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Statistical inference for first passage times of the Feller-CIR model

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Abstract

Boundary crossing problems of diffusion processes are of interest in many areas. For example, the Feller-Cox-Ingersoll-Ross model is used in neuroscience, in which a boundary crossing corresponds to the spiking of a neuron. In this paper, we study the problem of statistical inference for the unknown parameters of this model when only successive first passage times are observed. The probability density function of the first passage time is intractable, but through its Laplace transform, which is a ratio of confluent hypergeometric functions, we determine the identifiable parameter set, discuss the tail behavior of the probability density function and its partial derivatives, and propose a conditional version of maximum likelihood estimation.

Keywords: Confluent hypergeometric function, Cox-Ingersoll-Ross model, Laplace inversion, maximum likelihood estimation, modified Tauberian theorem

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1. Introduction

Stochastic modeling is a well-developed approach that has applications in many areas. An important example is diffusion processes, which have become a prominent tool for modeling continuous-time evolution of phenomena not only in the natural sciences – physics, neuroscience, epidemiology – but also finance and economics. In 1900 Bachelier [1] studied the Brownian motion model for stock markets. Later, Gerstein and Mandelbrot [11] and Stein [24] proposed diffusion models for neural activity. Black and Scholes [2] modeled stock prices using diffusions, and derived a formula that plays an important role in pricing certain financial instruments. In mathematical finance, the Cox-Ingersoll-Ross (CIR, [6]) model is often used to capture the dynamics of interest rates; earlier, Feller [10] had proposed this model, so we refer to it as the Feller-CIR model.

Applications of these diffusion processes require inference for parameters of the models. There are two main types of problems: one deals with discretely-observed diffusions, and the other deals with the first passage time (fpt) to a certain boundary. In both cases because of the intractability of the probability density function (pdf), it is hard to apply the classical maximum likelihood estimation (MLE) theory. The focus of this paper is on inference based on fpt data, because of its many applications. For example, in neuroscience, Gerstein and Mandelbrot [11] proposed the first integrate-and-fire diffusion model for a single neuron activity. They approximated it by a Brownian motion with constant drift and diffusion coefficients: $dX_t = \mu dt + \sigma dB_t$, with X_t hitting a constant boundary corresponding to the firing of the neuron; the first passage time pdf is the inverse Gaussian distribution.

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A year later, Stein [24] proposed a model that also considered the membrane potential decay due to leakage, which is akin to the much earlier deterministic Lapicque model [3]. Later, Ricciardi and Sato [21] considered the Ornstein-Uhlenbeck (OU) diffusion limit of Stein's model, for which the pdf of the fpt through a constant boundary is intractable. They studied the first passage time pdf and moments of the OU process using Darling and Siegert's [7] approach that gave the Laplace transform of the fpt. Since then, there has been considerable study of inference about the underlying OU process using moments and numerical methods to compute the pdf: see [8, 9, 14, 23] for results and reviews. Mullowney and Iyengar [18] and Iyengar [13] proved the validity of the asymptotics of maximum likelihood estimates by inverting the Laplace transform, presented the identifiable functions of the parameters, and gave both point estimates and approximate confidence intervals for them.

We follow a similar approach to the Feller-CIR model. The goal of this paper is to prove the validity of maximum likelihood estimation for unknown parameters of the Feller-CIR model given only its fpt observations. In Section 2 we introduce the model and present the main contributions of the paper: determining the identifiable functions of the unknown parameters, proving the consistency of the MLE, and the asymptotic normality of an approximate MLE. In Section 3 we summarize our work, motivate and state conjectures about relaxing the conditions we used, and sketch ongoing work on computational methods. In Section 4 we provide properties of confluent hypergeometric functions and proofs of our main results.

2. Main results

2.1. The Feller-CIR model and the first passage time Laplace transform

The Feller-CIR model is a diffusion process Y_t with state space $[0, \infty)$ starting at y which is governed by the following stochastic differential equation (SDE) and initial condition:

$$dY_t = [-\gamma Y_t + \beta]\, dt + k\sqrt{Y_t}\, dB_t, \text{ with initial condition } Y_0 = y,$$

where $\alpha > 0$, $\beta > 0$, $k > 0$, and B_t is standard Brownian motion. Consider the first passage time of Y_t through a constant boundary y_c:

$$T_{y_c} = \inf\{t : Y_t = y_c\}.$$

From [12] the Laplace transform of T_{y_c} is

$$Ee^{-sT_{y_c}} = \begin{cases} \dfrac{F(\frac{s}{\gamma}, \frac{2\beta}{k^2}, \frac{2\gamma y}{k^2})}{F(\frac{s}{\gamma}, \frac{2\beta}{k^2}, \frac{2\gamma y_c}{k^2})} & \text{if } 0 < y < y_c, \\[2ex] \dfrac{U(\frac{s}{\gamma}, \frac{2\beta}{k^2}, \frac{2\gamma y}{k^2})}{U(\frac{s}{\gamma}, \frac{2\beta}{k^2}, \frac{2\gamma y_c}{k^2})} & \text{if } 0 < y_c < y, \end{cases}$$

where F and U are confluent hypergeometric functions of the first and the second kind, respectively. In the neural firing context, the process Y_t models the neuron's membrane potential y, with the resting potential as the initial condition which is below the firing threshold y_c. Thus, henceforth, we only deal with crossings from below, which involve F; crossing from above involve U and can be handled similarly. We will see below that we can invert the Laplace transform to get the pdf $g(t|\alpha)$ of T_{y_c}.

2.2. Identifiability of parameters

We assume that the available data come from a renewal process with inter-arrival times having pdf $g(t|\alpha)$. For neural activity, this means that we observe the successive spike times, and that after each spike, the neuron's membrane potential returns to its resting potential. The model has five unknown parameters $(\gamma, \beta, k, y, y_c)$. However, the Laplace transform above indicates that the four ratios there are identifiable. We have the following lemma which formalizes this result.

Lemma 2.1. *Given only first passage time observations, the identifiable parameters are*

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\frac{1}{\gamma}, \frac{2\beta}{k^2}, \frac{2\gamma y}{k^2}, \frac{2\gamma y_c}{k^2}\right).$$

Thus, we will need additional information, perhaps from an auxiliary experiment for neural activity or from historical interest rate data for financial time series to estimate all five parameters of the model. For our purposes here, the parameter space is

$$\mathcal{A} = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) : \alpha_i > 0, \alpha_3 < \alpha_4\},$$

where α_1 is the scale parameter, α_2 is a standardized drift parameter, α_3 is a parameter related to the starting location and α_4 is a parameter associated with the constant boundary.

2.3. Validity of the inversion formula

By Lemma 2.1, our parameter space is

$$\mathcal{A} = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) : \alpha_i > 0 \text{ for all } i\}$$

and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\frac{1}{\alpha}, \frac{2\beta}{k^2}, \frac{2\alpha y}{k^2}, \frac{2\alpha y_c}{k^2}\right)$. We first extend s to the complex plane. Let $s = \frac{u^2}{4\alpha_1} + \frac{\alpha_2}{2\alpha_1}$, where $u = te^{i\theta}$. Notice that if we restrict $\Re s \geq \frac{\alpha_2}{2\alpha_1}$, we will have $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$. Then the expansion of F in 4.1 is valid. Thus, we can obtain for $\Re\alpha_2 \geq 1$:

$$\begin{aligned}
\frac{F(\alpha_1 s, \alpha_2, \alpha_3)}{F(\alpha_1 s, \alpha_2, \alpha_4)} &= e^{\frac{1}{2}(\alpha_3-\alpha_4)}\left(\frac{\alpha_3}{\alpha_4}\right)^{\frac{1-\alpha_2}{2}} \frac{I_{\alpha_2-1}(u\sqrt{\alpha_3})(1+O(t^{-2})) + \frac{1}{t}I_{\alpha_2}(u\sqrt{\alpha_3})(e^{-i\theta}\alpha_3^{\frac{3}{2}} + O(t^{-2}))}{I_{\alpha_2-1}(u\sqrt{\alpha_4})(1+O(t^{-2})) + \frac{1}{t}I_{\alpha_2}(u\sqrt{\alpha_4})(e^{-i\theta}\alpha_4^{\frac{3}{2}} + O(t^{-2}))} \\
&= e^{\frac{1}{2}(\alpha_3-\alpha_4)}\left(\frac{\alpha_3}{\alpha_4}\right)^{\frac{1-2\alpha_2}{4}} e^{u(\sqrt{\alpha_3}-\sqrt{\alpha_4})}\left[1 + O\left(t^{-1}\right)\right] \\
&= e^{\frac{1}{2}(\alpha_3-\alpha_4)}\left(\frac{\alpha_3}{\alpha_4}\right)^{\frac{1-2\alpha_2}{4}} e^{(t\cos\theta + it\sin\theta)(\sqrt{\alpha_3}-\sqrt{\alpha_4})}\left[1 + O\left(t^{-1}\right)\right].
\end{aligned} \tag{2.1}$$

