left\langle \begin{matrix} n-rk \\ \beta+\alpha k \end{matrix} \right\rangle_{(x,y,a_n,b_n)},$$

with $\mathcal{G}_0^{(\alpha,r,\beta)} = 0$. This sequence represents the sum of elements lying on the corresponding ray in the GAT. Note that $\mathcal{G}_1^{(\alpha,r,\beta)} = \dots = \mathcal{G}_\beta^{(\alpha,r,\beta)} = 0$ and $\mathcal{G}_n^{(\alpha,r,\beta)} = \binom{n-1}{\beta}$ for $\beta + 1 \leq n < \beta + \alpha + r$.

The following result provides a recurrence relation for the sum of elements lying along a finite ray in the GAT, generated by the sequences $(a_n)_n$ and $(b_n)_n$.

Theorem 3.44 (cf. [18]). For $\beta \geq 0$ and $\rho = (n - \beta)/(\alpha + r)$, we have

$$\mathcal{G}_{n+1}^{(\alpha,r,\beta)} = \sum_{k=0}^{\lfloor \rho \rfloor} x^{n-\beta-(\alpha+r)k} y^{\beta+\alpha k} \left(\sum_{i=1}^{n-\beta-(\alpha+r)k} \binom{n-rk-1-i}{\beta+\alpha k-1} \frac{a_i}{x^i} + \sum_{i=1}^{\beta+\alpha k} \binom{n-rk-1-i}{\beta+\alpha k-i} \frac{b_i}{y^i} \right) + \left\lfloor \frac{n-\beta}{n} \right\rfloor a_n + b_{\beta+\alpha \rho \lfloor \rho \in \mathbb{N} \rfloor},$$

where the bracketed notation $\lfloor S \rfloor$ stands for 1 if S is true, 0 otherwise.

The following proposition shows that there exists a grid point (m, β') and a direction $(\alpha', -r)$ with $r > 0$ that defines a transversal ray of the GAT containing the same elements in reverse order.

Proposition 3.45 (cf. [18]). Let $\alpha \in \mathbb{N}$, $r < 0$, $\beta = 0, \dots, \alpha - 1$ such that $\alpha + r > 0$. Then

$$\sum_{k=0}^{\lfloor (n-\beta)/(\alpha+r) \rfloor} \left\langle \begin{matrix} n-rk \\ \beta+\alpha k \end{matrix} \right\rangle_{(x,y,a_n,b_n)} = \sum_{k=0}^{\lfloor (n-\beta')/(\alpha'+r') \rfloor} \left\langle \begin{matrix} m-r'k \\ \beta'+\alpha'k \end{matrix} \right\rangle_{(y,x,b_n,a_n)},$$

with $m = n - r \lfloor (n - \beta)/(\alpha + r) \rfloor$, $r' = -r$, $\alpha' = \alpha + r$, $\beta' = n - \beta - (\alpha + r) \lfloor (n - \beta)/(\alpha + r) \rfloor$.

Establishing the recurrence relation

According to Proposition 3.45, we consider the case $\alpha > 0$. First, we set, for $m \geq 1$,

$$\lambda_0^{(m)} := (-1)^m \binom{\alpha - 1}{m}$$

and for $s \geq 1$,

$$\lambda_s^{(m)} := (-1)^{m-1} \binom{\alpha - s - 1}{m - 1}.$$

Note that the $\lambda_s^{(m)}$'s, $1 \leq s \leq \alpha - m$, are the binomial coefficients given by the extension of BCT to negative rows. By induction, we have the following.

Lemma 3.46 (cf. [18]). For $r \geq 2$ and $1 \leq m \leq \alpha - 1$,

$$\sum_{s=0}^{\alpha-m} \lambda_s^{(m)} = 0.$$

Let now be $\sigma_i = \frac{n-1-i-\beta}{\alpha+r}$, for $\leq i \leq \alpha - 1$. For $\sigma_i \in \mathbb{N}$ and $\alpha \geq 1$, we define

$$M_0 := -b_{\beta+\alpha\sigma_0} + y^\alpha b_{\beta+\alpha\sigma_0-r},$$

and, for $j \geq 1$,

$$M_j := -x^j \left(\lambda_0^{(j)} b_{\beta+\alpha\sigma_j} + \lambda_1^{(j)} b_{\beta+\alpha\sigma_{j-1}} y + \cdots + \lambda_{\alpha-j}^{(j)} b_{\beta+\alpha\sigma_{j-\alpha+j}} + y^{\alpha-j} \right).$$

We denote the sequence $(\mathcal{G}_n^{(\alpha,r,\beta)})_n$ by $(\mathcal{G}_n)_n$. In the following theorem, we provide the recurrence relation for the sequence $(\mathcal{G}_n)_n$.

Theorem 3.47 (cf. [18]). *The terms of the sequence $(\mathcal{G}_n)_n$ satisfy the recurrence relation*

$$\sum_{i=0}^{\alpha} (-x)^i \binom{\alpha}{i} \mathcal{G}_{n-j} - y^\beta \sum_{i=0}^{\alpha-\beta} (-x)^i \binom{\alpha-\beta}{i} a_{n-\beta-1-j} + \sum_{i=0}^{\alpha-1} M_i [\sigma_i \in \mathbb{N}] = y^\alpha \mathcal{G}_{n-\alpha-r}.$$

The theorem below provides the generating function for the sequence $(\mathcal{G}_n)_n$.

Theorem 3.48 (cf. [18]). *For $\alpha \in \mathbb{N}$, $0 \leq \beta < \alpha$, $r \in \mathbb{Z}$ and $y \neq 0$, the generating function $\mathcal{G}(z) = \sum_{n \geq 0} \mathcal{G}_{n+1} z^n$ associated to (\mathcal{G}_n) is given by*

$$\mathcal{G}(z) = \frac{(1-xz)^\alpha}{(1-xz)^\alpha - y^\alpha z^{\alpha+r}} C_0(z),$$

with

$$C_0(z) = b_\beta z^\beta + \left(\frac{yz}{1-xz} \right)^\beta \left(\sum_{n \geq 0} a_n z^n - \Omega + \frac{xz}{1-xz} \sum_{k=1}^{\beta} \left(\frac{1-xz}{y} \right)^k b_k \right).$$

In the next section, we explore the Bi^snomial Coefficient Triangle (B^sCT) as an extension of the Binomial Coefficient Triangle (BCT) where the Pascal rule is generalized to s terms in each row.

