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Some fixed point theorems for a class of contractive mappings over a complete b-multiplicative metric space

Krishna Bhattacharjee [a], Rakhal Das [b], Binod Chandra Tripathy [c]

[a]Department of Mathematics, The ICFAI University, Tripura 799210, India
[b]Department of Mathematics, The ICFAI University, Tripura 799210, India
[c]Department of Mathematics, Tripura University, Agartala, Tripura 799022, India

Abstract

In this study, we investigate a fixed point for a group of contractive mappings in the context of a complete b-multiplicative metric space. This research employs the Rus contraction to derive novel findings related to the complete b-multiplicative metric space. Furthermore, we establish a theorem on common fixed points for two mappings in complete b-multiplicative metric spaces. To validate our results, we provide several non-trivial examples.

Keywords: Contraction mapping, fixed point, common fixed point, multiplicative metric space, b-multiplicative metric space

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1. Introduction

The use of fixed point theory is essential in various scientific, engineering, applied science, and nonlinear analysis research areas.

A highly valuable tool in this field is the Banach contraction theorem. This theorem asserts that if T is a mapping on a complete metric space (X, d) into itself and it meets the condition $d(Tx, Ty) \leq Kd(x, y)$ where $K \in [0, 1)$ for all $x, y \in X$, then T possesses a unique fixed point on X. Banach presented this theorem in 1922 [6]. Since then, numerous authors have built upon this result by utilizing various contractions and mappings in diverse metric spaces.

In the year 1972, multiplicative calculus, also known as non-Newtonian calculus, was first introduced by Grossman and Katz [14] by substituting the functions of division and multiplication for those of subtraction and addition, respectively. The concept of the multiplicative metric was defined by Bashirov et al. [7] utilizing concepts from Grossman and Katz [14] in the year 2008.

In further research examining the topological characteristics of multiplicative metric spaces, Ozavsar and Cevikel [18] established several kinds of contraction mappings in the year 2012. The following have provided an effective contribution in this direction: He et al. [15] have introduced common fixed point results for weak commutative mappings in 2014; Abbas et al. [1] have studied common fixed points of generalized rational type concyclic mappings in 2015; Rome and Sarwar [21] for characterization of multiplicative metric completeness in the year 2016 etc.

†*Article ID:* MTJPAM-D-24-00029
Email addresses: bhattacharjeekrishna413@gmail.com (Krishna Bhattacharjee), rakhaldas95@gmail.com (Rakhal Das), tripathybc@gmail.com (Binod Chandra Tripathy)
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**Corresponding Author:* Rakhal Das

In the year 1989, the analog of the Banach contraction principle in b-metric space was established, which Bakhtin [5] suggested as a generalization of metric space. Usman et al. [4] introduce the concept of b-multiplicative metric space and discuss some of its topological properties based on the aforementioned observations in the year 2017.

Rus [19] first introduced the Rus contraction in the year 2006. Rus's primary area of study was contractions in metric space with fixed point properties. Recently Datta et al. [13] studied a common fixed point theorem in bi-complex valued b-metric space.

In the year 2021 Akkouchi [3] study "On sequences of certain contractive mappings and their fixed points". In the year 2022, "Nadler's fixed point theorem for set-valued mappings in b-metric spaces" established by Czerwik ([11], [12]), he also study "Generalized metric spaces". Recently Bhattacharjee et al. [19] study the "Fixed point theorems on complete b-metric space by using Rus contraction mapping".

Assuming these above remarks are accurate, we would like to introduce certain new fixed point theorems that can be used in b-multiplicative metric spaces by implementing the Rus contraction method in this case.

2. Preliminary and basic definition

This section gathers some fundamental definitions and findings.

Definition 2.1 (*cf.* [7])**.** Let $\bar{P} \neq \emptyset$ be any set and a mapping $\Phi : \bar{P} \times \bar{P} \to [1, \infty)$ is considered a multiplicative metric if all of the subsequent circumstances hold:

- **(m1)** $\Phi(\bar{\wp}, \bar{z}) > 1$ $for\ all$ $\bar{\wp}, \bar{z} \in \bar{P}$ $with$ $\bar{\wp} \neq \bar{z}$;
- **(m2)** $\Phi(\bar{\wp}, \bar{z}) = 1 \iff \bar{\wp} = \bar{z}$;
- **(m3)** $\Phi(\bar{\wp}, \bar{z}) = \Phi(\bar{z}, \bar{\wp})$;
- **(m4)** $\Phi(\bar{\wp}, \bar{z}) \leq \Phi(\bar{\wp}, \bar{w}) \cdot \Phi(\bar{w}, \bar{z})$ for all $\bar{\wp}, \bar{z}, \bar{w} \in \bar{P}$.

The pair $(\bar{P}, \Phi)$ is referred to as multiplicative metric space.

Example 2.2 (*cf.* [18])**.** Let R^n_+ be the collection of all n-tubles of positive real numbers. Let $\Phi^* : R^n_+ \times R^n_+ \to R$ be described as follows:

$$\Phi^*(x, y) = \left|\frac{x_1}{y_1}\right|^* \left|\frac{x_2}{y_2}\right|^* \cdots \left|\frac{x_n}{y_n}\right|^*$$

where $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in R^n_+$ and $|.|^* : R_+ \to R_+$ is defined as follows

$$|a|^* = \begin{cases} a & \text{if } a \geq 1; \\ \frac{1}{a} & \text{if } a < 1. \end{cases} \tag{2.1}$$

Then, it is clear that all the criteria of the multiplicative metric are fulfilled.

Definition 2.3 (*cf.* [4])**.** Let $\bar{P}$ be a non-empty set and $s \geq 1$ and a mapping $\Phi : \bar{P} \times \bar{P} \to [1, \infty)$ is referred to as b-multiplicative metric if the following conditions hold:

- **($\mathbf{m_b}$1)** $\Phi(\bar{\wp}, \bar{z}) > 1$ $for\ all$ $\bar{\wp}, \bar{z} \in \bar{P}$ $with$ $\bar{\wp} \neq \bar{z}$;
- **($\mathbf{m_b}$2)** $\Phi(\bar{\wp}, \bar{z}) = 1 \iff \bar{\wp} = \bar{z}$;
- **($\mathbf{m_b}$3)** $\Phi(\bar{\wp}, \bar{z}) = \Phi(\bar{z}, \bar{\wp})$;
- **($\mathbf{m_b}$4)** $\Phi(\bar{\wp}, \bar{z}) \leq \Phi(\bar{\wp}, \bar{w})^s \cdot \Phi(\bar{w}, \bar{z})^s$ for all $\bar{\wp}, \bar{z}, \bar{w} \in \bar{P}$.

The triplet $(\bar{P}, \Phi, s)$ is called a b-multiplicative metric space.

Example 2.4. Let $W = [0, \infty)$. Define a mapping $m_b : W \times W \to [1, \infty)$

$$m_b(m, n) = a^{|m^2 - n^2|},$$

where for any fixed real number $a > 1$ and $m \geq n$, m_b is a b-multiplicative metric on W for each a, with $s \geq 1$. It's important to note that m_b is also a multiplicative metric on W.

Remark 2.5. A b-multiplicative metric space is a multiplicative metric space if for all $s \geq 1$ the condition $(m_b 4)$ satisfy.

Example 2.6. Let $K = \{|v| \in \mathbb{N} : 0 \leq v \leq 5\}$. Define a mapping $\theta_b : K \times K \to [1, \infty)$

$$\theta_b(m, n) = a^{|m-n|^3},$$

where $a > 1$ is any fixed real number and $m \geq n$. Then for each a; θ_b is b-multiplicative metric on K with $s = 2$. Note that θ_b is not a multiplicative metric on K as the triangle inequality is not satisfied by this.

Definition 2.7 (*cf.* [22])**.** Let $\bar{P} \neq \emptyset$ be any random set and $p \geq 1$ be any real number. A mapping $d : \bar{P} \times \bar{P} \to [0, \infty)$ is regarded as b-metric with coefficient "p", if the following conditions hold:

- **(b1)** $d(\bar{\wp}, \bar{q}) > 0$;
- **(b2)** $d(\bar{\wp}, \bar{q}) = 0 \iff \bar{\wp} = \bar{q}$;
- **(b3)** $d(\bar{\wp}, \bar{q}) = d(\bar{q}, \bar{\wp})$;
- **(b4)** $d(\bar{\wp}, \bar{q}) \leq p[d(\bar{\wp}, \bar{w}) + d(\bar{w}, \bar{q})]$ for all $\bar{\wp}, \bar{q}, \bar{w} \in \bar{P}$.