Then if $|s| \to \infty$ with $\Re(s) \geq \frac{\alpha_2}{2\alpha_1}$, we know $t \to +\infty$, then $|\frac{F(\alpha_1 s,\alpha_2,\alpha_3)}{F(\alpha_1 s,\alpha_2,\alpha_4)}|$ (with $\alpha_3 < \alpha_4$) is exponentially decaying in t. By a result in [5], the Bromwich integral inversion formula is valid. Writing $\hat{g}(s|\alpha) = \frac{F(\alpha_1 s,\alpha_2,\alpha_3)}{F(\alpha_1 s,\alpha_2,\alpha_4)}$, we can apply the Bromwich integral which yields the continuous pdf

$$g(t|\alpha) = \frac{1}{2\pi i}\int_{\frac{\alpha_2}{2\alpha_1}-i\infty}^{\frac{\alpha_2}{2\alpha_1}+i\infty} e^{ts}\hat{g}(s|\alpha)\, ds. \tag{2.2}$$

The partial derivatives of $\hat{g}(s|\alpha)$ with respect to the parameters also exist and are bounded by exponentially decaying functions, and they can be computed by differentiating under the Bromwich integral. The proofs are similar, but longer, so we omit them: [27] contains the details.

2.4. An approximate MLE

Given the results of the tail behaviors of the pdf $g(t|\alpha)$ of the first passage time as $t \to +\infty$ and $t \to 0^+$ in Section 3, we can prove the existence and asymptotic properties of the approximate MLE; henceforth we drop the term approximate. Our study the MLE consists of two parts: consistency and asymptotic normality. As with most literature, and in order to make the argument rigorous, we restrict the parameter space to be a compact subset $\mathcal{K} \subset \mathcal{A}$. For other problems there are compactification techniques to handle non-compact parameter spaces, for example, for the Cauchy. However, the discussion of $\log g(t|\alpha)$ as $\alpha_i \to 0^+$ and $\alpha_3 \to \alpha_4$ is extremely difficult in the Feller-CIR case. Fortunately, in practice, researchers usually have sufficient knowledge to determine a smaller range of the parameters than $\mathcal{A}$. Thus, we assume that the parameter space is $\mathcal{K}$: for example, let $\mathcal{K}$ be $\{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) : 0 < \epsilon \leq \alpha_i \leq M, \alpha_3 + \epsilon \leq \alpha_4\}$, for some small $\epsilon > 0$ and large M. We first state our main result about the consistency of the MLE.

Lemma 2.2. *If $\alpha^{(0)}$ is in the compact parameter space $\mathcal{K}$, then the MLE $\hat{\alpha}_n$ for $\alpha^{(0)}$ exists and is strongly consistent: that is, $\hat{\alpha}_n \to \alpha^{(0)}$ almost surely as $n \to \infty$.*

Turning to the asymptotic normality of the MLE, we encounter several serious challenges. The classical regularity conditions in Lehmann's text [16], which include the pdf's differentiability and boundedness of partial derivatives up to the third order are too strong. van der Vaart [25] gives a weaker condition: the differentiability of the root density $\alpha \to \sqrt{g(t|\alpha)}$ in quadratic mean, which entails the existence of a vector of measurable functions $\dot{s_\alpha}$ such that

$$\int_0^\infty \left(\sqrt{g(t|\alpha + h)} - \sqrt{g(t|\alpha)} - \frac{1}{2} h^T \dot{s_\alpha}(t) \right)^2 dt = o(\|h\|^2).$$

In addition, there is a Lipschitz condition that there exists a measurable function $\dot{m}(t)$ with finite second moment under probability measure with pdf $g(t|\alpha^{(0)})$ such that for all $\alpha^{(1)}$ and $\alpha^{(2)}$ in a small closed neighborhood of $\alpha^{(0)}$ satisfying

$$|\log g(t|\alpha^{(1)}) - \log g(t|\alpha^{(2)})| \leq \dot{m}(t)\|\alpha^{(1)} - \alpha^{(2)}\|.$$

With only access to the Laplace transform of $g(t|\alpha)$, we will see from Lemmas 4.4 and 4.5 below, that when t is bounded away from zero, the two conditions are satisfied. But for $t \downarrow 0$, we only know that

$$\frac{\partial \log g(t|\alpha)}{\partial \alpha} = \frac{1}{t} \left(-\frac{\partial \{\alpha_1 (\sqrt{\alpha_3} - \alpha_4)^2\}}{\partial \alpha} + \frac{\partial l(t|\alpha)}{\partial \alpha} \right);$$

and because we have been unable to control $\frac{\partial l(t|\alpha)}{\partial \alpha}$, we consider instead a modification. In particular, because of the difficulty as when $t \downarrow 0$, we propose a conditional version of MLE. By conditioning on the event that the first passage time is greater than $\Delta > 0$, we can obtain a new class of conditional density functions:

$$\tilde{g}(t|\alpha, \Delta) = \frac{g(t|\alpha)}{1 - \int_0^\Delta g(l|\alpha) dl} \quad \text{for} \quad t \geq \Delta.$$

Instead of dealing with $g(t|\alpha)$, if we use $\tilde{g}(t|\alpha, \Delta)$ as our density function, we will have both consistency and asymptotic normality. We have the following theorem about our approximate MLE.

Theorem 2.3. *Suppose that the true parameter $\alpha^{(0)} = (\alpha_1^{(0)}, \alpha_2^{(0)}, \alpha_3^{(0)}, \alpha_4^{(0)})$ is in the interior of $\mathcal{K}$. Given first passage time observations $t_1, t_2, \cdots, t_n$ with density function $\tilde{g}(t|\alpha, \Delta)$, if the information matrix $\tilde{I}(\alpha^{(0)})$ is invertible, then there exists an MLE $\hat{\tilde{\alpha}}_n$ such that*

$$\sqrt{n}\left(\hat{\tilde{\alpha}}_n - \alpha^{(0)}\right) \xrightarrow{d} N\left(0, \tilde{I}(\alpha^{(0)})^{-1}\right).$$

The difference between $\hat{\alpha}_n$ and $\hat{\tilde{\alpha}}_n$ can be made arbitrarily small, because we can fix a tiny Δ. When Δ is extremely small, we expect $\hat{\alpha}_n \approx \hat{\tilde{\alpha}}_n$. In fact, our numerical results show that when $\Delta = 10^{-100}$, $\hat{\alpha}_n$ and $\hat{\tilde{\alpha}}_n$ are numerically indistinguishable. Our work on computation for the practical implementation of these results is ongoing, so we do not describe them here.

3. Summary

We have determined the identifiable parameters of the Feller-CIR model based on first passage time observations for the upcrossing case, and provided the theoretical justification for the use of the MLE. The down-crossing case using U instead of F follows similarly. Although the pdf of the first passage times is intractable, we show that by working with its Laplace transform, we can establish an approximate MLE, prove its consistency and asymptotic normality. We acknowledge that the two restrictions that we made – the parameter space to the compact $\mathcal{K} \subset \mathcal{A}$, and the inter-arrival time to be at least $\Delta > 0$ – are to simplify our proofs. These restrictions cause no difficulties in practice because the compact parameter space can easily be made large enough to contain the physically possible parameter values. For example, the inter-arrival times for a single neuron's spikes must be bounded away from 0 to account for the neuron's refractory period, or recovery after a spike.