4. Direction with Bi^snomial Coefficient Triangle (B^sCT)

In the Binomial Coefficient Triangle (BCT), the direction $(1, 1, 0)$ generates the Fibonacci sequence, while the direction $(1, r, 0)$ gives the r -Fibonacci sequence. However, the Tribonacci and more generally multibonacci sequences do not correspond to any direction in the BCT. In this context, for $s \in \mathbb{N}$, we introduce the Bi^snomial Coefficients Triangle, denoted $\binom{n}{k}_s$, in which the Pascal rule is extended to s terms in the same line

$$\binom{n}{k}_s = \sum_{i=0}^s \binom{n-1}{k-i}_s. \tag{4.1}$$

Consider two integers, $s \geq 1$ and $n \geq 0$. For an integer $k = 0, 1, \dots, sn$, the bi^snomial coefficient $\binom{n}{k}_s$ corresponds to the coefficient corresponding to the k -th term in the expansion

$$(1 + z + z^2 + \cdots + z^s)^n = \sum_{k \geq 0} \binom{n}{k}_s z^k, \tag{4.2}$$

with $\binom{n}{k}_s = 0$ for $k > ns$ or $k < 0$ and $\binom{n}{k}_1$ corresponds to $\binom{n}{k}$.

The relation (4.1) corresponds to a recurrence on n . We also have the following recurrence relation over s

$$\binom{n}{k}_s = \sum_{i=0}^n \binom{n}{i} \binom{i}{k-i}_{s-1}.$$

The absorption identity [8] is given by

$$k \binom{n}{k}_s = n \sum_{i=1}^s i \binom{n-1}{k-i}_s,$$

for a general case see [19].

The symmetric relation for the bi^snomial coefficient is given by

$$\binom{n}{k}_s = \binom{n}{sn-k}_s.$$

An expression for the classic binomial coefficients is provided by

$$\binom{n}{k}_s = \sum_{i_1+i_2+\dots+i_s=k} \binom{n}{i_1} \binom{i_1}{i_2} \dots \binom{i_{s-1}}{i_s}.$$

Using multinomial coefficient, see for instance, [11, 16], we get

$$\binom{n}{k}_s = \sum_{i_1+2i_2+\dots+si_s=k} \binom{n}{i_1, i_2, \dots, i_s}.$$

De Moivre’s Formula is established as (see [11, 16])

$$\binom{n}{k}_s = \sum_{j=0}^{\lfloor k/(s+1) \rfloor} (-1)^j \binom{n}{j} \binom{k-j(s+1)+n-1}{n-1}.$$

Recently, Belbachir and Benmezai [15] established the following relation

$$\binom{n}{k}_s = (-1)^k \sum_{j_1+j_2+\dots+j_s=k} \binom{n}{j_1} \binom{n}{j_2} \dots \binom{n}{j_s} a^{j_1+2j_2+\dots+sj_s},$$

where $a = \exp(-2i\pi/(s+1))$, with $i^2 = -1$.

From (4.2), the sum of the bi^snomial coefficients is given as

$$\sum_{k=0}^{sn} \binom{n}{k}_s = (s+1)^n.$$

Note that for $s = 1$, we obtain the classical binomial coefficients, see Figure 2. Similarly, for $s = 2$, we obtain the bi²nomial coefficients forming the Bi²nomial Coefficient Triangle (see [32, 36, 44]), as illustrated in Figure 15.

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1													
1	1	1	1											
2	1	2	3	2	1									
3	1	3	6	7	6	3	1							
4	1	4	10	16	19	16	10	4	1					
5	1	5	15	30	45	51	45	30	15	5	1			
6	1	6	21	50	90	126	141	126	90	50	21	6	1	

Figure 15: Bi²nomial Coefficient Triangle ($s = 2$)

For $s = 3$, we obtain the bi³nomial coefficients forming the Bi³nomial Coefficient Triangle, as illustrated in Figure 16.

$n \backslash k$	1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1															
1	1	1	1	1												
2	1	2	3	4	3	2	1									
3	1	3	6	10	12	12	10	6	3	1						
4	1	4	10	20	31	40	44	40	31	20	10	4	1			
5	1	5	15	35	65	101	135	155	155	135	101	65	35	15	5	1

Figure 16: Bi³nomial Coefficient Triangle ($s = 3$)

4.1. Direction $(1, r, 0)$

In this subsection, we consider the direction $(1, r, 0)$ in B^sCT .

Theorem 4.1 (cf. [34]). For $s \geq 1$ and $r \geq 1$, we have

$$S_{n+1} := \sum_{k=0}^{\lfloor \frac{sn}{sr+1} \rfloor} \binom{n-rk}{k}_s x^{sn-(sr+1)k} y^k$$

satisfies for $n \geq 0$ the relation

$$S_{n+1} = \sum_{i=0}^s x^{s-i} y^i S_{n-ir},$$

with $S_1 = 1$ and $S_0 = S_{-1} = \dots = S_{1-sr} = 0$.

The theorem below provides the general formula for the sum of the terms of the sequences $(S_n)_n$.

Theorem 4.2. For $n \geq 0$, we have

$$\sum_{j=0}^n S_j = \frac{(x-y)(S_{n+1+sr} - S_{sr})}{x^{s+1} - y^{s+1} - x + y} + \frac{x-y}{x^{s+1} - y^{s+1} - x + y} \sum_{j=0}^{sr-1} \left(1 - \sum_{i=1}^{sr-j} x^{s-\lfloor \frac{i-1}{r} \rfloor} y^{\lfloor \frac{i-1}{r} \rfloor} \mathbf{1}_{r|(i-1)} \right) (S_{n+1+j} - S_j)$$

where

$$\mathbf{1}_{rn} = \begin{cases} 1, & \text{if } r \text{ divides } n, \\ 0, & \text{otherwise.} \end{cases}$$

Here, $\mathbf{1}_P$ denotes the indicator function, which takes the value 1 if the condition P holds and 0 otherwise.