The pair $(\bar{P}, d)$ is said to be a b-metric space.

Remark 2.8 (*cf.* [4]). Every b-metric space $(\bar{P}, d)$ generates a b-multiplicative metric space $(\bar{P}, \Phi, s)$ defined as

$$\Phi(x, y) = e^{d(x,y)} \; \forall \, x, y \in \bar{P}.$$

Definition 2.9 (*cf.* [18])**.** Let $(\bar{P}, \Phi)$ be a multiplicative metric space, $x \in \bar{P}$ and $\varepsilon > 1$. We now define a set $B_\varepsilon(x) = \{y \in \bar{P} : \Phi(x, y) < \varepsilon\}$, which is called multiplicative open ball of radius ε with can describe multiplicative closed ball as $B_\varepsilon(x) = \{y \in \bar{P} : \Phi(x, y) \leq \varepsilon\}$.

Definition 2.10 (*cf.* [4])**.** Let $(\bar{P}, \Phi, s)$ be a b-multiplicative metric space and $\{\wp_n\}$ be a sequence in $\bar{P}$. Then the sequence $\{\wp_n\}$ is referred to as a convergent sequence and converges to some $\wp$ in $\bar{P}$ if for every multiplicative open ball $B_\epsilon(\wp)$, there exists a natural number $n \geq N \implies \wp_n \in \; B_\epsilon(\wp)$. It is denoted by $\wp_n \to \wp(n \to \infty)$.

Lemma 2.11 (*cf.* [18])**.** *Let $(\bar{P}, \Phi, s)$ be a multiplicative metric space and $\{\wp_n\}$ be a sequence in $\bar{P}$ and $\wp \in \bar{P}$. Then $\wp_n \to \wp \iff \Phi(\wp_n, \wp) \to 1 \; (n \to \infty)$.*

Definition 2.12 (*cf.* [4])**.** Let $(\bar{P}, \Phi, s)$ be a b-multiplicative metric space and $\{\wp_n\}$ be a sequence in $\bar{P}$. The sequence $\{\wp_n\}$ is termed as a multiplicative Cauchy sequence if it satisfies the condition that, for every $\varepsilon > 1$, there exists a natural number N_0 such that $\Phi(\wp_n, \wp_m) < \varepsilon$ for all $m, n \geq N_0$.

Lemma 2.13 (*cf.* [4])**.** *In a b-multiplicative metric space $(\bar{P}, \Phi, s)$, if a sequence $\{\wp_n\}$ is multiplicative convergent, then the multiplicative limit point is unique.*

Lemma 2.14 (*cf.* [18])**.** *Let $(\bar{P}, \Phi, s)$ be a multiplicative metric space and $\{\wp_n\}$ be a sequence in $\bar{P}$. The sequence $\{\wp_n\}$ is a multiplicative Cauchy sequence $\iff \Phi(\wp_n, \wp_m) \to 1$ (as $m, n \to \infty$).*

Definition 2.15 (*cf.* [4])**.** A b-multiplicative metric space $(\bar{P}, \Phi, s)$ is said to be complete if every multiplicative Cauchy sequence $\{\wp_n\}$ is convergent in $\bar{P}$.

Definition 2.16 (*cf.* [8])**.** Suppose there exist two self-mappings L and J that are defined on a multiplicative metric space $(\bar{P}, \Phi)$. Then a point $u \in \bar{P}$ is considered as a fixed point of L if $\Phi(L(u), u) = 1$, i.e., $L(u) = u$. Moreover, if $\Phi(L(u), u) = \Phi(J(u), u) = 1$, then u is referred to as a common fixed point of the mappings L and J.

Definition 2.17 (*cf.* [18])**.** Let $(\bar{P}, \Phi)$ be a multiplicative metric space. A self-map Þ on a multiplicative metric space $(\bar{P}, \Phi)$ is referred to as a Multiplicative contraction mapping if there exists a real constant $0 \leq \lambda < 1$ such that $\Phi(Þ\bar{\wp}, Þ\bar{u}) \leq \Phi(\bar{\wp}, \bar{u})^{\lambda}$ for all $\bar{\wp}, \bar{u} \in X$.

Theorem 2.18 (*cf.* [18])**.** *Let $(\bar{P}, \Phi)$ be a complete multiplicative metric space. Let Þ $: \bar{P} \to \bar{P}$ be a multiplicative contraction mapping. Then Þ has a unique fixed point on $\bar{P}$.*

3. Summary of the article

In this paper, we have studied fixed points for contractive mappings in the context of a complete b-multiplicative metric space. We obtain novel conclusions, such as a theorem on common fixed points for two mappings in this setting, using the Rus contraction. Nontrivial examples that have been shown the validity of the theoretical conclusions are used to support the research. In order to demonstrate the uniqueness of fixed points under different contractive conditions, we wrapped up our investigation with the overall framework of b-multiplicative metric spaces. We are optimistic that their results may be applied to additional metric spaces, including controlled b-metric spaces, cone b-metric spaces, and rectangular b-metric spaces, indicating that more study in this area may be possible.

4. Main results

In this part, we present the confirmed key results of this study and offer illustrations to illustrate those principles.

By combining the Banach contraction [6], Kannan contraction [9], and Chatterjee contraction [16], a new type of contraction mapping known as the multiplicative Rus contraction mapping has been established. This new contraction aims to provide a generalization of the three well-known contractions. The existence and uniqueness of the fixed point in the Rus contraction guarantee the existence and uniqueness of the fixed point in the aforementioned contractions.

Definition 4.1. Let $(\bar{P}, \Phi)$ be a multiplicative metric space.A self-map Þ on a multiplicative metric space $(\bar{P}, \Phi)$ is referred to as Multiplicative Rus contraction mapping if there exists real valued numbers $a, b, c \geq 0$, where $0 \leq a + 2b + 2c < 1$, such that, for each $\bar{\wp}, \bar{u} \in X$ satisfies

$$\Phi(Þ\bar{\wp}, Þ\bar{u}) \leq \Phi(\bar{\wp}, \bar{u})^{a} \cdot \left(\Phi(Þ\bar{\wp}, \bar{\wp}) \cdot \Phi(Þ\bar{u}, \bar{u})\right)^{b} \cdot \left(\Phi(Þ\bar{\wp}, \bar{u}) \cdot \Phi(\bar{\wp}, Þ\bar{u})\right)^{c}. \tag{4.1}$$

Example 4.2. Let $\bar{P} = \{\wp \in \mathbb{R} : 1 \leq \wp \leq 2\}$. Considering the standard metric of $\mathbb{R}$, we have $\Phi(\wp, y) = e^{|\wp - y|}$ where $\wp, y \in \bar{P}$. It is straightforward to confirm that $(\bar{P}, \Phi)$ forms a complete b-multiplicative metric space for any $s \geq 1$.
Let Þ represent a self mapping on $\bar{P}$ that is defined as follows:

$$Þ(\wp) = \begin{cases} \frac{1}{2} & \text{if} \quad \wp = 1, \\ 0 & \text{if} \quad 1 < \wp < 2, \\ -\frac{1}{2} & \text{if} \quad \wp = 2. \end{cases}$$

Let us select $a \geq 0.14, b \geq 0.15, c \geq 0.15$ ensuring that $a + 2b + 2c < 1$. As a result, the condition (4.1) holds for any selected values of a, b, and c. Therefore, Þ forms a Rus contraction mapping on $\bar{P}$.

Lemma 4.3. *Let $(\bar{P}, \Phi, s)$ be a b-multiplicative metric space and $\{\wp_n\}$ be a sequence in $\bar{P}$ and $\wp \in \bar{P}$. Then $\wp_n \to \wp$ iff $\Phi(\wp_n, \wp) \to 1$ $(n \to \infty)$.*

Proof. Assume that the sequence $(\wp_n)$ is b-multiplicative convergent to $\wp$. This means that for any $\varepsilon > 1$, there exists a natural number N such that $\Phi(\wp_n, \wp) < \varepsilon$ when $n \geq N$. Consequently, the following inequality holds:

$$\frac{1}{\varepsilon} < \Phi(\wp_n, \wp) < 1 \cdot \varepsilon \quad \text{for all} \quad n \geq N.$$

This means $|\Phi(\wp_n, \wp)| < \varepsilon$ for all $n \geq N$, which implies that the sequence $\Phi(\wp_n, \wp)$ is multiplicative convergent to 1.