Nevertheless, we believe that our results can be strengthened. There are alternative conditions for asymptotic normality of the MLE, for example [15]. We find that the differentiability in quadratic mean condition is crucial; however, it also requires other conditions for the uniform integrability in a neighborhood of the true parameter (Assumptions

A1 and A2 of [15]). Since those conditions also require some control over $\partial \log g(t|\alpha)/\partial\alpha$, they are still quite difficult to verify. However, because $l(t|\alpha)$ goes to 0 uniformly as shown in Lemma 4.7, we have the following conjectures, work on which is also ongoing.

CONJECTURE 1. In a closed neighborhood of $\alpha^{(0)}$, when $0 < t < \delta$, $\frac{\partial l(t|\alpha)}{\partial\alpha}$ can be bounded uniformly by some polynomial in t^{-1}.

CONJECTURE 2 Suppose that the true value of $\alpha^{(0)} = (\alpha_1^{(0)}, \alpha_2^{(0)}, \alpha_3^{(0)}, \alpha_4^{(0)})$ is in the interior of a compact set $\mathcal{K} \subset \mathcal{A}$ and Conjecture 1 is valid. Then given only first passage time observations t_1, t_2,$\cdots$, t_n, if the information matrix at $\alpha^{(0)}$ defined as $I(\alpha^{(0)})$ is invertible, there exits an MLE $\hat{\alpha}_n$ for $\alpha^{(0)}$ satisfying

$$\sqrt{n}\left(\hat{\alpha}_n - \alpha^{(0)}\right) \xrightarrow{d} N\left(0, I(\alpha^0)^{-1}\right).$$

4. Proofs

Preliminary for Lemma 2.1: Asymptotic expansion for F

We start with Volkmer [26], who worked with complex a in $F(a, b, c)$ to prove the following:

Lemma 4.1. *Suppose that $b \in \mathbb{C}$ is not 0 or a negative integer, $u = te^{i\theta}$ with $t > 0$, $\theta \in \mathcal{R}$, and $N \geq 1$, $R > 0$. Then*

$$\frac{2^{1-b}u^{b-1}}{\Gamma(b)} e^{-\frac{1}{2}z^2} z^b F\left(\frac{1}{4}u^2 + \frac{1}{2}b, b, z^2\right) = zI_{b-1}(uz)\left(\sum_{s=0}^{N-1} \frac{A_s(z)}{u^{2s}} + g_1(u,z)\right) + \frac{z}{u} I_b(uz)\left(\sum_{s=0}^{N-1} \frac{B_s(z)}{u^{2s}} + zh_1(u,z)\right), \tag{4.1}$$

where I_v is the modified Bessel function of the first kind of order v, $|g_1(u,z)| + |h_1(u,z)| \leq \frac{L_1}{t^{2N}}$ for $0 < |z| \leq R$, $t \geq t_1$. L_1, t_1 are positive constants independent of z and u (but possibly depending on b, θ, N, R). $A_s(z)$ and $B_s(z)$ are polynomials that can be obtained recursively. In fact, we will only use $A_0(z) = 1$ and $B_0(z) = \frac{1}{6}z^3$.

In Lemma 4.1, we find that L_1 might depend on θ, while we would like an expansion that makes g_1 and h_1 be bounded by a constant free of θ. So we prove the following lemma:

Lemma 4.2. *With the same conditions required for (2.1), we have that for $\Re b > 0$:*

$$\begin{aligned} F\left(\frac{1}{4}u^2 + \frac{1}{2}b, b, z^2\right) &= \frac{\Gamma(b)}{2^{1-b}u^{b-1}} e^{\frac{1}{2}z^2} z^{1-b} \\ &\times \left(I_{b-1}(uz)\left(\sum_{s=0}^{N-1} \frac{\tilde{A}_s(e^{i\theta}z)}{t^{2s}} + g_{21}(t, e^{i\theta}z)\right) + \frac{1}{t} I_b(uz)\left(\sum_{s=0}^{N-1} \frac{\tilde{B}_s(e^{i\theta}z)}{t^{2s}} + e^{i\theta}zh_{21}(t, e^{i\theta}z)\right)\right), \end{aligned} \tag{4.2}$$

where $u = te^{i\theta}$, $|g_{2i}| + |h_{2i}| \leq \frac{K_1}{t^{2N}}$ for $0 < |z| \leq R$, $t \geq t_1$. K_1 is a positive constant depending on R, N, b and t_1. $\tilde{A}_s$ and $\tilde{B}_s$ are some polynomials that can be obtained by a recursive relationship.

Proof. When $\Re\mu \geq 0$, t is real (we can make it positive), following the notation of Volkmer [26], write $W_3(te^{i\theta}, \mu, e^{-i\theta}x) = e^{i\theta}\tilde{W}_3(t, \mu, x)$, where W_3 is a solution to the ODE:

$$w'' = \frac{1}{z}w'(z) + (u^2 + \frac{\mu^2 - 1}{z^2} + f(z))w(z).$$

$\tilde{W}_3$ is a solution to the ODE

$$\tilde{w}'' = \frac{1}{x}\tilde{w}'(x) + \left[t^2 + \frac{\mu^2 - 1}{x^2} + e^{-2i\theta}f(e^{-i\theta}x)\right]\tilde{w}(x).$$

Because t is real in $\tilde{W}_3$, we can apply Olver's work [19], which gives us an expansion for $\tilde{W}_3$:

$$\tilde{W}_3(t, \mu, x) = xI_\mu(tx)\left[\sum_{s=0}^{N-1} \frac{\tilde{A}_s(x)}{t^{2s}} + g_2(t, x)\right] + \frac{x}{t} I_{\mu+1}\left(\sum_{s=0}^{N-1} \frac{\tilde{B}_s(x)}{t^{2s}} + xh_2(t, x)\right), \tag{4.3}$$

where $|g_2| + |h_2| \le \frac{K_1}{t^{2N}}$ for $0 < |x| \le R$, $t \ge t_1$. and K_1 is a positive constant depending on R, N, μ and t_1. Next, let $z = e^{-i\theta}x$, $u = te^{i\theta}$; then we have

$$W_3(te^{i\theta}, \mu, z) = zI_\mu(uz)\left(\sum_{s=0}^{N-1} \frac{\tilde{A}_s(e^{i\theta}z)}{t^{2s}} + g_2(t, e^{i\theta}z)\right) + \frac{z}{t}I_{\mu+1}\left(\sum_{s=0}^{N-1} \frac{\tilde{B}_s(e^{i\theta}z)}{t^{2s}} + e^{i\theta}zh_2(t, e^{i\theta}z)\right).$$

On the other hand, $F\left(\frac{1}{4}u^2 + \frac{1}{2}b, b, z^2\right) = \frac{\Gamma(b)}{2^{1-b}u^{b-1}} e^{\frac{1}{2}z^2} z^{-b} W_3(u, \mu, z)$ with $\mu = b - 1$ and $f = z^2$, so we have proved the case for $\Re b \ge 1$. When $0 < \Re b < 1$, we can not use Olver's work [19] directly, as $\Re\mu = \Re b - 1 < 0$. However, in the proof of Theorem 7.1 in [26], the author showed that (4.1) is still valid. Let $z = e^{-i\theta}x$, $u = te^{i\theta}$ again, the case for $0 < \Re b < 1$ is proved. □

Proof of Lemma 2.1

We show that distinct parameter vectors α correspond to distinct Laplace transforms when $y < y_c$. Consider $\frac{F(\alpha_1 s, \alpha_2, \alpha_3)}{F(\alpha_1 s, \alpha_2, \alpha_4)}$ with $\alpha_1 > 0$, $\alpha_2 \ge 1$, $0 < \alpha_3 < \alpha_4$. When $s \to +\infty$, notice all the numbers are real, with Lemma 4.2 ($\theta = 0$, $\tilde{A}_0(z) = 1$, $\tilde{B}_0(z) = \frac{1}{6}z^3$) we have:

$$\frac{F(\alpha_1 s, \alpha_2, \alpha_3)}{F(\alpha_1 s, \alpha_2, \alpha_4)} = e^{\frac{1}{2}(\alpha_3 - \alpha_4)} \left(\frac{\alpha_3}{\alpha_4}\right)^{\frac{1-\alpha_2}{2}} \times \frac{I_{\alpha_2 - 1}\left(\sqrt{(4\alpha_1 s - 2\alpha_2)\alpha_3}\right)(1 + O(\frac{1}{s})) + \frac{1}{\sqrt{4\alpha_1 s - 2\alpha_2}} I_{\alpha_2}\left(\sqrt{(4\alpha_1 s - 2\alpha_2)\alpha_3}\right)\left(\frac{1}{6}\alpha_3^{\frac{3}{2}} + O\left(\frac{1}{s}\right)\right)}{I_{\alpha_2 - 1}\left(\sqrt{(4\alpha_1 s - 2\alpha_2)\alpha_4}\right)(1 + O(\frac{1}{s})) + \frac{1}{\sqrt{4\alpha_1 s - 2\alpha_2}} I_{\alpha_2}\left(\sqrt{(4\alpha_1 s - 2\alpha_2)\alpha_4}\right)\left(\frac{1}{6}\alpha_4^{\frac{3}{2}} + O\left(\frac{1}{s}\right)\right)}. \tag{4.4}$$