Proof. The proof is established through induction. □

5. Catalan Triangle

In this section, we discuss the Catalan Triangle. The properties of this triangle and its relationship to the Catalan numbers are explored.

The entries of the Catalan Triangle are defined by the recurrence relation

$$\binom{n}{k}_c = \binom{n-1}{k+1}_c + 2 \binom{n-1}{k}_c + \binom{n-1}{k-1}_c.$$

The Catalan Triangle is illustrated in Figure 17.

n\k	1	2	3	4	5	6	...
1	1						
2	2	1					
3	5	4	1				
4	14	14	6	1			
5	42	48	27	8	1		
6	132	165	110	44	10	1	
⋮			⋮				

Figure 17: Catalan Triangle

The first column consists of the sequence of Catalan numbers, given by (see [43])

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

The explicit formula in terms of binomial coefficients is:

Theorem 5.1 (cf. [43]). *We have the following identity*

$$\binom{n}{k}_c = \frac{k}{n} \binom{2n}{n-k} = \frac{2k}{n-k} \binom{2n-1}{n+k}.$$

Theorem 5.2 (cf. [43]). *The sum of the rows of Catalan Triangle is expressed as*

$$\sum_{k=1}^n \binom{n}{k}_c = \frac{1}{2} \binom{2n}{n} \quad (n \geq 1).$$

Theorem 5.3 (cf. [43]). *The connection with Catalan numbers is given by:*

$$\binom{n}{k}_c = \sum_{i_1+i_2+\dots+i_k=n} C_{i_1} C_{i_2} \dots C_{i_k}$$

and

$$\sum_{k=1}^{\min n_i} \binom{n_1}{k}_c \binom{n_2}{k}_c \dots \binom{n_k}{k}_c = C_{n_1+n_2+\dots+n_k}.$$

6. Triangles with Delannoy rule

In the following section, we explore triangles that follow the Delannoy rule, where each entry is the sum of the three entries directly above it.

6.1. Direction in Delannoy Triangle

The number of lattice paths from $(0, 0)$ to (n, k) , where movements are restricted to the right (east) $(1, 0)$ and up (north) $(0, 1)$, is given by the binomial coefficient $\binom{n+k}{n}$ or equivalently $\binom{n+k}{k}$.

The Delannoy numbers are defined by the following recurrence relation

$$D_{n,k} = \begin{cases} 1, & \text{for } n = 0 \text{ or } k = 0, \\ D_{n-1,k} + D_{n,k-1} + D_{n-1,k-1}, & \text{otherwise.} \end{cases}$$

These numbers represent the number of paths from $(0, 0)$ to (n, k) using only steps of $(1, 0)$, $(0, 1)$, or $(1, 1)$ (right, up, or diagonal). The Delannoy numbers are also given by the following sums

$$D_{n,k} = \sum_{i=0}^{\min(k,n)} \binom{k}{i} \binom{n}{i} 2^i = \sum_{i=0}^{\min(k,n)} \binom{k+n-i}{k} \binom{k}{i}.$$

Alladi and Hoggat [3] have structured these numbers in the form of a BC-like Triangle known as Delannoy Triangle, as illustrated in Figure 18 below.

n/k	0	1	2	3	4	5	6	7	...	
0	1									
1	1	1								
2	1	3	1							
3	1	5	5	1						
4	1	7	13	7	1					
5	1	9	25	25	9	1				
6	1	11	41	63	41	11	1			
7	1	13	61	129	129	61	13	1		
8	1	15	85	231	321	231	85	15	1	
9	1	17	113	377	681	681	377	113	17	1
⋮					⋮					

Figure 18: Delannoy Triangle

Let $\binom{n}{k}_{[2]}$ denote the entry in the k -th column of the n -th row of the Delannoy Triangle. According to the definition of the triangle, we have (see [3])

$$\binom{n}{k}_{[2]} = \binom{n-1}{k}_{[2]} + \binom{n-1}{k-1}_{[2]} + \binom{n-2}{k-1}_{[2]},$$

where $\binom{n}{0}_{[2]} = \binom{n}{n}_{[2]} = 1$ and $\binom{n}{k}_{[2]} = 0$ for $k \notin \{0, 1, \dots, n\}$.

The connection between the Delannoy numbers $D_{n,k}$ and the entries of the Delannoy Triangle $\binom{n}{k}_{[2]}$ is expressed through the following formula:

$$\binom{n}{k}_{[2]} = D_{n-k,k}.$$

Barry [9] showed that

$$\binom{n}{k}_{[2]} = \sum_{j=0}^k \binom{k}{j} \binom{n-j}{k} \quad \text{and} \quad \binom{n}{k}_{[2]} = \sum_{j=0}^k 2^j \binom{k}{j} \binom{n-k}{j}.$$

These are two expressions of Delannoy coefficients in terms of binomial coefficients.

The sum of the elements on the rising diagonal is the Tribonacci sequence

$$t_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}_{[2]}.$$

The Tribonacci numbers, (t_n) , satisfy $t_0 = 0, t_1 = 1, t_2 = 1$, and for $n \geq 3$,

$$t_n = t_{n-1} + t_{n-2} + t_{n-3}.$$

6.1.1. Direction $(1, r, 0)$

Let T_n be the sum of elements laying over the transversal of direction $(1, r, 0)$

$$T_{n+1} := \sum_{k=0}^{\lfloor n/(r+1) \rfloor} \binom{n-rk}{k}_{[2]}.$$

In the following result, we present the linear recurrence relation for the sequence $(T_n)_n$.

Theorem 6.1 (cf. [7]). *The sequence $(T_n)_n$ satisfies the following linear recurrence relation*

$$T_n = T_{n-1} + T_{n-r-1} + T_{n-r-2},$$

with the initial values $T_1 = 1$, and $T_0 = T_{-1} = \dots = T_{-r} = 0$.