It is easy to confirm the contrary. □

Lemma 4.4. *Let $(\bar{P}, \Phi, s)$ be a b-multiplicative metric space and $\{\wp_n\}$ be a sequence in $\bar{P}$. The sequence $\{\wp_n\}$ is a multiplicative Cauchy sequence iff $\Phi(\wp_n, \wp_m) \to 1$ $(m, n \to \infty)$.*

Proof. Let $(\wp_n)$ be a b-multiplicative Cauchy sequence. Then, for every $\varepsilon > 1$, there exists N such that $\Phi(\wp_n, \wp_m) < \varepsilon$ for all $m, n \geq N$. Hence, we have $|\Phi(\wp_n, \wp_m)| < \varepsilon$ whenever $n, m \geq N$. This means $\Phi(\wp_n, \wp_m) \to 1(m, n \to \infty)$ in $(\mathbb{R}^+, |\cdot|)$.

It is simple to demonstrate Lemma's sufficiency side. □

4.1. Result on complete b-multiplicative metric spaces

In this subsection of we have established some new results on complete b-multiplicative metric spaces.

Theorem 4.5. *Let $(\bar{P}, \Phi, s)$ be a complete b-multiplicative metric space. If the mapping $Þ : \bar{P} \to \bar{P}$ satisfies*

$$\Phi(Þ\bar{\wp}, Þ\bar{u}) \leq \Phi(\bar{\wp}, \bar{u})^{\mu_1} \cdot (\Phi(Þ\bar{\wp}, \bar{\wp}) \cdot \Phi(Þ\bar{u}, \bar{u}))^{\mu_2} \cdot (\Phi(Þ\bar{\wp}, \bar{u}) \cdot \Phi(\bar{\wp}, Þ\bar{u}))^{\mu_3}, \tag{4.2}$$

where $\mu_1, \mu_2, \mu_3 \geq 0$ such that $0 \leq \mu_1 + 2\mu_2 + 2s\mu_3 < \frac{1}{s}$ then Þ has a unique fixed point on $\bar{P}$.

Proof. let $\bar{\wp}_0 \in \bar{P}$, then we take Picard's iteration such that $\bar{\wp}_1 = Þ\bar{\wp}_0, \bar{\wp}_2 = Þ\bar{\wp}_1 = Þ^2\bar{\wp}_0, \ldots, \bar{\wp}_{n+1} = Þ^n\bar{\wp}_0$. If $\bar{\wp}_n = \bar{\wp}_{n+1}$, then it is evident that $\bar{\wp}_n$ is a fixed point of Þ. However, if $\bar{\wp}_n \neq \bar{\wp}_{n+1}$ for all $n \geq 0$, then we need to prove that the sequence $\{\bar{\wp}_n\}$ is a Cauchy sequence. Firstly, we demonstrate that the sequence $\{\bar{\wp}_n\}$ is a Cauchy sequence. Now, we have

$$\begin{aligned}
\Phi(\bar{\wp}_{n+1}, \bar{\wp}_n) &= \Phi(Þ\bar{\wp}_n, Þ\bar{\wp}_{n-1}) \\
&\leq \Phi(\bar{\wp}_n, \bar{\wp}_{n-1})^{\mu_1} \cdot (\Phi(Þ\bar{\wp}_n, \bar{\wp}_n) \cdot \Phi(Þ\bar{\wp}_{n-1}, \bar{\wp}_{n-1}))^{\mu_2} \cdot (\Phi(Þ\bar{\wp}_n, \bar{\wp}_{n-1}) \cdot \Phi(\bar{\wp}_n, Þ\bar{\wp}_{n-1}))^{\mu_3} \\
&= \Phi(\bar{\wp}_n, \bar{\wp}_{n-1})^{\mu_1} \cdot (\Phi(\bar{\wp}_{n+1}, \bar{\wp}_n) \cdot \Phi(\bar{\wp}_n, \bar{\wp}_{n-1}))^{\mu_2} \cdot (\Phi(\wp_{n+1}, \wp_{n-1}) \cdot \Phi(\bar{\wp}_n, \bar{\wp}_n))^{\mu_3} \\
&= \Phi(\bar{\wp}_n, \bar{\wp}_{n-1})^{\mu_1} \cdot (\Phi(Þ\bar{\wp}_n, \bar{\wp}_n) \cdot \Phi(Þ\bar{\wp}_{n-1}, \bar{\wp}_{n-1}))^{\mu_2} \cdot (\Phi(\wp_{n+1}, \wp_{n-1}))^{\mu_3},
\end{aligned}$$

as $\Phi(\bar{\wp}_n, \bar{\wp}_n) = 1$ by (b2),

$$\begin{aligned}
\Phi(\bar{\wp}_{n+1}, \bar{\wp}_n) &\leq \Phi(\bar{\wp}_n, \bar{\wp}_{n-1})^{\mu_1} \cdot (\Phi(Þ\bar{\wp}_n, \bar{\wp}_n) \cdot \Phi(Þ\bar{\wp}_{n-1}, \bar{\wp}_{n-1}))^{\mu_2} \cdot \Phi(\bar{\wp}_{n+1}, \bar{\wp}_n)^{s\mu_3} \cdot \Phi(\bar{\wp}_n, \bar{\wp}_{n-1})^{s\mu_3} \\
&= \Phi(\bar{\wp}_n, \bar{\wp}_{n-1})^{(\mu_1+\mu_2+s\mu_3)} \cdot \Phi(\bar{\wp}_{n+1}, \bar{\wp}_n)^{(\mu_2+s\mu_3)}.
\end{aligned}$$

$$\implies \Phi(\bar{\wp}_{n+1}, \bar{\wp}_n)^{(1-\mu_2-\mu_3)} \leq \Phi(\bar{\wp}_n, \bar{\wp}_{n-1})^{(\mu_1+\mu_2+\mu_3)}.$$

$$\implies \Phi(\bar{\wp}_{n+1}, \bar{\wp}_n) \leq \Phi(\bar{\wp}_n, \bar{\wp}_{n-1})^{\frac{(\mu_1+\mu_2+s\mu_3)}{(1-\mu_2-s\mu_3)}}. \tag{4.3}$$

Now let $\frac{(\mu_1+\mu_2+s\mu_3)}{(1-\mu_2-s\mu_3)} = k$. So clearly $k < 1$ as $s(\mu_1 + 2\mu_2 + 2\mu_3) < 1$.

So from equation (4.3) it follows that:

$$\begin{aligned}
\Phi(\bar{\wp}_{n+1}, \bar{\wp}_n) &\leq \Phi(\bar{\wp}_n, \bar{\wp}_{n-1})^k \\
&\vdots \\
\implies \Phi(\bar{\wp}_{n+1}, \bar{\wp}_n) &\leq \Phi(\bar{\wp}_1, \bar{\wp}_0)^{k^n}.
\end{aligned} \tag{4.4}$$

Then by using triangular inequality and (4.2), for any positive integers $m > n$, we have

$$\begin{aligned}
\Phi(\bar{\wp}_n, \bar{\wp}_m) &\leq \Phi(\bar{\wp}_n, \bar{\wp}_{n+1})^{s^n} \cdot \Phi(\bar{\wp}_{n+1}, \bar{\wp}_{n+2})^{s^{(n+1)}} \cdot \ \ldots \ \cdot \Phi(\bar{\wp}_{m-1}, \bar{\wp}_m)^{s^{(m-1)}} \\
&\leq \Phi(\bar{\wp}_1, \bar{\wp}_0)^{(sk)^n} \cdot \Phi(\bar{\wp}_1, \bar{\wp}_0)^{(sk)^{n+1}} \cdot \ \ldots \ \cdot \Phi(\bar{\wp}_1, \bar{\wp}_0)^{(sk)^{m-1}} \\
&= \Phi(\bar{\wp}_1, \bar{\wp}_0)^{\left[(sk)^n + (sk)^{n+1} + \cdots + (sk)^{m-1}\right]} \\
&= \Phi(\bar{\wp}_1, \bar{\wp}_0)^{(sk)^n \left[1 + sk + \cdots + sk^{(m-n-1)}\right]} \\
&\leq \Phi(\bar{\wp}_1, \bar{\wp}_0)^{\left[(sk)^n \sum_0^\infty (sk)^n\right]} \\
&= \Phi(\bar{\wp}_1, \bar{\wp}_0)^{\left[\frac{(sk)^n}{1-sk}\right]} \rightarrow_b 1 \ \ as\ (k)^n \ \rightarrow \ 0 \text{ when n } \rightarrow \ \infty.
\end{aligned}$$

Therefore, the sequence $\{\bar{\wp}_n\}$ demonstrates multiplicative Cauchy properties in $\bar{P}$. As $\bar{P}$ represents a complete b-multiplicative metric space, there exists a $\bar{\wp} \in \bar{P}$ such that $\bar{\wp}_n$ converges to $\bar{\wp}^*$ $(as\ n \to \infty)$.

