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Alternative proofs of two mathematical propositions on two elementary functions

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Abstract

In the paper, with the aid of some properties of the extended mean values, the authors provide two alternative proofs of two lemmas employed in the paper (B.-Y. Xi and F. Qi, Necessary and sufficient conditions of Schur m-power convexity of a new mixed mean, Filomat 38 (19), 6937–6944, 2024).

Keywords: Extended mean value, integral representation, necessary and sufficient condition, alternative proof, decreasing property, inequality

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1. Introduction

In [30, Section 2], the authors proved the following two propositions.

Proposition 1.1 (*cf.* [30, Lemma 2]). *For $r \in (-1, 1)$ and $x \in (0, 1)$, let*

$$V_r(x) = \frac{1}{(x+1)^2} \frac{x^{r+1} - 1}{x^{r-1} - 1}. \tag{1.1}$$

Then the following conclusions are true:

1. *The function $V_r(x)$ is decreasing in $x \in (0, 1)$ for given $r \in (-1, 1)$.*
2. *The double inequality $\frac{r+1}{4(r-1)} < V_r(x) < 0$ is valid for $x \in (0, 1)$ and $r \in (-1, 1)$.*

Proposition 1.2 (*cf.* [30, Lemma 3]). *For $\omega \ge 0$ and $r \in \mathbb{R}$, let*

$$F_{r,\omega}(x) = (x+1)^2(1 - x^{1-r}) + 4\omega x^{1-r}(x^{r+1} - 1), \quad x \in (0, 1]. \tag{1.2}$$

Then the following conclusions are true:

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1. *The non-negativity $F_{r,\omega}(x) \ge 0$ holds for $x \in (0,1]$ if and only if*

$$(r,\omega) \in \left\{(r,\omega) : r < 1, \frac{r+1}{r-1}\omega \ge -1, \omega \ge 0\right\}.$$

2. *The non-positivity $F_{r,\omega}(x) \le 0$ holds for $x \in (0,1]$ if and only if*

$$(r,\omega) \in \{(r,\omega) : r \ge 1, \omega \ge 0\}.$$

Taking $x = e^{-2t}$ for $t \in (0,\infty)$ in (1.1) yields

$$V_r(x) = \frac{1}{4}\mathcal{V}_r(t) = \frac{1}{4}\frac{\sinh[(r+1)t]}{\cosh^2 t \sinh[(r-1)t]}, \quad t \in (0,\infty).$$

By Proposition 1.1, we derive that the function $\mathcal{V}_r(t)$ is negative and increasing in $t \in (0,\infty)$ for fixed $r \in (-1,1)$, as well as $\frac{r+1}{r-1} < \mathcal{V}_r(x) < 0$ for all $t \in (0,\infty)$ and $r \in (-1,1)$.

The aim of this paper is to provide two alternative proofs of Propositions 1.1 and 1.2, with the aid of some properties of the extended mean values $E(r,s;\alpha,\beta)$ defined by the formula (2.1) below.

2. Alternative proofs

We now start out to state our two alternative proofs of Propositions 1.1 and 1.2.

2.1. *An alternative proof of Proposition* 1.1

The function $V_r(x)$ defined in (1.1) can be rewritten as

$$V_r(x) = \frac{r+1}{r-1}\left[\frac{1}{x+1}\left(\frac{r-1}{r+1}\frac{x^{r+1}-1}{x^{r-1}-1}\right)^{1/2}\right]^2 = \frac{r+1}{r-1}\left[\frac{E(r-1,r+1;x,1)}{x+1}\right]^2,$$

where

$$E(r,s;\alpha,\beta) = \begin{cases} \left(\dfrac{r}{s}\dfrac{\beta^s-\alpha^s}{\beta^r-\alpha^r}\right)^{1/(s-r)}, & rs(r-s)(\alpha-\beta) \ne 0 \\ \left(\dfrac{1}{r}\dfrac{\beta^r-\alpha^r}{\ln\beta-\ln\alpha}\right)^{1/r}, & r(\alpha-\beta) \ne 0 \text{ and } s=0 \\ \dfrac{1}{e^{1/r}}\left(\dfrac{\alpha^{\alpha^r}}{\beta^{\beta^r}}\right)^{1/(\alpha^r-\beta^r)}, & \alpha \ne \beta \text{ and } r=s\ne 0 \\ \sqrt{\alpha\beta}, & \alpha\ne\beta \text{ and } r=s=0 \\ \alpha, & \alpha=\beta \text{ and } r,s\in\mathbb{R} \end{cases} \tag{2.1}$$

for $(r,s) \in \mathbb{R}^2$ and $(\alpha,\beta) \in (0,\infty)^2$ denotes the extended mean values. The mean values $E(r,s;\alpha,\beta)$ have been systematically studied, comprehensively reviewed, and briefly surveyed in [1]-[18], [22]-[28], [31, 32], for example. As a result, it suffices to prove that the function $\frac{E(r-1,r+1;x,1)}{x+1}$ is increasing in $x \in (0,1)$ for given $r \in (-1,1)$.