The next result presents the generating function for the sequence $(T_n)_n$.

Theorem 6.2 (cf. [4]). *The generating function of the sequence $(T_{n+1})_{n \geq 0}$ is*

$$\sum_{n \geq 0} T_{n+1} z^n = \frac{1}{1 - z - z^{r+1} - z^{r+2}}.$$

The sum of the terms of the sequence $(T_n)_n$ is given by the following formula.

Theorem 6.3. *For $n \geq 0$, we have*

$$\sum_{j=0}^n T_j = \frac{1}{2} (T_{n+r+1} + T_n - 1).$$

Proof. The proof is established through induction. □

For $r = 1$, we obtain the sum of the Tribonacci sequence (see [24]),

$$\sum_{j=0}^n t_j = \frac{1}{2} (t_{n+2} + t_n - 1).$$

6.1.2. General case

Consider the sequence $(T_n^{(\alpha, r, \beta)})_n$ defined by the sum on the elements lying on the transversal of direction (α, r, β) , i.e.,

$$T_{n+1}^{(\alpha, r, \beta)} = \sum_{k=0}^{\lfloor (n-\beta)/(\alpha+r) \rfloor} \binom{n-rk}{\beta + \alpha k}_{[2]}.$$

In the subsequent result, we present the linear recurrence relation for the sequence $(T_n^{(\alpha, r, \beta)})_n$.

Theorem 6.4 (cf. [4]). *The sequence $(T_n^{(\alpha, r, \beta)})_n$ satisfies, for $n \geq 2\alpha + r$, the following linear recurrence relation*

$$\sum_{s=0}^{\alpha} (-1)^s \binom{\alpha}{s} T_{n-s}^{(\alpha, r, \beta)} = \sum_{s=0}^{\alpha} \binom{\alpha}{s} T_{n-\alpha-r-s}^{(\alpha, r, \beta)}.$$

In the subsequent result, we determine the generating function for the sequence $(T_n^{(\alpha, r, \beta)})_n$.

Theorem 6.5 (cf. [4]). *The generating function of the sequence $(T_n^{(\alpha, r, \beta)})_n$ is given by*

$$\sum_{n \geq 0} T_{n+1}^{(\alpha, r, \beta)} z^n = \frac{(1-z)^{\alpha-\beta-1} (1+z)^\beta z^\beta}{(1-z)^\alpha - (1+z)^\alpha z^{\alpha+r}}.$$

The sum of the terms of the sequences associated with the Delannoy Triangle is given by the following formula.

Theorem 6.6. For $n \geq 0$, we have

$$\sum_{j=0}^n T_j^{(\alpha,r,\beta)} = \frac{1}{2^\alpha} \sum_{j=0}^{2\alpha+r-1} \left(\sum_{i=0}^{2\alpha+r-1-j} (-1)^i \binom{\alpha}{i} - \binom{\alpha}{i-\alpha-r} \right) (T_{n+1+j}^{(\alpha,r,\beta)} - T_j^{(\alpha,r,\beta)}).$$

Proof. The proof is established through induction. □

6.2. Direction in generalized Delannoy Triangle

The generalized Delannoy numbers, denoted as $D_{x,y,z}(n, k)$, represent the number of weighted lattice paths from $(0, 0)$ to (n, k) using only the east $(1, 0)$, north $(0, 1)$, and northeast $(1, 1)$ steps (i.e., \rightarrow , \uparrow , and \nearrow). These steps are assigned weights x , y , and z , respectively, where x , y and z are real or complex variables [25]. They are given by the recurrence relation, see [7], for $n, k \geq 1$

$$D_{x,y,z}(n, k) = xD_{x,y,z}(n - 1, k) + yD_{x,y,z}(n, k - 1) + zD_{x,y,z}(n - 1, k - 1),$$

with the legs values $D_{x,y,z}(0, k) = y^k$ and $D_{x,y,z}(n, 0) = x^n$. They are also given by the sum, [30],

$$D_{x,y,z}(n, k) = \sum_{i=0}^k \binom{k}{i} \binom{n+k-i}{k} x^{k-i} y^{n-i} z^i.$$

Note that when $x = y = z = 1$, the sequence $D_{1,1,1}(n, k)$ reduces to the ordinary Delannoy numbers $D_{n,k}$.

Cheon et al. [25] introduced the generalized Delannoy Triangle with entries $\left[\binom{n}{k}^{(x,y,z)} \right]_{n,k \in \mathbb{N}}$, where

$$\binom{n}{k}^{(x,y,z)}_{[2]} = \begin{cases} D_{x,y,z}(n - k, k) & \text{for } n \geq k, \\ 0 & \text{for } n < k, \end{cases}$$

which gives the following recurrence relation

$$\binom{n}{k}^{(x,y,z)}_{[2]} = y \binom{n-1}{k}^{(x,y,z)}_{[2]} + x \binom{n-1}{k-1}^{(x,y,z)}_{[2]} + z \binom{n-2}{k-1}^{(x,y,z)}_{[2]},$$

with initial condition $\binom{0}{0}^{(x,y,z)}_{[2]} = 1$. The initial terms of the generalized Delannoy Triangle are shown in Figure 19.

n/k	0	1	2	3	4	5	...
0	1						
1	y	x					
2	y^2	$z + 2xy$	x^2				
3	y^3	$y(2z + 3xy)$	$x(2z + 3xy)$	x^3			
4	y^4	$y^2(3z + 4xy)$	$z^2 + 6xy(z + xy)$	$x^2(3z + 4xy)$	x^4		
5	y^5	$y^3(4z + 5xy)$	$10x^2y^3 + 12xy^2z + 3yz^2$	$10x^3y^2 + 12x^2yz + 3xz^2$	$x^3(4z + 5xy)$	y^5	
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	

Figure 19: Generalized Delannoy Triangle

It is clear that when $x = y = z = 1$, we obtain the classical Delannoy Triangle [3].