Again we have,

$$\begin{aligned}
\Phi(\text{Þ}\bar{\wp}^*, \bar{\wp}^*) &\leq \Phi(\text{Þ}\bar{\wp}^*, \bar{\wp}_{n+1})^s \cdot \Phi(\bar{\wp}_{n+1}, \bar{\wp}^*)^s \\
&\leq \Phi(\bar{\wp}^*, \bar{\wp}_n)^{s\mu_1} \cdot \big(\Phi(\text{Þ}\bar{\wp}^*, \bar{\wp}^*) \cdot \Phi(\text{Þ}\bar{\wp}_n, \bar{\wp}_n)\big)^{s\mu_2} \cdot \ \big(\Phi(\text{Þ}\bar{\wp}^*, \bar{\wp}_n)^* \cdot \Phi(\bar{\wp}^*, \text{Þ}\bar{\wp}_n)\big)^{s\mu_3} \cdot \Phi(\text{Þ}\bar{\wp}_n, \bar{\wp}^*)^s \\
&\leq \Phi(\bar{\wp}^*, \bar{\wp}_n)^{s\mu_1} \cdot (\Phi(\text{Þ}\bar{\wp}^*, \bar{\wp}^*) \cdot \Phi(\bar{\wp}_{n+1}, \bar{\wp}_n))^{s\mu_2} \cdot \ (\Phi(\text{Þ}\bar{\wp}^*, \bar{\wp}_n) \cdot \Phi(\bar{\wp}^*, \bar{\wp}_{n+1}))^{s\mu_3} \cdot \Phi(\bar{\wp}_{n+1}, \bar{\wp}^*)^s,
\end{aligned}$$

when $n \to \infty$, $\bar{\wp}_n \to \bar{\wp}^*$,

$$\Phi(\text{Þ}\bar{\wp}^*, \bar{\wp}^*) \ \leq \ \Phi(\bar{\wp}^*, \bar{\wp}^*)^{s\mu_1} \cdot \big(\Phi(\text{Þ}\bar{\wp}^*, \bar{\wp}^*) \cdot \Phi(\bar{\wp}^*, \bar{\wp}^*)\big)^{s\mu_2} \cdot \ \big(\Phi(\text{Þ}\bar{\wp}^*, \bar{\wp}^*) \cdot \Phi(\bar{\wp}^*, \bar{\wp}^*)\big)^{s\mu_3} \cdot \Phi(\bar{\wp}^*, \bar{\wp}^*)^s.$$

$$\implies \Phi(\text{Þ}\bar{\wp}^*, \bar{\wp}^*)^{(1-s\mu_2-s\mu_3)} \leq \Phi(\bar{\wp}^*, \bar{\wp}^*)^{(s\mu_1+s\mu_2+s\mu_3+s)}.$$

$$\implies \Phi(\text{Þ}\bar{\wp}^*, \bar{\wp}^*) \leq \Phi(\bar{\wp}^*, \bar{\wp}^*)^{\frac{(s\mu_1+s\mu_2+s\mu_3+s)}{(1-s\mu_2-s\mu_3)}}. \tag{4.5}$$

Since $\Phi(\bar{\wp}^*, \bar{\wp}^*) = 1$ as $n \to \infty$ then $\Phi(\text{Þ}\bar{\wp}^*, \bar{\wp}^*) \leq 1$ which leads to a contradiction as $\Phi(\text{Þ}\bar{\wp}^*, \bar{\wp}^*) > 1$. Therefore we conclude that $\Phi(\text{Þ}\bar{\wp}^*, \bar{\wp}^*) = 1 \implies \text{Þ}\bar{\wp}^* = \bar{\wp}^*$. Therefore, $\bar{\wp}^*$ is a fixed point of Þ.

Eventually, we exposition that $\bar{\wp}^* \in \bar{P}$ is a unique fixed point of Þ. Let's assume the presence of another fixed point $\bar{u}^* \in \bar{P}$ for which $\Phi(\text{Þ}\bar{u}^*, \bar{u}^*) = 1$. We get

$$\begin{aligned}
\Phi(\bar{\wp}^*, \bar{u}^*) &= \Phi(\text{Þ}\bar{\wp}^*, \text{Þ}\bar{u}^*) \\
&\leq \Phi(\bar{\wp}^*, \bar{u}^*)^{\mu_1} \cdot \big(\Phi(\text{Þ}\bar{\wp}^*, \bar{\wp}^*) \cdot \Phi(\text{Þ}\bar{u}^*, \bar{u}^*)\big)^{\mu_2} \cdot \big(\Phi(\text{Þ}\bar{\wp}^*, \bar{u}^*) \cdot \Phi(\bar{\wp}^*, \text{Þ}\bar{u}^*)\big)^{\mu_3} \\
&= \Phi(\bar{\wp}^*, \bar{u}^*)^{\mu_1} \cdot \big(\Phi(\text{Þ}\bar{\wp}^*, \bar{u}^*) \cdot \ \Phi(\bar{\wp}^*, \text{Þ}\bar{u}^*)\big)^{\mu_3} \\
&= \Phi(\bar{\wp}^*, \bar{u}^*)^{\mu_1} \cdot \big(\Phi(\bar{\wp}^*, \bar{u}^*) \cdot \Phi(\bar{\wp}^*, \bar{u}^*)\big)^{\mu_3} \\
&= \Phi(\bar{\wp}^*, \bar{u}^*)^{(\mu_1+2\mu_3)} < \Phi(\bar{\wp}^*, \bar{u}^*)
\end{aligned}$$

which is a contradiction as $(\mu_1 + 2\mu_3) < 1$. So we conclude that $\Phi(\bar{\wp}^*, \bar{u}^*) = 1$, that implies $\bar{\wp}^* = \bar{u}^*$. Hence $\bar{\wp}^*$ is a unique fixed point of Þ.Hence Þ has a unique fixed point on $\bar{P}$. □

Example 4.6. Let $W = \{x \in \mathbb{Z}^+ : 1 \leq x < \infty\}$. Define a mapping $m_b : W \times W \to [1, \infty)$

$$m_b(m, n) = a^{|m^2-n^2|},$$

where $a > 1$ is any fixed real number. Then $(W, m_b\, s)$ be a complete b-multiplicative metric space with $s = 2$. Let $\text{Þ} : W \to W$ be a function defined by

$$\text{Þ}(x) = \begin{cases} \frac{1}{x} & \text{if } \ x \text{ is } \ \text{even} \\ 1 & \text{if } \ x \text{ is } \ \text{odd.} \end{cases}$$

Then for $\mu_1 = 0.1$, $\mu_2 = 0.01$, $\mu_3 = 0.01$, we get $0 \leq \mu_1 + 2\mu_2 + 4\mu_3 = 0.16 < 0.5 = \frac{1}{2}$ and the condition (4.2) is satisfied. Here the unique fixed point Þ is $x = 1$.