On the other hand, modified Bessel functions have these expansions [20]:

$$K_v = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}\left(\sum_{s=0}^{n-1} \frac{a_s(v)}{z^s} + \gamma_n\right) \quad \text{and}$$

$$I_v(z) = \frac{e^z}{(2\pi z)^{\frac{1}{2}}}\left(\sum_{s=0}^{n-1} (-1)^s \frac{a_s(v)}{z^s} + \delta_n\right) - ie^{-v\pi i} \frac{e^{-z}}{(2\pi z)^{\frac{1}{2}}}\left(\sum_{s=0}^{n-1} \frac{a_s(v)}{z^s} + \gamma_n\right),$$

where

$$|\gamma_n| \le 2e^{|(v^2 - \frac{1}{4})z^{-1}|}|a_n(v)z^{-n}| \quad \text{if } |arg(z)| \le \frac{1}{2}\pi$$

and

$$|\delta_n| \le 2\chi(n)e^{\frac{1}{2}\pi|(v^2 - \frac{1}{4})z^{-1}|}|a_n(v)z^{-n}| \text{ if } -\frac{1}{2}\pi \le |arg(z)| \le 0.$$

Note also that

$$2\chi(n)e^{\frac{1}{2}\pi|(v^2 - \frac{1}{4})(\Re(z))^{-1}|}|a_n(v)(Re(z))^{-n}| \text{ if } 0 \le |arg(z)| < \frac{1}{2}\pi,$$

where $\chi(n) = \pi^{\frac{1}{2}}\Gamma(\frac{n}{2} + 1)/\Gamma(\frac{1}{2}n + \frac{1}{2})$, $a_n(v) = \frac{\Pi_{k=0}^n (4v^2 - (2k+1)^2)}{(n+1)!} \times (\sum_{k=0}^n \frac{1}{4v^2 - (2k+1)^2})$. Thus,

$$\begin{aligned} \frac{F(\alpha_1 s, \alpha_2, \alpha_3)}{F(\alpha_1 s, \alpha_2, \alpha_4)} &= e^{\frac{1}{2}(\alpha_3 - \alpha_4)} \left(\frac{\alpha_3}{\alpha_4}\right)^{\frac{1-2\alpha_2}{4}} \frac{e^{\sqrt{(4\alpha_1 s - 2\alpha_2)\alpha_3}}(1 + O(s^{-1/2}))}{e^{\sqrt{(4\alpha_1 s - 2\alpha_2)\alpha_4}}(1 + O(s^{-1/2}))} \\ &= e^{\frac{1}{2}(\alpha_3 - \alpha_4)} \left(\frac{\alpha_3}{\alpha_4}\right)^{\frac{1-2\alpha_2}{4}} e^{\sqrt{4\alpha_1 s - 2\alpha_2}(\sqrt{\alpha_3} - \sqrt{\alpha_4})}\left(1 + O\left(\frac{1}{\sqrt{s}}\right)\right). \end{aligned} \tag{4.5}$$

For the case $0 < \alpha_2 < 1$, it is not hard to see that Eq. (2.2) is still valid. On the other hand, from [17], the survival function of T_{y_c} can be written as $P(T_{y_c} > t) = \sum_{n=1}^\infty c_n e^{-\lambda_n t}$ with $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$, where for large n,

$$\lambda_n \sim \frac{\alpha\pi^2}{\frac{8\alpha y_c}{k^2}}\left(n + \frac{\beta}{2k^2} - \frac{3}{4}\right)^2 - \frac{\alpha\beta}{2k^2} = \frac{\pi^2}{4\alpha_1\alpha_4}\left(n + \frac{\alpha_2}{2} - \frac{3}{4}\right)^2 - \frac{\alpha_2}{2\alpha_1}, \tag{4.6}$$

$$c_n \sim \frac{(-1)^{n+1} 2\pi(n+\frac{\alpha_2}{2}-\frac{3}{4})}{\pi^2(n+\frac{\alpha_2}{2}-\frac{3}{4})^2 - 2\alpha_2\alpha_4} e^{\frac{1}{2}(\alpha_3-\alpha_4)} (\frac{\alpha_3}{\alpha_4})^{\frac{1}{4}-\frac{\alpha_2}{2}} \cos\left[\pi\left(n+\frac{\alpha_2}{2}-\frac{3}{4}\right)\sqrt{\frac{\alpha_3}{\alpha_4}} - \frac{\pi\alpha_2}{2} + \frac{\pi}{2}\right]. \tag{4.7}$$

To proceed, we need another result.

Lemma 4.3. *If c_n and λ_n satisfy the asymptotic properties* (4.4) *and* (4.5) *(we allow different α_1, α_2, α_3, α_4), the expansion is unique.*

Proof. If we have two expansions with coefficients (c_n, λ_n) and (c_n', λ_n'):

$$\sum_{n=1}^{\infty} c_n e^{-\lambda_n t} = \sum_{n=1}^{\infty} c_n' e^{-\lambda_n' t}.$$

Notice we can assume all the c_n, λ_n, c_n' and λ_n' are not zero. If $\lambda_1 \neq \lambda_1'$, assume $\lambda_1 < \lambda_1'$ then:

$$\frac{c_1}{c_1'} e^{(\lambda_1'-\lambda_1)t} = \frac{1+\sum_{n=2}^{\infty}\frac{c_n'}{c_1'}e^{-(\lambda_n'-\lambda_1')t}}{1+\sum_{n=2}^{\infty}\frac{c_n}{c_1}e^{-(\lambda_n-\lambda_1)t}}.$$

Because of equation (4.1) and (4.2), we know there exists $N > 3$, $c > 0$, $d > 0$ such that:

$$\left|\sum_{n=N}^{\infty} \frac{c_n}{c_1} e^{-(\lambda_n-\lambda_1)t}\right| \le \sum_{n=N}^{\infty} d e^{-c(n-1)^2 t} < d\int_{N-2}^{\infty} e^{-cl^2 t} dl = d\int_{\frac{N-2}{\sqrt{ct}}}^{\infty} \frac{e^{-s^2}}{\sqrt{ct}} ds.$$

By which we know as $t \to \infty$, $\sum_{n=N}^{\infty} \frac{c_n}{c_1} e^{-(\lambda_n-\lambda_1)t} \to 0$. Then $t \to \infty$, $\sum_{n=2}^{N} \frac{c_n}{c_1} e^{-(\lambda_n-\lambda_1)t} \to 0$. So $\sum_{n=2}^{\infty} \frac{c_n}{c_1} e^{-(\lambda_n-\lambda_1)t} \to 0$. With the same method, $\sum_{n=2}^{\infty} \frac{c_n'}{c_1'} e^{-(\lambda_n'-\lambda_1')t} \to 0$, Therefore $\frac{1+\sum_{n=2}^{\infty}\frac{c_n'}{c_1'}e^{-(\lambda_n'-\lambda_1')t}}{1+\sum_{n=2}^{\infty}\frac{c_n}{c_1}e^{-(\lambda_n-\lambda_1)t}} \to 1$. However, $\frac{c_1}{c_1'} e^{(\lambda_1'-\lambda_1)t} \to \infty$, we have a contradiction. So $\lambda_1 = \lambda_1'$, $c_1 = c_1'$. For $n \ge 2$, use induction. □