6.2.1. Direction (1, r, 0)

Consider the sequence $\mathcal{T}_n := \mathcal{T}_n^{(1,r,\beta)}(x, y, z)$ given by

$$\mathcal{T}_{n+1} := \sum_{k=0}^{\lfloor n/(r+1) \rfloor} \binom{n-rk}{k}_{[2]}^{(x,y,z)}, \text{ with } \mathcal{T}_0 = 0.$$

The recurrence relation for the sequence $(\mathcal{T}_n)_n$ is as follows.

Theorem 6.7 (cf. [7]). *The sequence $(\mathcal{T}_n)_n$ satisfies the following linear recurrence relation*

$$\mathcal{T}_n = y\mathcal{T}_{n-1} + x\mathcal{T}_{n-r-1} + z\mathcal{T}_{n-r-2},$$

with the initial values $\mathcal{T}_1 = 1$, and $\mathcal{T}_0 = \mathcal{T}_{-1} = \dots = \mathcal{T}_{-r} = 0$.

In the following result, we derive the generating function for the sequence $(\mathcal{T}_n)_n$.

Theorem 6.8 (cf. [7]). *The generating function of the sequence $(\mathcal{T}_n)_{n \geq 1}$ is*

$$\sum_{n \geq 0} \mathcal{T}_{n+1} t^n = \frac{1}{1 - yt - xt^{r+1} - zt^{r+2}}.$$

The sum of the terms associated with the sequence $(\mathcal{T}_n)_n$ is given by the following formula.

Theorem 6.9. *For $n \geq 0$, we have*

$$\sum_{j=0}^n \mathcal{T}_j = \frac{1}{x + y + z - 1} \left(y\mathcal{T}_{n+r+1} + z\mathcal{T}_n - \mathcal{T}_{r+1} + (1 - y) \sum_{j=0}^r (\mathcal{T}_{n+1+j} - \mathcal{T}_j) \right).$$

Proof. The proof is established through induction. □

6.2.2. General case

Consider the sequence $\mathcal{T}_n^{(\alpha,r,\beta)} := \mathcal{T}_n^{(\alpha,r,\beta)}(x, y, z)$ defined by

$$\mathcal{T}_{n+1}^{(\alpha,r,\beta)} := \sum_{k=0}^{\lfloor (n-\beta)/(r+\alpha) \rfloor} \binom{n-rk}{\beta + \alpha k}_{[2]}^{(x,y,z)}.$$

The recurrence relation for the sequence $(\mathcal{T}_n^{(\alpha,r,\beta)})_n$ is as follows.

Theorem 6.10 (cf. [7]). *The sequence $(\mathcal{T}_n^{(\alpha,r,\beta)})_n$, satisfies, for $n \geq 2\alpha + r$, the following linear recurrence relation*

$$\sum_{s=0}^{\alpha} (-y)^s \binom{\alpha}{s} \mathcal{T}_{n-s}^{(\alpha,r,\beta)} = \sum_{s=0}^{\alpha} x^{\alpha-s} z^s \binom{\alpha}{s} \mathcal{T}_{n-\alpha-r-s}^{(\alpha,r,\beta)},$$

with $\mathcal{T}_0^{(\alpha,r,\beta)} = \mathcal{T}_1^{(\alpha,r,\beta)} = \dots = \mathcal{T}_\beta^{(\alpha,r,\beta)} = 0$ and $\mathcal{T}_k^{(\alpha,r,\beta)} = \binom{k-1}{\beta} y^{k-1}$ for $\beta + 1 \leq k \leq \beta + \alpha + r - 1$

The next result presents the generating function for the sequence $(\mathcal{T}_n^{(\alpha,r,\beta)})_n$.

Theorem 6.11 (cf. [7]). *The generating function of the sequence $(\mathcal{T}_n^{(\alpha,r,\beta)})_{n \geq 0}$ is given by*

$$\sum_{n \geq 0} \mathcal{T}_{n+1}^{(\alpha,r,\beta)} t^n = \frac{(1 - yt)^{\alpha-\beta-1} (x + zt)^\beta t^\beta}{(1 - yt)^\alpha - (x + zt)^\alpha t^{\alpha+r}}.$$

When $x = y = z = 1$, we obtain the sequence $(T_n^{(\alpha,r,\beta)})_n$ associated with the classical Delannoy Triangle.

Note that for $u = yt, v = \frac{x^\alpha}{y^{\alpha+r}}$, and $w = \frac{z}{xz}$, we obtain the following result.

Corollary 6.12. *The trivariate generating function of the sequence $\left\{ \binom{\beta+\alpha k}{j} \binom{n-rk-j}{\beta+\alpha k} \right\}_{n,k,j}$ is given by*

$$\sum_{n \geq 0} \sum_{k \geq 0} \sum_{j \geq 0} \binom{n-rk-j}{\beta+\alpha k-j, j, n-(\alpha+r)k-\beta-j} u^n v^k w^j = \frac{u^\beta (1-u)^{\alpha-\beta-1} (1+uw)^\beta}{(1-u)^\alpha - v(1+uw)^\alpha u^{\alpha+r}}.$$

In the following theorem, we give the general formula of the sum of the sequences associated to the generalized Delannoy Triangle.

Theorem 6.13. *For $n \geq 0$, we have*

$$\sum_{j=0}^n \mathcal{T}_j^{(\alpha,r,\beta)} = \frac{1}{(x+z)^\alpha - (1-y)^\alpha} \sum_{j=0}^{2\alpha+r-1} \left(\sum_{i=0}^{2\alpha+r-1-j} (-y)^i \binom{\alpha}{i} - \binom{\alpha}{i-\alpha-r} \right) x^{2\alpha+r-i} z^{i-\alpha-r} \left(\mathcal{T}_{n+1+j}^{(\alpha,r,\beta)} - \mathcal{T}_j^{(\alpha,r,\beta)} \right).$$

Proof. The proof is established through induction. □

7. Quasi s -Delannoy Triangle (Q^sDT)

In this section, we propose a generalization of the BCT and Delannoy Triangle, called the Quasi s -Delannoy Triangle, where the sum of the elements along the direction $(1, 1, 0)$ gives the terms of the s -Fibonacci sequence.