Theorem 4.7. *Let $(\bar{P}, \Phi, s)$ be a complete b-multiplicative metric space. If the mapping $\{Þ_j\}$ be a sequence of self mappings of $\bar{P}$ satisfying*

$$\Phi(Þ_j\bar{\wp}, Þ_j\bar{u}) \leq \Phi(\bar{\wp}, \bar{u})^{\mu_1} \cdot (\Phi(Þ_j\bar{\wp}, \bar{\wp}) \cdot \Phi(Þ_j\bar{u}, \bar{u}))^{\mu_2} \cdot (\Phi(Þ_j\bar{\wp}, \bar{u}) \cdot \Phi(\bar{\wp}, Þ_j\bar{u}))^{\mu_3},$$

where $\mu_1, \mu_2, \mu_3 \geq 0$ with $0 \leq \mu_1 + 2\mu_2 + 2s\mu_3 < \frac{1}{s}$ and possess a unique fixed points $\bar{u}_j \in \bar{P}$. Suppose $\{Þ_j\}$ converges uniformly to a mapping $Þ : \bar{P} \to \bar{P}$ then $\bar{u}_j$ converges to a unique fixed point of Þ.

Proof. Since $\{Þ_j\}$ converges uniformly to $Þ : \bar{P} \to \bar{P}$, Þ also satisfies

$$\Phi(Þ\bar{\wp}, Þ\bar{u}) \leq \Phi(\bar{\wp}, \bar{u})^{\mu_1} \cdot (\Phi(Þ\bar{\wp}, \bar{\wp}) \cdot \Phi(Þ\bar{u}, \bar{u}))^{\mu_2} \cdot (\Phi(Þ\bar{\wp}, \bar{u}) \cdot \Phi(\bar{\wp}, Þ\bar{u}))^{\mu_3}. \tag{4.6}$$

According to theorem (4.5), we assert that Þ possesses a unique fixed point $\bar{u}$ in X.
Utilizing (4.7), we obtain

$$\begin{aligned}
\Phi(\bar{u}_j, \bar{u}) &= \Phi(Þ_j\bar{u}_j, Þ\bar{u}) \\
&\leq \Phi(Þ_j\bar{u}_j, Þ_j\bar{u})^s \cdot \Phi(Þ_j\bar{u}, Þ\bar{u})^s \\
&\leq \Phi(\bar{u}_j, \bar{u})^{s\mu_1} \cdot \big(\Phi(Þ_j\bar{u}_j, \bar{u}_j) \cdot \Phi(Þ_j\bar{u}, \bar{u})\big)^{s\mu_2} \cdot \big(\Phi(Þ_j\bar{u}_j, \bar{u}) \cdot \Phi(\bar{u}_j, Þ\bar{u})\big)^{s\mu_3} \cdot \Phi(Þ_j\bar{u}, Þ\bar{u})^s. \\
&= \Phi(\bar{u}_j, \bar{u})^{s\mu_1} \cdot \big(\Phi(\bar{u}_j, \bar{u}_j) \cdot \Phi(Þ_j\bar{u}, Þ\bar{u})\big)^{s\mu_2} \cdot \big(\Phi(\bar{u}_j, \bar{u}) \cdot \Phi(\bar{u}_j, \bar{u})\big)^{s\mu_3} \cdot \Phi(Þ_j\bar{u}, Þ\bar{u})^s.
\end{aligned}$$

$$\implies \Phi(\bar{u}_j, \bar{u})^{(1-s\mu_1-2s^2\mu_3)} \leq \Phi(Þ_j\bar{u}, Þ\bar{u})^{(s+s\mu_1+2s^2\mu_3)}.$$

$$\implies \Phi(\bar{u}_j, \bar{u}) \leq \Phi(Þ_j\bar{u}, Þ\bar{u})^{\frac{(s+s\mu_1+2s^2\mu_3)}{(1-s\mu_1-2s^2\mu_3)}}. \tag{4.7}$$

So, $\Phi(Þ_j\bar{u}, Þ\bar{u}) \to 1$ as $j \to \infty$. Since $\Phi(\bar{u}_j, \bar{u}) \not< 1$. So we conclude $\Phi(\bar{u}_j, \bar{u}) \to 1$ as $j \to \infty$, that implies, $\lim_{j\to\infty} \bar{u}_j = \bar{u}$.
Hence, we can confidently assert that $\bar{u}_j$ converges to the unique fixed point $\bar{u}$ of Þ. □

Theorem 4.8. *Let $(\bar{P}, \Phi, s)$ be a complete b-multiplicative metric space with $s \geq 1$. If the mapping $Þ, \Psi : \bar{P} \to \bar{P}$ satisfies*

$$\Phi(Þ\bar{\wp}, \psi\bar{u}) \leq \Phi(\bar{\wp}, \bar{u})^{\mu_1} \cdot (\Phi(Þ\bar{\wp}, \bar{\wp}) \cdot \Phi(\psi\bar{u}, \bar{u}))^{\mu_2} \cdot (\Phi(Þ\bar{\wp}, \bar{u}) \cdot \Phi(\bar{\wp}, \psi\bar{u}))^{\mu_3}, \tag{4.8}$$

where $\mu_1, \mu_2, \mu_3 \geq 0$, such that $0 \leq \mu_1 + 2\mu_2 + 2s\mu_3 < \frac{1}{s}$, then Þ and ψ have a unique common fixed point on $\bar{P}$.

Proof. Let $\bar{u}_0 \in X$ then we set

$$\begin{aligned}
\bar{u}_1 = Þ(\bar{u}_0) \quad &, \quad \bar{u}_2 = \psi(\bar{u}_1) \\
\bar{u}_3 = Þ(\bar{u}_2) \quad &, \quad \bar{u}_4 = \psi(\bar{u}_3) \\
\vdots \qquad & \qquad \vdots \\
\bar{u}_{2n+1} = Þ(\bar{u}_{2n}) \quad &, \quad \bar{u}_{2n+2} = \psi(\bar{u}_{2n+1}).
\end{aligned}$$

Then we get

$$\begin{aligned}
\Phi(\bar{u}_{2n+1}, \bar{u}_{2n+2}) &= \Phi(Þ\bar{u}_{2n}, \psi\bar{u}_{2n+1}) \\
&\leq \Phi(\bar{u}_{2n}, \bar{u}_{2n+1})^{\mu_1} \cdot \big(\Phi(Þ\bar{u}_{2n}, \bar{u}_{2n}) \cdot \Phi(\bar{u}_{2n+1}, \psi\bar{u}_{2n+1})\big)^{\mu_2} \\
&\quad \cdot \big(\Phi(Þ\bar{u}_{2n}, \bar{u}_{2n+1}) \cdot \Phi(\bar{u}_{2n}, \psi\bar{u}_{2n+1})\big)^{\mu_3} \\
&\leq \Phi(\bar{u}_{2n}, \bar{u}_{2n+1})^{\mu_1} \cdot \big(\Phi(Þ\bar{u}_{2n}, \bar{u}_{2n}) \cdot \Phi(\psi\bar{u}_{2n+1}, \bar{u}_{2n+1})\big)^{\mu_2} \\
&\quad \cdot \big(\Phi(Þ\bar{u}_{2n}, \bar{u}_{2n+1}) \cdot \Phi(\bar{u}_{2n}, \psi\bar{u}_{2n+1})\big)^{\mu_3}.
\end{aligned} \tag{4.9}$$

Further, we have

$$\left.\begin{aligned}\Phi(\psi\bar{u}_{2n+1},\bar{u}_{2n+1}) \leq \Phi(\text{Þ}\bar{u}_{2n},\bar{u}_{2n})^{\left(\frac{\mu_1+\mu_2+s\mu_3}{1-\mu_2-s\mu_3}\right)}\\ \Phi(\text{Þ}\bar{u}_{2n},\bar{u}_{2n}) \leq \Phi(\bar{u}_{2n},\bar{u}_{2n-1})^{\left(\frac{\mu_1+\mu_2+s\mu_3}{1-\mu_2-s\mu_3}\right)}\end{aligned}\right\} \tag{4.10}$$

$$\left.\begin{aligned}\Phi(\text{Þ}\bar{u}_{2n},\bar{u}_{2n+1}) = 1\\ \Phi(\psi\bar{u}_{2n-1},\bar{u}_{2n}) = 1\end{aligned}\right\} \tag{4.11}$$

$$\Phi(\bar{u}_{2n},\psi\bar{u}_{2n+1}) = \Phi(\bar{u}_{2n},\bar{u}_{2n+2}) \leq \Phi(\bar{u}_{2n},\bar{u}_{2n+1})^s.\Phi(\bar{u}_{2n+1},\bar{u}_{2n+2})^s. \tag{4.12}$$