So if we have two sets of parameters $(\alpha_{11}, \alpha_{21}, \alpha_{31}, \alpha_{41})$ and $(\alpha_{12}, \alpha_{22}, \alpha_{32}, \alpha_{42})$ that makes the Laplace transform the same in s, we must have:

$$\begin{cases} \sqrt{\alpha_{11}}(\sqrt{\alpha_{31}} - \sqrt{\alpha_{41}}) = \sqrt{\alpha_{12}}(\sqrt{\alpha_{32}} - \sqrt{\alpha_{42}}) \\ \sqrt{\alpha_{21}}(\sqrt{\alpha_{31}} - \sqrt{\alpha_{41}}) = \sqrt{\alpha_{22}}(\sqrt{\alpha_{32}} - \sqrt{\alpha_{42}}) \\ \alpha_{11}\alpha_{41} = \alpha_{12}\alpha_{42}. \end{cases}$$

From the first two equations, we know $\frac{\alpha_{11}}{\alpha_{21}} = \frac{\alpha_{12}}{\alpha_{22}}$, along with the third Equation of (4.3), we can have $\frac{\alpha_{31}}{\alpha_{41}} = \frac{\alpha_{32}}{\alpha_{42}}$. On the other hand, if we take $s = \frac{\alpha_{21}}{\alpha_{11}} = \frac{\alpha_{22}}{\alpha_{12}}$, we will have $\frac{F(\alpha_{11}s,\alpha_{21},\alpha_{31})}{F(\alpha_{11}s,\alpha_{21},\alpha_{41})} = e^{\alpha_{31}-\alpha_{41}} = \frac{F(\alpha_{12}s,\alpha_{22},\alpha_{32})}{F(\alpha_{12}s,\alpha_{22},\alpha_{42})} = e^{\alpha_{32}-\alpha_{42}}$. So $\alpha_{31} - \alpha_{41} = \alpha_{32} - \alpha_{42}$. Therefore, we must have $\alpha_{31} = \alpha_{32}$, $\alpha_{41} = \alpha_{42}$. Moreover, by the first and second equation in (4.6) again we have that $\alpha_{11} = \alpha_{12}$, $\alpha_{21} = \alpha_{22}$.

Preliminary for Lemma 2.2*:*

In order to check the regularity conditions of maximum likelihood estimation, we would like to investigate those asymptotics when $t \downarrow 0$. An intuitive solution is to use the Tauberian theorem, which links the asymptotic of Laplace transform and probability measure. However, the Tauberian theorem only deals with a single measure, but we need a neighborhood argument. We therefore give an extension of de Bruijn's Tauberian's theorem, but first we need three lemmas.

Lemma 4.4. *If $\forall\rho > 0$, $\limsup_n \rho x_n \log \mu_n(0, \frac{1}{\phi(\rho x_n)}) \le -B$, $\lambda_0 = (\frac{B}{-\alpha})^{\frac{1}{\alpha-1}} > 0$, then $\forall\rho > 0$*

$$\limsup_n \rho x_n \log \int_{\frac{1}{\phi(\frac{\rho x_n}{\xi})}}^{+\infty} e^{-x\psi(\rho x_n)} d\mu_n(x) \le -B\xi - \xi^{\alpha} \quad 0 < \xi < \lambda_0, \tag{4.8}$$

$$\limsup_n \rho x_n \log \int_{-\infty}^{\frac{1}{\phi(\frac{\rho x_n}{\xi})}} e^{-x\psi(\rho x_n)} d\mu_n(x) \le -B\xi - \xi^{\alpha} \quad \xi > \lambda_0. \tag{4.9}$$

Proof. We prove inequality (4.9) for the $\xi > \lambda_0$ case. Let $\xi_k = \xi + k\epsilon$, $k = 1 \cdots K$. Assume $\frac{1}{\phi\left(\frac{\rho x_n}{\xi_k}\right)} \geq \frac{1}{\phi\left(\frac{\rho x_n}{\xi_{k+1}}\right)}$ then

$$\int_{\frac{1}{\phi\left(\frac{\rho x_n}{\xi_{k+1}}\right)}}^{\frac{1}{\phi\left(\frac{\rho x_n}{\xi_k}\right)}} \exp(-x\psi(\rho x_n))d\mu_n(x) \leq \mu_n\left(0, \frac{1}{\phi\left(\frac{\rho x_n}{\xi_k}\right)}\right)\exp\left(-\frac{1}{\rho x_n} \times \frac{\phi(\rho x_n)}{\phi\left(\frac{\rho x_n}{\xi_{k+1}}\right)}\right).$$

By Potter's theorem, when n large, $\forall k \in (0, \cdots, K-1)$, $\exists \delta \in (0,1)$ such that

$$\frac{\phi(\rho x_n)}{\phi(\frac{\rho x_n}{\xi_{k+1}})} \geq \delta \min\left(\xi_{k+1}^{\alpha'}, \xi_{k+1}^{\alpha''}\right),$$

where $\alpha'' - \alpha = \alpha - \alpha' > 0$. In addition, by assumption, we can choose $-B' > -B$ such that

$$\mu_n\left(0, \frac{1}{\phi\left(\frac{\rho x_n}{\xi_k}\right)}\right) \leq \exp\left(-\frac{1}{\rho x_n} B'\xi_k\right).$$

Next, we have

$$\int_{\frac{1}{\phi\left(\frac{\rho x_n}{\xi_{k+1}}\right)}}^{\frac{1}{\phi\left(\frac{\rho x_n}{\xi_k}\right)}} \exp\left(-x\psi\left(\rho x_n\right)\right) d\mu_n(x) \leq \exp\left(-\frac{1}{\rho x_n}\left(B'\xi_k + \delta \min\left(\xi_{k+1}^{\alpha'}, \xi_{k+1}^{\alpha''}\right)\right)\right).$$

Let $p(x) = -B'x - \delta \min(x^{\alpha'}, x^{\alpha''})$, then

$$\int_{\frac{1}{\phi\left(\frac{\rho x_n}{\xi_{k+1}}\right)}}^{\frac{1}{\phi\left(\frac{\rho x_n}{\xi_k}\right)}} \exp(-x\psi(\rho x_n))d\mu_n(x) \leq \exp\left(\frac{1}{\rho x_n}\left(p(\xi_k) + B'\epsilon\right)\right).$$

When $x > \xi$,

$$\begin{aligned}\frac{p(x) - p(\xi)}{x - \xi} &= -B' - \left(\frac{\delta \min\left(x^{\alpha'}, x^{\alpha''}\right) - \delta \min\left(\xi^{\alpha'}, \xi^{\alpha''}\right)}{x - \xi}\right) \\ &\leq -B' - \min\left(\alpha''\xi^{\alpha''-1}, \alpha'\xi^{\alpha'-1}\right).\end{aligned}$$

Because $(-\alpha\lambda_0)^{\alpha-1} = B$, we know $(-\alpha\xi)^{\alpha-1} < B$. Then B', α' and α'' can be chosen s.t.

$$\max\left(-\alpha''\xi^{\alpha''-1}, -\alpha'\xi^{\alpha'-1}\right) < -B'.$$

Thus

$$\frac{p(x) - p(\xi)}{x - \xi} \leq -B' + \max(-\alpha''\xi^{\alpha''-1}, -\alpha'\xi^{\alpha'-1}) \leq C < 0.$$

So we can have:

$$\int_{\frac{1}{\phi\left(\frac{\rho x_n}{\xi_{k+1}}\right)}}^{\frac{1}{\phi\left(\frac{\rho x_n}{\xi_k}\right)}} \exp(-x\psi(\rho x_n))d\mu_n(x) \leq \exp\left(\frac{1}{\rho x_n}\left(p(\xi) + C(k+1)\epsilon + B'\epsilon\right)\right).$$

Sum up the $k-1$ terms:

$$\begin{aligned}\int_{\frac{1}{\phi\left(\frac{\rho x_n}{\xi_{k+1}}\right)}}^{\frac{1}{\phi\left(\frac{\rho x_n}{\xi_k}\right)}} \exp(-x\psi(\rho x_n))d\mu_n(x) &\leq \frac{\exp\left(\frac{1}{\rho x_n}(p(\xi) + (B' + C)\epsilon)\right)}{1 - \exp\left(\frac{C\epsilon}{\rho x_n}\right)} \\ &\leq (1 + o(1))\exp\left(\frac{1}{\rho x_n}(p(\xi) + B'\epsilon)\right).\end{aligned}$$