7.1. The quasi $B^s C$ Triangle

Definition 7.1 (cf. [5]). The quasi-bi^snomial coefficient $\binom{n}{k}_{[s]}$ is defined by the number of lattice path from $(0, 0)$ to (n, k) with steps in $(1, 0), (1, 1), (2, 1), \dots, (s, 1)$, with $\binom{n}{0}_{[s]} = \binom{n}{n}_{[s]} = 1$ and the convention $\binom{n}{k}_{[s]} = 0$ for $k > n$ or $k < 0$.

The quasi-bi^snomial coefficient $\binom{n}{k}_{[s]}$ satisfies the following recurrence relation

$$\binom{n}{k}_{[s]} = \binom{n-1}{k}_{[s]} + \binom{n-1}{k-1}_{[s]} + \binom{n-2}{k-1}_{[s]} + \dots + \binom{n-s}{k-1}_{[s]}.$$

Here the extended Pascal rule is generated vertically.

Theorem 7.2 (cf. [5]). *The explicit formula for the quasi-bi^snomial coefficient is given by*

$$\binom{n}{k}_{[s]} = \sum_{j_1} \sum_{j_2} \dots \sum_{j_{s-1}} \binom{k}{j_1} \binom{j_1}{j_2} \dots \binom{j_{s-2}}{j_{s-1}} \binom{n - \sum_{i=1}^{s-1} j_i}{k},$$

the multinomial version is

$$\binom{n}{k}_{[s]} = \sum_{k_1, k_2, \dots, k_s} \binom{k}{k_1, k_2, \dots, k_s} \binom{n+k - \sum_{i=1}^s i k_i}{k},$$

where $\binom{k}{k_1, k_2, \dots, k_s} = \frac{k!}{k_1! k_2! \dots k_s!}$ for $k_1 + k_2 + \dots + k_s = k$ and $k_i \geq 0$, and $\binom{k}{k_1, k_2, \dots, k_s} = 0$ else.

For $s = 3$, we obtain the quasi-bi³nomial coefficient triangle, as shown in Figure 20, which is given by the the following recurrence

$$\binom{n}{k}_{[3]} = \binom{n-1}{k}_{[3]} + \binom{n-1}{k-1}_{[3]} + \binom{n-2}{k-1}_{[3]} + \binom{n-3}{k-1}_{[3]},$$

with $\binom{n}{0}_{[3]} = \binom{n}{n}_{[3]} = 1$.

n\k	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	3	1							
3	1	6	5	1						
4	1	9	15	7	1					
5	1	12	33	28	9	1				
6	1	15	60	81	45	11	1			
7	1	18	96	189	66	33	13	1		
8	1	21	141	378	459	281	91	15	1	
9	1	24	195	675	1107	946	449	120	17	1

Figure 20: Quasi-bi³nomial coefficient triangle

7.1.1. The s-Fibonacci sequence

Let $(\mathcal{F}_{n,s})_n$ be the terms of the sequence obtained by summing the elements located on the direction $(1, 1, 0)$ of the quasi-bi^snomial coefficient Triangle, then

$$\mathcal{F}_{n+1} \equiv \mathcal{F}_{n+1,s} := \sum_k \binom{n-k}{k}_{[s]}, \text{ with } \mathcal{F}_0 = 0.$$

Theorem 7.3 (cf. [5]). For $n \geq 0$, $(\mathcal{F}_n)_n$ satisfies the following recurrence relation

$$\mathcal{F}_{n+1} = \mathcal{F}_n + \mathcal{F}_{n-1} + \dots + \mathcal{F}_{n-s},$$

with $\mathcal{F}_1 = 1$ and $\mathcal{F}_{-i} = 0$ for $i \in \{0, -1, \dots, -(s-1)\}$.

Note that when $s = 1$ and $s = 2$, we obtain the terms of the Fibonacci and Tribonacci sequences, respectively.

The relationship between the quasi-bi^snomial coefficient and the bi^snomial coefficient is given by the following formula (see [5])

$$\binom{n}{k}_{[s]} = \sum_i \binom{n-i}{k} \binom{k}{i}_{s-1}, \quad s \geq 1.$$

The quasi Bi^snomial Coefficient Triangle (quasi B^sCT) is a kind of convolution of the Binomial Coefficient Triangle (BCT) and the Bi^snomial Coefficient Triangle (B^sCT).

7.1.2. Direction (α, r, β)

Let $(\mathcal{F}_n^{(\alpha,r,\beta)})_n$ be a sequence defined by

$$\mathcal{F}_{n+1}^{(\alpha,r,\beta)} = \sum_k \binom{n-rk}{\beta+\alpha k}_{[s]}, \text{ with } \mathcal{F}_0^{(\alpha,r,\beta)} = 0.$$

The following theorem describes the recurrence relation satisfied by the sequence $(\mathcal{F}_n^{(\alpha,r,\beta)})_n$.

Theorem 7.4 (cf. [5]). For $n \geq \alpha s + r$, the sequence $(\mathcal{F}_n^{(\alpha,r,\beta)})_n$, satisfies the following linear recurrence relation

$$\sum_{i=0}^{\alpha} (-1)^i \binom{\alpha}{i} \mathcal{F}_{n-i}^{(\alpha,r,\beta)} = \sum_{i=0}^{\alpha(s-1)} \binom{\alpha}{i}_{s-1} \mathcal{F}_{n-\alpha-r-i}^{(\alpha,r,\beta)}$$

with the initial conditions $\mathcal{F}_0^{(\alpha,r,\beta)} = 0$ and $\mathcal{F}_j^{(\alpha,r,\beta)} = \sum_k \binom{j-rk}{\beta+\alpha k}_{[s]}$ for $j = 1, \dots, \alpha s + r - 1$.