Let $f : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ be defined by

$$f(t,s) = \begin{cases} \dfrac{s^t-1}{t}, & t \ne 0; \\ \ln s, & t = 0. \end{cases}$$

The function $f(t,s)$ can be reformulated as the integral representation

$$f(t,s) = \int_1^s u^{t-1}\,\mathrm{d}u. \tag{2.2}$$

The integral representation (2.2) has been investigated and applied in [7, 8], [16]-[18], [22, 23], for example. Since

$$E(r-1, r+1; x, 1) = \left[\frac{f(r+1, x)}{f(r-1, x)}\right]^{1/2},$$

taking the logarithm and differentiating with respect to x lead to

$$\begin{aligned}\ln\frac{E(r-1, r+1; x, 1)}{x+1} &= \frac{1}{2}\ln\frac{f(r+1, x)}{f(r-1, x)} - \ln(x+1)\\ &= \frac{1}{2}\int_{r-1}^{r+1}\frac{\partial f(s, x)/\partial s}{f(s, x)}\,\mathrm{d}s - \ln(x+1)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial x}\ln\frac{E(r-1, r+1; x, 1)}{x+1} &= \frac{1}{2}\int_{r-1}^{r+1}\frac{\partial}{\partial x}\left[\frac{\partial f(s, x)/\partial s}{f(s, x)}\right]\mathrm{d}s - \frac{1}{x+1}\\ &= \frac{1}{2}\int_{r-1}^{r+1}\frac{\partial^2[\ln f(s, x)]}{\partial x\partial s}\,\mathrm{d}s - \frac{1}{x+1}\\ &= \frac{1}{2}\int_{r-1}^{r+1}\frac{\partial^2}{\partial s\partial x}\left(\ln\frac{x^s-1}{s}\right)\mathrm{d}s - \frac{1}{x+1}\\ &= \frac{1}{2}\int_{r-1}^{r+1}\frac{\partial}{\partial s}\left(\frac{sx^{s-1}}{x^s-1}\right)\mathrm{d}s - \frac{1}{x+1}\\ &= \frac{1}{2x}\int_{r-1}^{r+1}\frac{\partial}{\partial s}\left(s+\frac{s}{x^s-1}\right)\mathrm{d}s - \frac{1}{x+1}\\ &= \frac{1}{2x}\int_{r-1}^{r+1}\left[1+\frac{\partial}{\partial s}\left(\frac{s}{x^s-1}\right)\right]\mathrm{d}s - \frac{1}{x+1}\\ &= \frac{1}{x} - \frac{1}{x+1} + \frac{1}{2x}\int_{r-1}^{r+1}\frac{\partial}{\partial s}\left(\frac{s}{x^s-1}\right)\mathrm{d}s.\end{aligned}$$

From the relation

$$\frac{s}{x^s-1} = -\frac{s}{1-(1/x)^{-s}} = -\frac{1}{\ln(1/x)}\frac{s\ln(1/x)}{1-\mathrm{e}^{-s\ln(1/x)}},$$

we derive that

$$\frac{\partial}{\partial s}\left(\frac{s}{x^s-1}\right) = -\frac{1}{\ln(1/x)}\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{v}{1-\mathrm{e}^{-v}}\right)\frac{\partial[s\ln(1/x)]}{\partial s} = -\mathcal{H}'(v),$$

where $v = s\ln\frac{1}{x}$ and

$$\mathcal{H}(t) = -\frac{1}{f(t, \mathrm{e}^{-1})} = \frac{1}{f(-t, \mathrm{e})} = \begin{cases}\dfrac{t}{1-\mathrm{e}^{-t}}, & t \ne 0\\ 1, & t = 0\end{cases} \tag{2.3}$$

with the first derivative

$$\mathcal{H}'(t) = \begin{cases}\dfrac{\mathrm{e}^t(\mathrm{e}^t - 1 - t)}{(\mathrm{e}^t - 1)^2}, & t \ne 0;\\ \dfrac{1}{2}, & t = 0.\end{cases} \tag{2.4}$$

As a result, we obtain

$$\frac{\partial}{\partial x}\ln\frac{E(r-1, r+1; x, 1)}{x+1} = \frac{1}{x}\left[1 - \frac{1}{2\ln(1/x)}\int_{(r-1)\ln(1/x)}^{(r+1)\ln(1/x)}\mathcal{H}'(v)\,\mathrm{d}v\right] - \frac{1}{x+1}.$$