On the other hand,

$$\int_{-\infty}^{\frac{1}{\phi\left(\frac{\rho x_n}{\xi_K}\right)}} \exp(-x\psi(\rho x_n))d\mu_n(x) \leq \mu_n\left(0, \frac{1}{\phi(\frac{\rho x_n}{\xi_K})}\right) \leq \exp\left(\frac{1}{\rho x_n}(-B')\xi_K\right).$$

By making K large, we have:

$$\int_{-\infty}^{\frac{1}{\phi\left(\frac{\rho x_n}{\xi}\right)}} \exp(-x\psi(\rho x_n))d\mu_n(x) \leq (1+o(1))\exp\left(\frac{1}{\rho x_n}(p(\xi)+B'\epsilon)\right).$$

So

$$\limsup_n \rho x_n \log \int_{-\infty}^{\frac{1}{\phi\left(\frac{\rho x_n}{\xi}\right)}} \exp(-x\psi(\rho x_n))d\mu_n(x) \leq p(\xi)+B'\epsilon = -B'\xi - \delta\min(\xi^{\alpha'}, \xi^{\alpha''}).$$

Let $B' \uparrow B$, $\delta \uparrow 1$, $\alpha' \uparrow \alpha$, $\alpha'' \downarrow \alpha$ and $\epsilon \downarrow 0$, we know:

$$\limsup_n \rho x_n \log \int_{-\infty}^{\frac{1}{\phi\left(\frac{\rho x_n}{\xi}\right)}} \exp(-x\psi(\rho x_n))d\mu_n(x) \leq -B\xi - \xi^{\alpha}.$$

As to inequality (4.8), use an analogous proof with $\xi_k = \xi - k\epsilon$. □

Lemma 4.5. *If* $\forall\rho > 0$, $\limsup_n \rho x_n \log \mu_n \left(0, \frac{1}{\phi(\rho x_n)}\right) \leq -B$, *then* $\limsup_n \rho x_n \log M_n(\psi(\rho x_n)) \leq h(B)$.

Proof. With the same λ_0, choose $0 < \xi_1 < \lambda_0 < \xi_2 < +\infty$. Then

$$\begin{aligned}\limsup_n \rho x_n \log \int_{\frac{1}{\phi\left(\frac{\rho x_n}{\xi_2}\right)}}^{\frac{1}{\phi\left(\frac{\rho x_n}{\xi_1}\right)}} \exp(-x\psi(\rho x_n))d\mu_n(x) &\leq \limsup_n \rho x_n \log\left(\exp\left(-\frac{\phi(\rho x_n)}{\rho x_n \phi\left(\frac{\rho x_n}{\xi_2}\right)}\right)\times \mu_n\left(0, \frac{1}{\phi\left(\frac{\rho x_n}{\xi_1}\right)}\right)\right)\\ &\leq -B\xi_1 - \xi_2^{\alpha}.\end{aligned}$$

From Lemma 4.4 we have

$$\limsup_n \rho x_n \log \int_{\frac{1}{\phi\left(\frac{\rho x_n}{\xi_1}\right)}}^{+\infty} e^{-x\psi(\rho x_n)}d\mu_n(x) \leq -B\xi_1 - \xi_1^{\alpha} < -B\xi_1 - \xi_2^{\alpha},$$

$$\limsup_n \rho x_n \log \int_{-\infty}^{\frac{1}{\phi\left(\frac{\rho x_n}{\xi_2}\right)}} e^{-x\psi(\rho x_n)}d\mu_n(x) \leq -B\xi_2 - \xi_2^{\alpha} < -B\xi_1 - \xi_2^{\alpha}.$$

Thus for large n

$$\int_{-\infty}^{+\infty} \exp(-x\psi(\rho x_n))d\mu_n(x) \leq 3\exp\left(-\frac{-B\xi_1 - \xi_2^{\alpha}}{\rho x_n}\right).$$

Let $\xi_1 \uparrow \lambda_0$ and $\xi_2 \downarrow \lambda_0$, it is not hard to see:

$$\limsup_n \rho x_n \log(M_n(\psi(\rho x_n))) \leq h(B).$$

□

Lemma 4.6. *If* $\forall\rho > 0$, $\limsup_n \rho x_n \log \mu_n \left(0, \frac{1}{\phi(\rho x_n)}\right) \leq -B$, *with* $\liminf_n \rho x_n \log M_n(\psi(\rho x_n)) \geq C > -\infty$. *Then* $\liminf_n \rho x_n \log \mu_n \left(0, \frac{1}{\phi(\rho x_n)}\right) \geq -B\frac{\lambda_2}{\lambda_1}$. *Where* $\lambda_2 \geq \lambda_1$ *are two roots of* $-B\lambda - \lambda^{\alpha} = C$.

Proof. From Lemma 4.5, we have $C \leq h(B) = \sup(-B\lambda - \lambda^{\alpha})$, so there are two roots $\lambda_1 \leq \lambda_2$ in $(0, +\infty)$, which coincides if and only if $h(B) = C$.

Choose $0 < \eta_1 < \lambda_1 \leq \lambda_2 < \eta_2 < \infty$, by Lemma 4.4:

$$\limsup_n \rho x_n \log \int_{\frac{1}{\phi\left(\frac{\rho x_n}{\eta_1}\right)}}^{+\infty} e^{-x\psi(\rho x_n)}d\mu_n(x) \leq -B\eta_1 - \eta_1^{\alpha} < -B\lambda_1 - \lambda_1^{\alpha} = C,$$

$$\limsup_n \rho x_n \log \int_{-\infty}^{\frac{1}{\phi\left(\frac{\rho x_n}{\eta_2}\right)}} e^{-x\psi(\rho x_n)}d\mu_n(x) \leq -B\eta_2 - \eta_2^{\alpha} < -B\lambda_2 - \lambda_2^{\alpha} = C.$$

So for ϵ small and n large, we know:

$$\int_{\frac{1}{\phi(\frac{\rho x_n}{\eta_1})}}^{+\infty} e^{-x\psi(\rho x_n)} d\mu_n(x) \leq \exp\left(\frac{1}{\rho x_n}(C-2\epsilon)\right),$$

$$\int_{-\infty}^{\frac{1}{\phi(\frac{\rho x_n}{\eta_2})}} e^{-x\psi(\rho x_n)} d\mu_n(x) \leq \exp\left(\frac{1}{\rho x_n}(C-2\epsilon)\right),$$

where both of e^x and $\exp(x)$ refer to the natural exponential of x, as usual.
By hypothesis, we have: when n is large

$$\int_{-\infty}^{+\infty} e^{-\psi(\rho x_n)x} d\mu_n(x) \geq \exp(\rho x_n(C-\epsilon)).$$

Then $\int_{\frac{1}{\phi\left(\frac{\rho x_n}{\eta_2}\right)}}^{\frac{1}{\phi\left(\frac{\rho x_n}{\eta_1}\right)}} e^{-\psi(\rho x_n)x} d\mu_n(x) \geq (1+o(1))\exp(\rho x_n(C-\epsilon))$. By which we have:

$$\liminf_n \rho x_n \log \int_{\frac{1}{\phi(\frac{\rho x_n}{\eta_2})}}^{\frac{1}{\phi(\frac{\rho x_n}{\eta_1})}} e^{-\psi(\rho x_n)x} d\mu_n(x) \geq C.$$

On the other hand,

$$\begin{aligned}\liminf_n \rho x_n \log \int_{\frac{1}{\phi\left(\frac{\rho x_n}{\eta_2}\right)}}^{\frac{1}{\phi\left(\frac{\rho x_n}{\eta_1}\right)}} e^{-\psi(\rho x_n)x} d\mu_n(x) &\leq \liminf_n \rho x_n \log\left(\exp\left(-\frac{\phi(\rho x_n)}{\phi\left(\frac{\rho x_n}{\eta_2}\right)}\frac{1}{\rho x_n}\right)\times \mu_n\left(0, \frac{1}{\phi(\frac{\rho x_n}{\eta_1})}\right)\right) \\ &\leq -\eta_2^{\alpha} + \eta_1 \liminf_n \frac{\rho x_n}{\eta_1} l \log\left(\mu_n\left(0, \frac{1}{\phi\left(\frac{\rho x_n}{\eta_1}\right)}\right)\right).\end{aligned}$$