Putting the values of (4.10), (4.11) and (4.12) in (4.9) we obtain,

$$\begin{aligned}\Phi(\bar{u}_{2n+1},\bar{u}_{2n+2}) &\leq \Phi(\bar{u}_{2n},\bar{u}_{2n+1})^{\mu_1}\cdot\left(\Phi(\text{Þ}\bar{u}_{2n},\bar{u}_{2n})^{\mu_2}\cdot\Phi(\text{Þ}\bar{u}_{2n},\bar{u}_{2n})^{\left(\frac{\mu_2(\mu_1+\mu_2+s\mu_3)}{1-\mu_2-s\mu_3}\right)}\right.\\ &\quad \cdot\Phi(\bar{u}_{2n},\bar{u}_{2n+1})^{s\mu_3}.\Phi(\bar{u}_{2n+1},\bar{u}_{2n+2})^{s\mu_3}\\ &= \Phi(\bar{u}_{2n+1},\bar{u}_{2n})^{\mu_1}\cdot\left(\Phi(\bar{u}_{2n+1},\bar{u}_{2n})^{\mu_2}\cdot\Phi(\bar{u}_{2n+1},\bar{u}_{2n})^{\left(\frac{\mu_2(\mu_1+\mu_2+s\mu_3)}{1-\mu_2-s\mu_3}\right)}\right.\\ &\quad \cdot\Phi(\bar{u}_{2n+1},\bar{u}_{2n})^{s\mu_3}.\Phi(\bar{u}_{2n+1},\bar{u}_{2n+2})^{s\mu_3}.\end{aligned}$$

$$\implies \Phi(\bar{u}_{2n+1},\bar{u}_{2n+2})^{(1-s\mu_3)} \leq \Phi(\bar{u}_{2n},\bar{u}_{2n+1})^{\left(\frac{(\mu_1+\mu_2+s\mu_3)(1-s\mu_3)}{1-\mu_2-s\mu_3}\right)}.$$

$$\implies \Phi(\bar{u}_{2n+1},\bar{u}_{2n+2}) \leq \Phi(\bar{u}_{2n+1},\bar{u}_{2n})^{\left(\frac{\mu_1+\mu_2+s\mu_3}{1-\mu_2-s\mu_3}\right)}. \tag{4.13}$$

Next let $\frac{(\mu_1+\mu_2+s\mu_3)}{(1-\mu_2-s\mu_3)} = k'$. So clearly $k' < 1$ as $\mu_1+2\mu_2+2\mu_3 < \frac{1}{s}$.
So from (4.13) it follows:

$$\begin{aligned}\Phi(\bar{u}_{2n+1},\bar{u}_{2n+2}) &\leq \left(\Phi(\bar{u}_{2n},\bar{u}_{2n+1})\right)^{k'}\\ &\vdots\\ \implies \Phi(\bar{u}_{2n+1},\bar{u}_{2n+2}) &\leq \left(\Phi(\bar{u}_0,\bar{u}_1)\right)^{(k')^{2n+1}}.\end{aligned} \tag{4.14}$$

So it follows from the above that

$$\Phi(\bar{u}_n,\bar{u}_{n+1}) \leq \left(\Phi(\bar{u}_0,\bar{u}_1)\right)^{(k')^n}. \tag{4.15}$$

By applying (m_b4) and referencing equation (4.15), we can establish the following for any positive integers m where $m \geq n$, we have

$$\begin{aligned}\Phi(\bar{u}_n,\bar{u}_{n+p}) &\leq \Phi(\bar{u}_n,\bar{u}_{n+1})^{s^n}\cdot\Phi(\bar{u}_{n+1},\bar{u}_{n+2})^{s^{(n+1)}}\cdot\ \dots\ \cdot\Phi(\bar{u}_{n+p-1},\bar{u}_{n+p})^{s^{(n+p-1)}}\\ &\leq \Phi(\bar{u}_1,\bar{u}_0)^{(sk')^n}\cdot\Phi(\bar{u}_1,\bar{u}_0)^{(sk')^{n+1}}\cdot\dots\cdot\Phi(\bar{u}_1,\bar{u}_0)^{(sk')^{n+p-1}}\\ &= \Phi(\bar{u}_1,\bar{u}_0)^{\left[(sk')^n+(sk')^{n+1}+\cdots+(sk')^{n+p-1}\right]}\\ &= \Phi(\bar{u}_1,\bar{u}_0)^{(sk')^n\left[1+sk'+\cdots+sk'^{(p-1)}\right]}\\ &\leq \Phi(\bar{u}_1,\bar{u}_0)^{(sk')^n\left[\sum_{m=0}^{\infty}(sk')^m\right]}\\ &= \Phi(\bar{u}_1,\bar{u}_0)^{\left[\frac{(sk')^n}{1-sk'}\right]} \to 1\ \ as\ n\ \to\ \infty,\ (k')^n\ \to\ 0.\end{aligned}$$

Hence, $\{\bar{u}_n\}$ is a multiplicative Cauchy sequence in X. Given that X is a complete b-multiplicative metric space, we can conclude that there is a value $\bar{u}^* \in X$ for which $\bar{u}_n$ converges to $\bar{u}^*$ as n approaches infinity.
Again we have,

$$\begin{aligned}
\Phi(\text{Þ}\bar{u}^*, \bar{u}^*) &\leq \Phi(\text{Þ}\bar{u}^*, \bar{u}_{2n+2})^s \cdot \Phi(\bar{u}_{2n+2}, \bar{u}^*)^s \\
&= \Phi(\text{Þ}\bar{u}^*, \psi\bar{u}_{2n+1})^s \cdot \Phi(\psi\bar{u}_{2n+1}, \bar{u}^*)^s \\
&\leq \Phi(\bar{u}^*, \bar{u}_{2n+1})^{s\mu_1} \cdot \big(\Phi(\text{Þ}\bar{u}^*, \bar{u}^*) \cdot \Phi(\psi\bar{u}_{2n+1}, \bar{u}_{2n+1})\big)^{s\mu_2} \\
&\quad \cdot \big(\Phi(\text{Þ}\bar{u}^*, \bar{u}_{2n+1}) \cdot \Phi(\bar{u}^*, \psi\bar{u}_{2n+1})\big)^{s\mu_3} \cdot \Phi(\psi\bar{u}_{2n+1}, \bar{u}^*)^s \\
&\leq \Phi(\bar{u}^*, \bar{u}_{2n+1})^{s\mu_1} \cdot \big(\Phi(\text{Þ}\bar{u}^*, \bar{u}^*) \cdot \Phi(\bar{u}_{2n+2}, \bar{u}_{2n+1})\big)^{s\mu_2} \\
&\quad \cdot \big(\Phi(\text{Þ}\bar{u}^*, \bar{u}_{2n+1}) \cdot \Phi(\bar{u}^*, \bar{u}_{2n+2})\big)^{s\mu_3} \cdot \Phi(\bar{u}_{2n+2}, \bar{u}^*)^s
\end{aligned}$$

when $n \to \infty, \bar{u}_n \to \bar{u}^*$,

$$\Phi(\text{Þ}\bar{u}^*, \bar{u}^*) = \Phi(\bar{u}^*, \bar{u}^*)^{s\mu_1} \cdot \big(\Phi(\text{Þ}\bar{u}^*, \bar{u}^*) \cdot \Phi(\bar{u}^*, \bar{u}^*)\big)^{s\mu_2} \cdot \big(\Phi(\text{Þ}\bar{u}^*, \bar{u}^*) \cdot \Phi(\bar{u}^*, \bar{u}^*)\big)^{s\mu_3} \cdot \Phi(\bar{u}^*, \bar{u}^*)^s$$

$$\implies \Phi(\text{Þ}\bar{u}^*, \bar{u}^*)^{(1-s\mu_2-s\mu_3)} \leq \Phi(\bar{u}^*, \bar{u}^*)^{(s+s\mu_1+s\mu_2+s\mu_3)}.$$

$$\implies \Phi(\text{Þ}\bar{u}^*, \bar{u}^*) \leq \Phi(\bar{u}^*, \bar{u}^*)^{\left(\frac{s+s\mu_1+s\mu_2+s\mu_3}{1-s\mu_2-w\mu_3}\right)}. \tag{4.16}$$

Therefore $\Phi(\bar{u}^*, \bar{u}^*) = 1$ *by* $(m_b2) \implies \Phi(\text{Þ}\bar{u}, \bar{u}) \leq 1$, which contradict the property (m_b1). So, we conclude that

$$\Phi(\text{Þ}\bar{u}^*, \bar{u}^*) = 1 \implies \text{Þ}\bar{u}^* = \bar{u}^* \text{ for every } \bar{u}^* \in \bar{P}. \tag{4.17}$$

So, $\bar{u}^*$ is a fixed point of Þ. Similarly, we can prove that

$$\Phi(\psi\bar{u}^*, \bar{u}^*) = 1 \implies \psi\bar{u}^* = \bar{u}^*, \text{ as } \Phi(\psi\bar{u}^*, \bar{u}^*) \not< 1 \text{ for every } \bar{u}^* \in \bar{P}.$$

So, $\bar{u}^*$ is a fixed point of ψ. Therefore $\bar{u}^*$ is a common fixed point of Þ and ψ and uniqueness of $\bar{u}^*$ is obvious.