Since the function $\mathcal{H}(t)$ in (2.3) satisfies the identity $\mathcal{H}(t) - \mathcal{H}(-t) = t$ on $(-\infty, \infty)$, see [20, Lemma 2.3] and [21, Lemma 9], then the first derivative $\mathcal{H}'(t)$ in (2.4) satisfies the identity $\mathcal{H}'(t) + \mathcal{H}'(-t) = 1$ on $(-\infty, \infty)$, see [19, Lemma 2.3] and [21, Lemma 1]. This means that the function $\mathcal{H}'(t) - \frac{1}{2}$ is odd on $(-\infty, \infty)$. Then the integral

$$\begin{aligned}\int_{(r-1)\ln(1/x)}^{(r+1)\ln(1/x)} \mathcal{H}'(v)\,\mathrm{d}v &= \ln\frac{1}{x} + \int_{(r-1)\ln(1/x)}^{(r+1)\ln(1/x)} \biggl[\mathcal{H}'(v) - \frac{1}{2}\biggr]\mathrm{d}v\\ &= \ln\frac{1}{x} + \biggl[\int_{(r-1)\ln(1/x)}^{(1-r)\ln(1/x)} + \int_{(1-r)\ln(1/x)}^{(1+r)\ln(1/x)}\biggr]\biggl[\mathcal{H}'(v) - \frac{1}{2}\biggr]\mathrm{d}v\\ &= \ln\frac{1}{x} + \int_{(1-r)\ln(1/x)}^{(1+r)\ln(1/x)} \biggl[\mathcal{H}'(v) - \frac{1}{2}\biggr]\mathrm{d}v.\end{aligned}$$

In [19, Lemma 2.3], see also [21, Lemma 1], the first derivative $\mathcal{H}'(t)$ in (2.4) was proved to be concave in $t \in (0, \infty)$. Thus, by the Hermite–Hadamard integral inequality for concave functions [5, 9, 29], we find

$$\begin{aligned}\int_{(1-r)\ln(1/x)}^{(1+r)\ln(1/x)} \biggl[\mathcal{H}'(v) - \frac{1}{2}\biggr]\mathrm{d}v &= \int_{(1-r)\ln(1/x)}^{(1+r)\ln(1/x)} \mathcal{H}'(v)\,\mathrm{d}v - \int_{(1-r)\ln(1/x)}^{(1+r)\ln(1/x)} \frac{1}{2}\,\mathrm{d}v\\ &\le 2r\mathcal{H}'\biggl(\ln\frac{1}{x}\biggr)\ln\frac{1}{x} - r\ln\frac{1}{x}\\ &= 2r\biggl[\mathcal{H}'\biggl(\ln\frac{1}{x}\biggr) - \frac{1}{2}\biggr]\ln\frac{1}{x}.\end{aligned}$$

Accordingly, we arrive at

$$\begin{aligned}\frac{\partial}{\partial x}\ln\frac{E(r-1, r+1; x, 1)}{x+1} &= \frac{1}{x}\left(\frac{1}{2} - \frac{\int_{(1-r)\ln(1/x)}^{(1+r)\ln(1/x)}[\mathcal{H}'(v) - \frac{1}{2}]\,\mathrm{d}v}{2\ln(1/x)}\right) - \frac{1}{x+1}\\ &\ge \frac{1}{x}\biggl(\frac{1}{2} - r\biggl[\mathcal{H}'\biggl(\ln\frac{1}{x}\biggr) - \frac{1}{2}\biggr]\biggr) - \frac{1}{x+1}\\ &= \frac{1}{x}\biggl[\frac{1+r}{2} - r\mathcal{H}'\biggl(\ln\frac{1}{x}\biggr)\biggr] - \frac{1}{x+1}\\ &= \frac{r(1+x)(x^2 - 2x\ln x - 1) + 1 - 3x + 3x^2 - x^3}{2x(x+1)(x-1)^2}\end{aligned}$$

for $r \in (-1, 1)$ and $x \in (0, 1)$.

In [19, Lemma 2.3], see also [21, Lemma 1], the first derivative $\mathcal{H}'(t)$ in (2.4) was proved to be increasing in $t \in (-\infty, \infty)$. This implies that

$$\frac{1}{2} < \mathcal{H}'\biggl(\ln\frac{1}{x}\biggr) < 1, \quad x \in (0, 1).$$

Consequently, we attain

$$\frac{\partial}{\partial x}\ln\frac{E(r-1, r+1; x, 1)}{x+1} \ge \frac{1}{x}\biggl[\frac{1+r}{2} - r\mathcal{H}'\biggl(\ln\frac{1}{x}\biggr)\biggr] - \frac{1}{x+1} \ge \frac{1}{2x} - \frac{1}{x+1} \ge 0$$

for $r \in (-1, 0]$ and $x \in (0, 1)$.