So $\liminf_n \frac{\rho x_n}{\eta_1} l \log\left(\mu_n\left(0, \frac{1}{\phi(\frac{\rho x_n}{\eta_1})}\right)\right) \geq \frac{C+\eta_2^{\alpha}}{\eta_1}$. Because of the arbitrary choice of ρ:

$$\liminf_n \rho x_n \log\left(\mu_n\left(0, \frac{1}{\rho x_n}\right)\right) \geq \frac{C+\eta_2^{\alpha}}{\eta_1}.$$

Finally, let $\eta_2 \downarrow \lambda_2$ and $\eta_1 \uparrow \lambda_1$ to finish the proof. □

Moving towards Lemma 4.8 below, we can study the asymptotic property of $g(t|\alpha)$ in a neighbourhood of parameters as $t \downarrow 0$. Recall that $\log(s\hat{g}(s|\alpha)) \sim 2\sqrt{\alpha_1}(\sqrt{\alpha_3} - \sqrt{\alpha_4})s^{\frac{1}{2}}$, as $s \to +\infty$, then from the de Bruijn's Tauberian theorem, for fixed α, we have:

$$\log g(t|\alpha) \sim -\alpha_1\left(\sqrt{\alpha_3} - \sqrt{\alpha_4}\right)^2 \frac{1}{t} \quad \text{as} \quad t \downarrow 0.$$

Thus we can write $g(t|\alpha) = \exp\left(-\frac{\alpha_1(\sqrt{\alpha_3}-\sqrt{\alpha_4})^2}{t} + \frac{l(t|\alpha)}{t}\right)$. Since we only know for fixed α, $l(t|\alpha) \to 0$, we would like to extend it to a neighbourhood argument:

Lemma 4.7. *For all α^0 is in the parameter space, there exits a closure of α^0, such that $l(t|\alpha)$ goes to zero uniformly as $t \downarrow 0$.*

Proof. We prove the lemma by contradiction. In a small closed neighborhood of α^0, if $l(t|\alpha) \not\to 0$ uniformly, then $\exists$ $\epsilon > 0$, $\alpha^n \to \tilde{\alpha}$ are in the neighborhood, $t_n \to 0$ such that $|l(\alpha^n, t_n)| > \epsilon$. Next, from the Laplace transform of $g(t|\alpha)$, we know that $g(t|\alpha)$ is differentiable with respect to t, and we have

$$\int_0^{+\infty} \exp(-st)\frac{\partial g(t|\alpha)}{\partial t} dt = \frac{sF(s\alpha_1, \alpha_2, \alpha_3)}{F(s\alpha_1, \alpha_2, \alpha_4)}.$$

By Theorem 1.2 in [22], the density function of first passage time of diffusion is unimodal (we can also prove this by constructing a sequence of birth-and-death processes that converge to the Feller-CIR model weakly). Then $\exists\, t_0$ such that, when $0 < t \leq t_0$, $\frac{\partial g(t|\tilde{\alpha})}{\partial t} > 0$. By the continuity of $\frac{\partial g(t|\alpha)}{\partial t}|_{t=t_0}$ in α, we know that $\exists\, N$ such that, when $n > N$, $\frac{\partial g(t|\alpha^n)}{\partial t}|_{t=t_0} > 0$. By unimodality again, we know that if $n > N$, $t \leq t_0$, $\frac{\partial g(t|\alpha^n)}{\partial t} > 0$. So we can treat $g(t|\alpha^n)$ as a sequence of measures with Radon-Nikodym derivative $\frac{\partial g(t|\alpha^n)}{\partial t}$ on $(0, t_0)$.

On the other hand, we know that $\exists\, k$ and N_1 such that when $n > N_1$, $t > t_0$

$$\left|\frac{\partial g(t|\alpha^n)}{\partial t}\right| \leq k,$$

from which we know that

$$\frac{sF(s\alpha_1, \alpha_2, \alpha_3)}{F(s\alpha_1, \alpha_2, \alpha_4)} - k\frac{\exp(-st_0)}{s} \leq \int_0^{t_0} \exp(-st)\frac{\partial g(t|\alpha^n)}{\partial t}dt \leq \frac{sF(s\alpha_1, \alpha_2, \alpha_3)}{F(s\alpha_1, \alpha_2, \alpha_4)} + k\frac{\exp(-st_0)}{s}. \tag{4.10}$$

When $s \to \infty$, $\frac{sF(s\alpha_1,\alpha_2,\alpha_3)}{F(s\alpha_1,\alpha_2,\alpha_4)}$ dominates $k\frac{\exp(-st_0)}{s}$, then $\forall\, \rho > 0$, take $s = \frac{1}{t_n}$:

$$\lim_n \rho t_n \log \int_0^{t_0} \exp\left(-\frac{t}{\rho^2 t_n^2}\right)\frac{\partial g(t|\alpha^n)}{\partial t}dt = -2\sqrt{\tilde{\alpha}_1}\left(\sqrt{\tilde{\alpha}_4} - \sqrt{\tilde{\alpha}_3}\right). \tag{4.11}$$

With Lemma 4.8, take $\phi(x) = \frac{1}{x}$, $\psi(x) = \frac{1}{x^2}$, we know that:

$$\lim_n t_n \log g(t_n|\alpha_n) = -\tilde{\alpha}_1\left(\sqrt{\tilde{\alpha}_4} - \sqrt{\tilde{\alpha}_3}\right)^2 \tag{4.12}$$

which contradicts with our assumption. $\square$

Lemma 4.8. *Let μ_n be a sequence of measures supported on $(0, \infty)$, $M_n(\lambda) = \int_0^\infty e^{-\lambda x}d\mu_n(x)$. $\phi \in R_\alpha(0^+)$ (function of regular variation with index $\alpha < 0$.$\psi(\lambda) = \phi(\lambda)/\lambda \in R_{\alpha-1}(0^+)$. Then for a sequence of $x_n \to 0^+$ and $B > 0$: $\forall\rho > 0$, $\lim_n \rho x_n \log(M_n(\psi(\rho x_n))) = h(B) = -(1-\alpha)(\frac{B}{-\alpha})^{\frac{\alpha}{\alpha-1}}$ if and only if $\forall\rho > 0$, $\lim_n \rho x_n \log \mu_n\left(0, \frac{1}{\phi(\rho x_n)}\right) = -B$.*

Proof. If $\forall\rho > 0$, $\lim_n \rho x_n \log \mu_n\left(0, \frac{1}{\phi(\rho x_n)}\right) = -B$, by Lemma 4.5, we know that

$$\limsup_n \rho x_n \log M_n(\psi(\rho x_n)) \leq h(B).$$

However, $\forall\xi > 0$, $\liminf_n \rho x_n \log M_n(\psi(\rho x_n)) \geq -\xi^\alpha + \xi \liminf_n (\frac{\rho x_n}{\xi}) \log \mu_n\left(0, \frac{1}{\phi(\frac{\rho x_n}{\xi})}\right)$, so we must have:

$$\liminf_n \rho x_n \log M_n(\psi(\rho x_n)) \geq h(B).$$

Then $\liminf_n \rho x_n \log M_n(\psi(\rho x_n)) = h(B)$.