□

Example 4.9. In example (4.6) if we take $\psi : W \to W$ be a function defined by

$$\text{Þ}(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } x \text{ is even} \\ 1 & \text{if } x \text{ is odd} \end{cases}$$

Then for $\mu_1 = 0.1, \mu_2 = 0.1, \mu_3 = 0.01$, we get $0 \leq \mu_1 + 2\mu_2 + 4\mu_3 = 0.34 < 0.5 = \frac{1}{s}$ and the condition (4.8) is satisfied. Here the unique fixed point Þ is $x = 1$.

Theorem 4.10. *Let $(\bar{Q}, \Phi, s)$ be a complete b-multiplicative metric space. If the mapping $\{\bar{\text{Þ}}\}$ be a sequence of self mappings of $\bar{Q}$ satisfying*

$$\Phi(\bar{\text{Þ}}\bar{\wp}, \bar{\text{Þ}}\bar{u}) \leq \Phi(\bar{\wp}, \bar{u})^{\mu_1} \cdot (\Phi(\bar{\text{Þ}}\bar{\wp}, \bar{\wp}) \cdot \Phi(\bar{\text{Þ}}\bar{u}, \bar{u}))^{\mu_2} \cdot (\Phi(\bar{\text{Þ}}\bar{\wp}, \bar{u}) \cdot \Phi(\bar{\wp}, \bar{\text{Þ}}\bar{u}))^{\mu_3}, \tag{4.18}$$

where $\mu_1, \mu_2, \mu_3 \geq 0$ with $0 \leq \mu_1 + 2\mu_2 + 2s\mu_3 < \frac{1}{s}$. If the sequence $\{\bar{\text{Þ}}^n(\wp_0)\}$ for some $\wp_0 \in \bar{Q}$, has a subsequence $\{\bar{\text{Þ}}^{n_k}(\wp_0)\}$ with $\lim_{n\to\infty} \{\bar{\text{Þ}}^{n_k}(\wp_0)\} = \bar{\wp} \in \bar{Q}$, then $\wp$ is the unique fixed points of $\bar{\text{Þ}}$ and $\lim_{n\to\infty} \{\bar{\text{Þ}}^n(\wp_0)\} = \bar{\wp}$.

Proof. Since $\lim_{n\to\infty} \{\bar{\text{Þ}}^{n_k}(\wp_0)\} = \bar{\wp}$ and $\bar{\wp} \in \bar{Q}$. Therefore,

$$\begin{aligned}
\Phi(\bar{\text{Þ}}\bar{\wp}, \bar{\wp}) &\leq \Phi\big(\bar{\text{Þ}}\bar{\wp}, \bar{\text{Þ}}^{n_k+1}(\wp_0)\big)^s \cdot \Phi\big(\bar{\text{Þ}}^{n_k+1}(\wp_0), \bar{\wp}\big)^s \\
&\leq \Phi\big(\bar{\text{Þ}}\bar{\wp}, \bar{\text{Þ}}(\bar{\text{Þ}}^{n_k}(\wp_0))\big)^s \cdot \Phi\big(\bar{\text{Þ}}^{n_k+1}(\wp_0), \bar{\wp}\big)^s \\
&\leq \Phi(\bar{\wp}, \bar{\text{Þ}}^{n_k}(\wp_0))^{s\mu_1} \cdot \big[\Phi(\bar{\text{Þ}}\bar{\wp}, \bar{\wp}) \cdot \Phi(\bar{\text{Þ}}^{n_k+1}(\wp_0), \bar{\text{Þ}}^{n_k}(\wp_0))\big]^{s\mu_2} \\
&\quad \cdot \big[\Phi(\bar{\text{Þ}}\bar{\wp}, \bar{\text{Þ}}^{n_k}(\wp_0)) \cdot \Phi(\bar{\wp}, \bar{\text{Þ}}^{n_k+1}(\wp_0))\big]^{s\mu_3} \cdot \Phi\big(\bar{\text{Þ}}^{n_k+1}(\wp_0), \bar{\wp}\big)^s \\
&\leq \Phi(\bar{\wp}, \bar{\text{Þ}}^{n_k}(\wp_0))^{s\mu_1} \cdot \big[\Phi(\bar{\text{Þ}}\bar{\wp}, \bar{\wp}) \cdot \Phi(\bar{\text{Þ}}^{n_k+1}(\wp_0), \bar{\wp})^s \cdot \Phi(\bar{\wp}, \bar{\text{Þ}}^{n_k}(\wp_0))^s\big]^{s\mu_2} \\
&\quad \cdot \big[\Phi(\bar{\text{Þ}}\bar{\wp}, \bar{\wp})^s \cdot \Phi(\bar{\wp}, \bar{\text{Þ}}^{n_k}(\wp_0))^s \cdot \Phi(\bar{\wp}, \bar{\text{Þ}}^{n_k+1}(\wp_0))\big]^{s\mu_3} \cdot \Phi\big(\bar{\text{Þ}}^{n_k+1}(\wp_0), \bar{\wp}\big)^s.
\end{aligned}$$

$$\implies \Phi(\bar{\text{P}}\bar{\wp}, \bar{\wp})^{(1-s\mu_2-s^2\mu_3)} \leq \Phi(\bar{\text{P}}^{n_k+1}(\wp_0), \bar{\wp})^{s(\mu_1+s\mu_2+\mu_3)} \cdot \Phi(\bar{\wp}, \bar{\text{P}}^{n_k}(\wp_0))^{s(\mu_1+s\mu_2+s\mu_3)}.$$

$$\implies \Phi(\bar{\text{P}}\bar{\wp}, \bar{\wp}) \leq \Phi(\bar{\text{P}}^{n_k+1}(\wp_0), \bar{\wp})^{\frac{s(\mu_1+s\mu_2+\mu_3)}{(1-s\mu_2-s^2\mu_3)}} \cdot \Phi(\bar{\wp}, \bar{\text{P}}^{n_k}(\wp_0))^{\frac{s(\mu_1+s\mu_2+s\mu_3)}{(1-s\mu_2-s^2\mu_3)}}. \tag{4.19}$$

On taking limit as $n \to \infty$ we get, $\bar{\text{P}}^{n_k}(\wp_0) = \bar{\wp}$. So from (4.19) we obtain, $\Phi(\bar{\text{P}}\bar{\wp}, \bar{\wp}) \to 1$. Hence $\bar{\wp}$ be a fixed point of $\bar{\text{P}}$ and it is obvious that $\bar{\wp}$ be the unique fixed point.
Since, $\bar{\wp}_1 = \bar{\text{P}}\wp_0, \bar{\wp}_2 = \bar{\text{P}}\bar{\wp}_1 = \bar{\text{P}}^2\wp_0, \ldots, \bar{\wp}_{n+1} = \bar{\text{P}}^n\wp_0 \cdots$.
Now

$$\begin{aligned}
\Phi(\bar{\text{P}}^n(\wp_0), \bar{\wp}) &= \Phi(\bar{\text{P}}^n(\wp_0), \bar{\text{P}}^n\bar{\wp}) \\
&\leq \Phi(\bar{\text{P}}^n(\wp_0), \bar{\text{P}}^{n+1}(\bar{\wp}_0))^s \cdot \Phi(\bar{\text{P}}^{n+1}(\bar{\wp}_0), \bar{\text{P}}^n\bar{\wp})^s \\
&\leq \Phi(\bar{\text{P}}^n(\wp_0), \bar{\text{P}}^{n+1}(\wp_0))^s \cdot \Phi\big(\bar{\text{P}}^{n+1}(\wp_0), \bar{\text{P}}^{n+1}(\bar{\wp})\big)^{s^2} \cdot \Phi(\bar{\text{P}}^{n+1}(\bar{\wp}), \bar{\text{P}}^n\bar{\wp})^{s^2} \\
&\leq \Phi(\bar{\text{P}}^n(\wp_0), \bar{\text{P}}^{n+1}(\wp_0))^s \cdot \Phi\big(\bar{\text{P}}^{n+1}(\wp_0), \bar{\text{P}}^{n+1}(\bar{\wp})\big)^{s^2}.
\end{aligned} \tag{4.20}$$