It is standard to prove that the function $x^2 - 2x\ln x - 1$ is negative in $x \in (0, 1)$. As a result, we obtain

$$\begin{aligned}\frac{\partial}{\partial x}\ln\frac{E(r-1, r+1; x, 1)}{x+1} &\ge \frac{r(1+x)(x^2 - 2x\ln x - 1) + 1 - 3x + 3x^2 - x^3}{2x(x+1)(x-1)^2}\\ &\ge \frac{1}{(x-1)^2}\biggl[\frac{2(x-1)}{x+1} - \ln x\biggr]\\ &\ge 0\end{aligned}$$

for $r \in (-1, 1)$ and $x \in (0, 1)$, where

$$\left[\frac{2(x-1)}{x+1} - \ln x\right]' = -\frac{(x-1)^2}{x(x+1)^2} < 0, \quad x \in (0,1)$$

and

$$\frac{2(x-1)}{x+1} - \ln x \ge \lim_{x\to 1^-}\left[\frac{2(x-1)}{x+1} - \ln x\right] = 0, \quad x \in (0,1).$$

This means that the function $\frac{E(r-1,r+1;x,1)}{x+1}$ is positive and increasing in $x \in (0, 1)$ for fixed $r \in (-1, 1)$. Consequently, the function $V_r(x)$ is negative and decreasing in $x \in (0, 1)$ for given $r \in (-1, 1)$.

From the definition in (2.1), it is easy to see that

$$\lim_{x\to 0^+} E(r-1, r+1; x, 1) = 0 \quad \text{and} \quad E(r-1, r+1; 1, 1) = 1.$$

Consequently, we acquire

$$V_r(x) \le \lim_{x\to 0^+} V_r(x) = \frac{r+1}{r-1}\left[\lim_{x\to 0^+} E(r-1, r+1; x, 1)\right]^2 = 0$$

and

$$V_r(x) \ge \lim_{x\to 1^-} V_r(x) = \frac{r+1}{r-1}\left[\frac{E(r-1, r+1; 1, 1)}{2}\right]^2 = \frac{1}{4}\frac{r+1}{r-1}.$$

The alternative proof of Proposition 1.1 is thus complete.

2.2. *An alternative proof of Proposition* 1.2

For $\omega \ge 0$, $r \in \mathbb{R}$, and $x \in (0, 1]$, we reformulate

$$F_{r,\omega}(x) = \frac{x^{r+1}-1}{r+1}\frac{r-1}{x^{r-1}}\left(\left[\frac{x+1}{E(r-1, r+1; 1, x)}\right]^2 + 4\omega\frac{r+1}{r-1}\right). \tag{2.5}$$

Observing the expressions (1.2) and (2.5), we easily see that,

1. if $r-1 \ge 0$, the non-positivity $F_{r,\omega}(x) \le 0$ in $x \in (0, 1)$ for all $\omega \ge 0$ is trivial;
2. if $r+1 \le 0$, the non-negativity $F_{r,\omega}(x) \ge 0$ in $x \in (0, 1)$ for all $\omega \ge 0$ is trivial;
3. the value $F_{r,\omega}(1) = 0$ for all $\omega \ge 0$ and $r \in \mathbb{R}$ is also trivial.

From the alternative proof of Proposition 1.1, we see that the function $\frac{E(r-1,r+1;x,1)}{x+1}$ is increasing in $x \in (0, 1)$ for any given $r \in (-1, 1)$. Therefore, we obtain

$$\frac{E(r-1, r+1; x, 1)}{x+1} < \frac{E(r-1, r+1; 1, 1)}{2} = 1$$

and

$$\frac{E(r-1, r+1; x, 1)}{x+1} > \lim_{x\to 0^+}\frac{E(r-1, r+1; x, 1)}{x+1} = 0$$

for $x \in (0, 1)$ and $r \in (-1, 1)$. Accordingly, when $1 + \omega\frac{r+1}{r-1} \ge 0$, the non-negativity $F_{r,\omega}(x) \ge 0$ is valid for $x \in (0, x)$, $r \in (-1, 1)$, and $\omega \ge 0$; but the non-positivity $F_{r,\omega}(x) \le 0$ is not valid for $x \in (0, x)$, $r \in (-1, 1)$, and $\omega \ge 0$. The alternative proof of Proposition 1.2 is thus complete.

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