If $\forall\rho > 0$, $\liminf_n \rho x_n \log M_n(\psi(\rho x_n)) = h(B)$. First, $\forall\xi > 0$,

$$\limsup_n \rho x_n \log M_n(\psi(\rho x_n)) \geq -\xi^\alpha + \xi \limsup_n \left(\frac{\rho x_n}{\xi}\right) \log \mu_n\left(0, \frac{1}{\phi\left(\frac{\rho x_n}{\xi}\right)}\right).$$

Because of the arbitrary choice of ρ and ξ,

$$h(B) \geq -\xi^\alpha + \xi \limsup_n \rho x_n \log \mu_n\left(0, \frac{1}{\phi(\rho x_n)}\right).$$

Then $\limsup_n \rho x_n \log \mu_n\left(0, \frac{1}{\phi(\rho x_n)}\right) \leq -B$. By Lemmas 4.5 and 4.6, we know

$$\liminf_n \rho x_n \log \mu_n\left(0, \frac{1}{\phi(\rho x_n)}\right) \geq -B.$$

So we must have:

$$\lim_n \rho x_n \log \mu_n \left(0, \frac{1}{\phi(\rho x_n)}\right) = -B.$$

□

Proof of Lemma 2.2. Define $m_\alpha(t) = \log g(t|\alpha)$, we first show that $\mathcal{M} = \{m_\alpha : \|\alpha - \alpha^0\| \leq \delta\}$ is a Donsker class under probability measure P_{α^0} (the measure with density function $g(t|\alpha)$). By Lemmas 4.7 and 4.8, $\exists\, A_1 > 0, A_2 > 0$ such that $\forall \epsilon > 0$, $\forall K > 0$, $\forall m_\alpha \in \mathcal{M}$, $0 < t \leq K\epsilon$:

$$m_\alpha^2(t) \leq \frac{A_1^2}{t^2}, \quad g(t|\alpha^0) \leq \exp\left(-\frac{A_2}{t}\right)$$

by which we have $\forall m_\alpha, m_\beta \in \mathcal{M}$:

$$\begin{aligned}\int_0^{K\epsilon} (m_\alpha(t) - m_\beta(t))^2 g(t|\alpha^0)dt &\leq \int_0^{K\epsilon} \frac{4A_1^2}{t^2} \exp\left(-\frac{A_2}{t}\right)dt \\ &= \int_{\frac{1}{K\epsilon}}^{+\infty} 4A_1^2 \exp(-A_2 t)dt = 4\frac{A_1^2}{A_2} \exp\left(-\frac{A_2}{K\epsilon}\right).\end{aligned}$$

So we can find a universal $K_1 > 0$, such that $\forall \epsilon > 0$, $\forall m_\alpha \in \mathcal{M}$,

$$\int_0^{K_1\epsilon} (m_\alpha(t) - m_\beta(t))^2 g(t|\alpha^0)dt \leq \frac{\epsilon^2}{2}. \tag{4.13}$$

So when $0 < t \leq K_1\epsilon$, we can choose the only one bracket to be $[-\frac{A_1}{t}, \frac{A_1}{t}]$, which can cover $m_\alpha(t)$. When $K_1\epsilon < t < t_1$ where t_1 is by the continuity of $\frac{\partial \log g(t|\alpha)}{\partial\alpha} = \frac{\frac{\partial g(t|\alpha)}{\partial\alpha}}{g(t|\alpha)}$, $\exists A_3 > 0, A_4 > 0$:

$$|m_\alpha(t) - m_\beta(t)| \leq \frac{A_3}{\exp(\frac{-A_4}{t})}\|\alpha - \beta\| \leq A_3 \exp\left(\frac{A_4}{K_1\epsilon}\right)\|\alpha - \beta\|. \tag{4.14}$$

When $t \geq t_1$, $\exists A_5 > 0, A_6 > 0$ such that:

$$|m_\alpha(t) - m_\beta(t)| \leq (A_5 + A_6 t)\|\alpha - \beta\|. \tag{4.15}$$

Define

$$m(t) = \begin{cases} A_3 \exp\left(\dfrac{A_4}{t}\right) & \text{if } K_1\epsilon < t < t_1 \\ (A_5 + A_6 t) & \text{if } t \leq t_1. \end{cases}$$

Then we can cover the class of functions using a class of brackets

$$\left\{\left[m_\alpha - \frac{A_7\epsilon m}{\sqrt{2}}, m_\alpha + \frac{A_7\epsilon m}{\sqrt{2}}\right] : \|\alpha - \alpha^0\| \leq \delta\right\}.$$

Next, the number of brackets we need is at most $(\frac{\sqrt{2}\delta}{A_7\epsilon})^4$. Now we give an value for A_7:

$$\begin{aligned}\int_{K_1\epsilon}^{+\infty} 2A_7^2\epsilon^2 m^2(t)g(t|\alpha^0)dt &\leq \int_{K_1\epsilon}^{t_1} 2A_7^2\epsilon^2 A_3^2 \exp(\frac{2A_4}{t})g(t|\alpha^0)dt + \int_{t_1}^{\infty} 2A_7^2\epsilon^2(A_5 + A_6 t)^2 g(t|\alpha^0)dt \\ &\leq \int_{K_1\epsilon}^{t_1} 2A_7^2\epsilon^2 A_3^2 \exp(\frac{2A_4}{K_1\epsilon})g(t|\alpha^0)dt + \int_{t_1}^{\infty} 2A_7^2\epsilon^2(A_5 + A_6 t)^2 g(t|\alpha^0)dt \\ &\leq A_7^2\epsilon^2 \exp\left(\frac{2A_4}{K_1\epsilon}\right)C_1.\end{aligned}$$

Since we confine ourselves with $\int_{K_1\epsilon}^{+\infty} 2A_7^2\epsilon^2 m^2(t)g(t|\alpha^0)dt \leq \frac{\epsilon^2}{2}$, thus: $A_7 \leq C\exp\left(-\frac{A_4}{K_1\epsilon}\right)$. The number of brackets can be bounded by $\left(\frac{\sqrt{2}\delta}{C\epsilon}\exp\left(\frac{A_4}{K_1\epsilon}\right)\right)^4$. Combine the two parts, using the notation in [25]:

$$N_{[\ \ ]}(\epsilon, \mathcal{M}, L^2(P_{\alpha^0})) \leq \left(\frac{\sqrt{2}\delta}{C\epsilon}\exp\left(\frac{A_4}{K_1\epsilon}\right)\right)^4.$$

While $J_{[\ \ ]}(\xi, \mathcal{M}, L^2(P_{\alpha^0})) = \int_0^\xi \sqrt{\log N_{[\ \ ]}(\xi, \mathcal{M}, L^2(P_{\alpha^0}))}d\xi$, we know $J_{[\ \ ]}(\xi, \mathcal{M}, L^2(P_{\alpha^0})) < +\infty$, by Theorem 19.5 in [25], $\mathcal{M}$ is Donsker. Actually, we only need that $\mathcal{M}$ is a Glivenko-Cantelli class. Apply Theorem 5.7 in [25] to the log likelihood function, the MLE $\hat{\alpha}_n$ converges in probability to the true parameter $\alpha^{(0)}$. □

Proof of Theorem 2.3. By a similar method used to prove the identifiability of parameters in Lemma 2.1, given only the observations, the identifiable parameters are α. With the definition of $\tilde{g}(t|\alpha, \Delta)$, we know that:

$$\frac{\partial \log \tilde{g}(t|\alpha, \Delta)}{\partial \alpha_i} = \frac{g_i(t|\alpha)}{g(t|\alpha)} + \frac{\int_0^\Delta g_i(s|\alpha)ds}{1 - \int_0^\Delta g(s|\alpha)ds}.$$

Because the support of t is bounded away from zero, by the continuity of $\frac{\partial \log \tilde{g}(t|\alpha,\Delta)}{\partial \alpha_i}$, and the mean value theorem for the multivariate case, we know that for all α and β in a neighborhood of the true parameter α_0, $\exists\ q(t) = a + bt$ such that:

$$\left|\frac{\partial \log \tilde{g}(t|\alpha, \Delta)}{\partial \alpha_i}\right| \leq q(t) \quad \text{for} \quad t \geq \Delta,$$

$$|\log \tilde{g}(t|\alpha, \Delta) - \log \tilde{g}(t|\beta, \Delta)| \leq q(t) \quad \text{for} \quad t \geq \Delta.$$

By the dominated convergence theorem, the information matrix $\tilde{I}(\alpha)$ is continuous at α^0. Using Theorem 7.6 in [25], $\alpha \to \sqrt{\tilde{g}(t|\alpha, \Delta)}$ is differentiable in quadratic mean. The class of log density functions also satisfies the required Lipschitz condition. As with Example 19.7 in [25], it is a Donsker class. Applying Theorem 5.7 in [25], we know that the consistency is valid. By Lipschitz condition along with differentiability in quadratic mean, we can apply Theorem 5.49 in [25] to finally get asymptotic normality. □

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