As $\bar{\wp}$ is a fixed point of $\bar{\text{P}}$ then $\Phi(\bar{\text{P}}^{n+1}\bar{\wp}, \bar{\text{P}}^n\bar{\wp}) = 1$.
Again we get,

$$\begin{aligned}
\Phi\big(\bar{\text{P}}^{n+1}(\wp_0), \bar{\text{P}}^{n+1}(\bar{\wp})\big) &= \Phi\big(\bar{\text{P}}(\bar{\text{P}}^n\wp_0), \bar{\text{P}}(\bar{\text{P}}^n\bar{\wp})\big) \\
&\leq \Phi(\bar{\text{P}}^n(\wp_0), \bar{\text{P}}^n\bar{\wp})^{\mu_1} \cdot \big[\Phi(\bar{\text{P}}^{n+1}(\wp_0), \bar{\text{P}}^n(\wp_0)) \cdot \Phi(\bar{\text{P}}^{n+1}\bar{\wp}, \bar{\text{P}}^n\bar{\wp})\big]^{\mu_2} \\
&\quad \cdot \big[\Phi(\bar{\text{P}}^{n+1}(\wp_0), \bar{\text{P}}^n\bar{\wp}) \cdot \Phi(\bar{\text{P}}^n(\wp_0), \bar{\text{P}}^{n+1}\bar{\wp})\big]^{\mu_3} \\
&\leq \Phi(\bar{\text{P}}^n(\wp_0), \bar{\text{P}}^n\bar{\wp})^{\mu_1} \cdot \big[\Phi(\bar{\text{P}}^{n+1}(\wp_0), \bar{\text{P}}^n(\wp_0)) \cdot \Phi(\bar{\text{P}}^{n+1}\bar{\wp}, \bar{\text{P}}^n\bar{\wp})\big]^{\mu_2} \\
&\quad \cdot \big[\Phi(\bar{\text{P}}^{n+1}(\wp_0), \bar{\text{P}}^n(\bar{\wp}_0)) \cdot \Phi(\bar{\text{P}}^n(\wp_0), \bar{\text{P}}^n\bar{\wp}) \cdot \Phi(\bar{\text{P}}^n(\wp_0), \bar{\text{P}}^n\bar{\wp}) \cdot \Phi(\bar{\text{P}}^n\bar{\wp}, \bar{\text{P}}^{n+1}\bar{\wp})\big]^{\mu_3}.
\end{aligned}$$

Since $\Phi(\bar{\text{P}}^{n+1}\bar{\wp}, \bar{\text{P}}^n\bar{\wp}) = \Phi(\bar{\text{P}}(\bar{\text{P}}^n\bar{\wp}), \bar{\text{P}}^n\bar{\wp}) = \Phi(\bar{\text{P}}\bar{\wp}, \bar{\wp}) = 1$ as $\bar{\wp}$ is a fixed point of $\bar{\text{P}}$ and $\bar{\text{P}}^n(\bar{\wp}) = \bar{\wp}$, so we get,

$$\begin{aligned}
\Phi\big(\bar{\text{P}}^{n+1}(\wp_0), \bar{\text{P}}^{n+1}(\bar{\wp})\big) &\leq \Phi(\bar{\text{P}}^n(\wp_0), \bar{\wp})^{\mu_1} \cdot \big[\Phi(\bar{\text{P}}^{n+1}(\wp_0), \bar{\text{P}}^n(\wp_0))\big]^{\mu_2} \\
&\quad \cdot \big[\Phi(\bar{\text{P}}^{n+1}(\wp_0), \bar{\text{P}}^n(\bar{\wp}_0)) \cdot \Phi(\bar{\text{P}}^n(\wp_0), \bar{\wp}) \cdot \Phi(\bar{\text{P}}^n(\wp_0), \bar{\wp}\big]^{\mu_3}
\end{aligned}$$

$$\implies \Phi\big(\bar{\text{P}}^{n+1}(\wp_0), \bar{\text{P}}^{n+1}(\bar{\wp})\big) \leq \Phi(\bar{\text{P}}^n(\wp_0), \bar{\wp})^{(\mu_1+2\mu_3)} \cdot \Phi(\bar{\text{P}}^{n+1}(\wp_0), \bar{\text{P}}^n(\bar{\wp}_0))^{(\mu_2+\mu_3)}. \tag{4.21}$$

Then from (4.20) we obtain,

$$\begin{aligned}
\Phi(\bar{\text{P}}^n(\wp_0), \bar{\wp}) &\leq \Phi(\bar{\text{P}}^n(\wp_0), \bar{\text{P}}^{n+1}(\wp_0))^s \cdot \Phi(\bar{\text{P}}^n(\wp_0), \bar{\wp})^{s^2(\mu_1+2\mu_3)} \cdot \Phi(\bar{\text{P}}^{n+1}(\wp_0), \bar{\text{P}}^n(\bar{\wp}_0))^{s^2(\mu_2+\mu_3)} \\
&\leq \Phi(\bar{\text{P}}^n(\wp_0), \bar{\wp})^{s^2(\mu_1+2\mu_3)} \cdot \Phi(\bar{\text{P}}^{n+1}(\wp_0), \bar{\text{P}}^n(\bar{\wp}_0))^{s+s^2(\mu_2+\mu_3)}.
\end{aligned}$$

$$\implies \Phi(\bar{\text{P}}^n(\wp_0), \bar{\wp})^{(1-s^2\mu_1-2s^2\mu_3)} \leq \Phi(\bar{\text{P}}^{n+1}(\wp_0), \bar{\text{P}}^n(\bar{\wp}_0))^{(s+s^2\mu_2+s^2\mu_3)}.$$

$$\implies \Phi(\bar{\text{P}}^n(\wp_0), \bar{\wp}) \leq \Phi(\bar{\text{P}}^{n+1}(\wp_0), \bar{\text{P}}^n(\bar{\wp}_0))^{\frac{(s+s^2\mu_2+s^2\mu_3)}{(1-s^2\mu_1-2s^2\mu_3)}}.$$

$$\implies \Phi(\bar{\text{P}}^n(\wp_0), \bar{\wp}) \leq \Phi(\bar{\wp}^{n+1}, \bar{\wp}^n)^{\frac{(s+s^2\mu_2+s^2\mu_3)}{(1-s^2\mu_1-2s^2\mu_3)}}. \tag{4.22}$$

Since $\bar{\wp}$ is the unique fixed point of $\bar{\text{P}}$. Therefore $\bar{\wp}^{n+1} = \bar{\wp}^n = \bar{\wp}$ $(n \to \infty) \implies \Phi(\bar{\wp}^{n+1}, \bar{\wp}^n) = 1$ (as $n \to \infty$). So from (4.22) we conclude that $\Phi(\bar{\text{P}}^n(\wp_0), \bar{\wp}) = 1$ (as $n \to \infty$) $\implies \lim_{n\to\infty}\{\bar{\text{P}}^n(\wp_0)\} = \bar{\wp}$. Hence proved. □

5. Conclusion

The general framework of b-multiplicative metric space has been updated, and we have illustrated various established fixed points outcomes in complete b-multiplicative metric space in the sections mentioned earlier. Our intention is that our findings will provide clarity on the uniqueness of a fixed point under diverse contractive conditions and support further exploration in this area. If we apply the concepts outlined in this paper in the future and engage in similar research on different metric spaces, such as controlled b-metric space, cone b-metric space, rectangular b-metric space, etc., we may discover some valuable insights.

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