Theorem 7.5 (cf. [5]). *The generating function is given by*

$$\sum_{n \geq 0} \mathcal{F}_{n+1}^{(\alpha,r,\beta)} z^n = \frac{(1-z)^{\alpha-\beta-1} (z+z^2+\dots+z^s)^\beta}{(1-z)^\alpha - z^{r+\alpha}(1+z+\dots+z^{s-1})^\alpha} = \frac{z^\beta(1-z)^{2\alpha-2\beta-1}(1-z^s)^\beta}{(1-z)^{2\alpha} - z^{r+\alpha}(1-z^s)^\alpha}.$$

In the following theorem, we give the general formula of the sum of the terms of the sequence associated to the quasi-bi^snomial coefficient Triangle.

Theorem 7.6. *For $n \geq 0$, we have*

$$\sum_{j=0}^n \mathcal{F}_j^{(\alpha,r,\beta)} = \frac{1}{s^\alpha} \sum_{j=0}^{s\alpha+r-1} \sum_{i=0}^{s\alpha+r-1-j} \left((-1)^i \binom{\alpha}{i} - \binom{\alpha}{i-\alpha-r}_{s-1} \right) (\mathcal{F}_{n+1+j}^{(\alpha,r,\beta)} - \mathcal{F}_j^{(\alpha,r,\beta)}).$$

Proof. The proof is established through induction. □

7.2. The quasi s -Lucas Triangle

Let's start by defining the quasi- s -Lucas coefficient.

Definition 7.7 (cf. [6]). The quasi- s -Lucas coefficient $L^{(s)}(n, k)$ is defined the following recurrence relation

$$L^{(s)}(n, k) = \binom{n}{k}_{[s]} + \binom{n-1}{k-1}_{[s]} + 2 \binom{n-2}{k-1}_{[s]} + \dots + s \binom{n-s}{k-1}_{[s]},$$

with the initial conditions $L^{(s)}(0, 0) = s + 1$, $L^{(s)}(n, 0) = 1$, and $L^{(s)}(n, n) = 2$.

The following theorem provides the explicit formula for the quasi- s -Lucas coefficient $L^{(s)}(n, k)$.

Theorem 7.8 (cf. [6]). *The explicit formula for $L^{(s)}(n, k)$ is given, for $n > 0$, by*

$$L^{(s)}(n, k) = \sum_{j_1} \sum_{j_2} \dots \sum_{j_{s-1}} \binom{k}{j_1} \binom{j_1}{j_2} \dots \binom{j_{s-2}}{j_{s-1}} \binom{n - \sum_{i=1}^{s-1} j_i}{k} \frac{n+k}{n-j_1-\dots-j_{s-1}},$$

with the summation is taken for values such that $j_1 + \dots + j_{s-1} < n$.

7.2.1. The s -Lucas sequence

Definition 7.9. Let us consider the sequence $(L_{n,s})_{n \geq 0}$ given by $L_{n,s} = \sum_k L^{(s)}(n-k, k)$.

Theorem 7.10 (cf. [6]). *The sequence $(L_{n,s})_{n \geq 0}$ satisfies the following recurrence relation*

$$L_{n,s} = \mathcal{F}_{n+1,s} + \mathcal{F}_{n-1,s} + 2\mathcal{F}_{n-2,s} + \dots + s\mathcal{F}_{n-s,s},$$

and for $n \geq s + 1$, we have

$$L_{n,s} = L_{n-1,s} + L_{n-2,s} + L_{n-3,s} + \dots + L_{n-(s+1),s}.$$

with the initial conditions $L_{0,s} = s + 1$, $L_{1,s} = 1$, $L_{2,s} = 2^2 - 1, \dots, L_{s,s} = 2^s - 1$.

7.2.2. Direction (α, r, β)

Let the sequence $(L_{n,s}^{(\alpha,r,\beta)})_n$ be defined by

$$L_{n,s}^{(\alpha,r,\beta)} := \sum_k L^{(s)}(n-rk, \beta + \alpha k).$$

The following theorem gives the recurrence relation of the sequence $(L_{n,s}^{(\alpha,r,\beta)})_n$.

Theorem 7.11 (cf. [6]). For $n \geq rs + \alpha$, the sequence $(L_{n,s}^{(\alpha,r,\beta)})_{n \geq 0}$ satisfies the following recurrence relation

$$\sum_{i=0}^{\alpha} (-1)^i \binom{\alpha}{i} L_{n-i,s}^{(\alpha,r,\beta)} = \sum_{i=0}^{\alpha(s-1)} \binom{\alpha}{i}_{s-1} L_{n-\alpha-r-i,s}^{(\alpha,r,\beta)}.$$

In the following theorem, we give the general formula of the sum of the terms of the sequence associated to the quasi- s -Lucas coefficient Triangle.

Theorem 7.12. For $n \geq 0$, we have

$$\sum_{j=0}^n L_{j,s}^{(\alpha,r,\beta)} = \frac{1}{s^\alpha} \sum_{j=0}^{s\alpha+r-1} \sum_{i=0}^{s\alpha+r-1-j} \left((-1)^i \binom{\alpha}{i} - \binom{\alpha}{i-\alpha-r}_{s-1} \right) (L_{n+1+j,s}^{(\alpha,r,\beta)} - L_{j,s}^{(\alpha,r,\beta)}).$$

Proof. The proof is established through induction. □

The explicit formula for the quasi- s -Lucas coefficient $L^{(s)}(n, k)$ is provided in the following theorem.

Theorem 7.13 (cf. [6]). For fixed integers n, k , and s , with $s \geq 1$, we have

$$L^{(s)}(n, k) = \sum_i \binom{n-i}{k} \binom{k}{i}_{s-1} \frac{n+k}{n-i}.$$

This identity is also a kind of weighted convolution between the BC and the B^sC.

In the following section, we introduce the BC Rhombus Triangle as an arithmetic triangle merging the Bi²nomial Coefficient Triangle and the Delannoy rule. We explore the directions $(1, 1, 0)$ and $(1, r, 0)$, establish the recurrence sequences associated with these directions, and derive the generating function and sum of the terms of these sequences.

8. BC Rhombus Triangle

The BC rhombus Triangle is an arithmetic triangle introduced by Klostermeyer et al. in 1997 [33] as a merge of Bi²nomial Coefficient Triangle and Delannoy rule. Unlike the classical Delannoy Triangle, where each entry is the sum of two entries from the previous row, the BC rhombus coefficient constructs each entry by summing four specific earlier terms. It is defined by the following recurrence

$$\binom{n}{k}_R = \binom{n-1}{k}_R + \binom{n-1}{k-1}_R + \binom{n-1}{k-2}_R + \binom{n-2}{k-2}_R \quad \text{for } n \geq 2, \text{ all } k, \tag{8.1}$$

with $\binom{0}{0}_R = 1$ and $\binom{n}{0}_R = \binom{n}{2n}_R = 1$ for all n , and the convention $\binom{n}{k}_R = 0$ for $k < 0$ or $k > 2n$. The BC rhombus Triangle is illustrated in Figure 21.

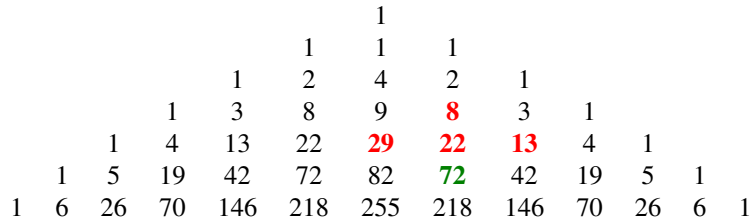


Figure 21: BC Rhombus Triangle

Ramirez [42] provided an explicit expression as follows

$$\binom{n}{k}_R = \sum_{m=0}^n \sum_{l=0}^{n-k-2m} \binom{2m+k}{m} \binom{2m+k+l}{l} \binom{l}{n-k-2m-l}.$$

8.1. Direction (1, 1, 0)

The sum of the elements on the rising diagonal lines in BC Rhombus Triangle is denoted

$$A_{n+1} := \sum_{k=0}^{\lfloor 2n/3 \rfloor} \binom{n-k}{k}_R.$$

The next result provides the recurrence relation for the sequence $(A_n)_n$.

Theorem 8.1. *The sequence $(A_n)_n$ satisfies for $n \geq 4$ the following recurrence relation,*

$$A_n = A_{n-1} + A_{n-2} + A_{n-3} + A_{n-4},$$

with initial conditions $A_0 = 0, A_1 = 1, A_2 = 1, A_3 = 2$.

Proof. From (8.1), we get result. □

This sequence, known as the Tetranacci sequence ([A000078](#)), is a generalization of the Fibonacci numbers, and it has the following generating function

$$\sum_{n \geq 0} A_n z^n = \frac{z}{1 - z - z^2 - z^3 - z^4}. \tag{8.2}$$

8.2. Direction (1, r, 0)

Let $r \in \mathbb{N}$ and $(A_{n,r})_n$ be the sequence associated with the direction $(1, r, 0)$, defined by

$$A_{n+1} \equiv A_{n+1,r} = \sum_{k=0}^{\lfloor 2n/(2r+1) \rfloor} \binom{n-rk}{k}_R.$$

We give the recurrence relation for the sequence $(A_{n,r})_n$ in the following theorem.

Theorem 8.2. *The sequence $(A_n)_n$ satisfies a linear recurrence relation, given by*

$$A_{n+1} = A_n + A_{n-r} + A_{n-2r} + A_{n-2r-1}, \quad (n \geq 2r + 1), \tag{8.3}$$

with $A_0 = 0$ and $A_1 = \dots = A_r = 1$, and $A_n = n - r$ for $n = r + 1, \dots, 2r$.

Proof. From (8.1) and (8.2), we obtain

$$\begin{aligned} A_n + A_{n-r} + A_{n-2r} + A_{n-2r-1} &= \sum_{k=0}^{n-rk-1} \binom{n-rk-1}{k}_R + \sum_{k=0}^{n-1-r(k+1)} \binom{n-1-r(k+1)}{k}_R \\ &\quad + \sum_{k=0}^{n-1-r(k-2)} \binom{n-1-r(k-2)}{k}_R + \sum_{k=0}^{n-2-r(k-2)} \binom{n-2-r(k-2)}{k}_R \\ &= \sum_{k \geq 0} \left(\binom{n-rk-1}{k}_R + \binom{n-1-rk}{k-1}_R + \binom{n-1-rk}{k-2}_R + \binom{n-2-rk}{k-2}_R \right) \\ &= \sum_{k \geq 0} \binom{n-rk}{k}_R. \end{aligned}$$

□

For $r = 1$, we obtain the Tetranacci sequence, and for $r = 2$, we get the sequence ([A079962](#)), in Sloane [44].

The next result establishes the generating function for the sequence $(A_n)_n$.

Theorem 8.3. *The generating function associated to the sequence $(A_n)_n$ is given by*

$$\sum_{n \geq 0} A_{n+1} z^n = \frac{1}{1 - z - z^{r+1} - z^{2r+1} - z^{2r+2}}.$$

Proof. Using the recurrence relation (8.3), we get the result. □

In the following theorem, we give the general formula of the sum of the terms of the sequence $(A_n)_n$

Theorem 8.4. For $n \geq 0$, we have

$$\sum_{j=0}^n A_j = \frac{1}{3} \left(A_{n+2r+1} + A_n - r - \sum_{j=0}^{r-1} (A_{n+1+j} - A_j) \right).$$

Proof. We use the principle of mathematical induction. □

For example the sum of the terms of the Tetranacci numbers is

$$\sum_{j=0}^n A_j = \frac{1}{3} (A_{n+3} - A_{n+1} + A_n - 1).$$

9. Conclusion

This survey provides a comprehensive study of linear recurrence sequences derived from sums of elements along transversals in classical arithmetic triangles. We established their recurrence relations and generating functions, uncovered new combinatorial identities, derived explicit formulas for the sums of their general terms, and provided a general formula for their partial sums. These results contribute to the broader study of recurrence sequences and their combinatorial properties, offering potential directions for further research.

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