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On entropies and Fisher information matrix for an extended Beta distribution

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Abstract

The extended beta-type 1 distribution has the probability density function proportional to $u^{\eta-1}(1-u)^{\zeta-1}\exp\left[-\frac{\kappa}{u}-\frac{\tau}{(1-u)}\right], 0<u<1$. In this exploratory article, the exact forms of Rényi and Shannon entropies for the extended beta-type 1 distribution are discussed. Furthermore, the Fisher information matrix of the above-mentioned distribution is pointed out systematically.

Keywords: Beta function, extended beta function, entropy, information matrix, probability distribution

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1. Introduction and preliminaries

Throughout this exploratory article, we will suppose $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ to be the arrangements of integer numbers, real numbers and complex numbers, respectively, and that

$$\mathbb{N} := \{1, 2, 3, \ldots\},\ \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\} = \mathbb{N} \cup \{0\}.$$

The Euler's classical beta function $\mathrm{B}(\eta, \zeta)$ is defined by (*cf.* [19, 24])

$$\mathrm{B}(\eta,\zeta) = \int_0^1 u^{\eta-1}\,(1-u)^{\zeta-1}\,\mathrm{d}u\,, \quad \Re(\eta) > 0,\ \Re(\zeta) > 0\,. \tag{1.1}$$

In 1997, Chaudhry *et al.* (*cf.* [7, Eq. (1.7)]) introduced the following extension of the Euler beta function by adding an exponential factor $\exp\left[\frac{-\kappa}{u(1-u)}\right]$ in the integrand in (1.1) by

$$\mathrm{B}_\kappa(\eta,\zeta) = \int_0^1 u^{\eta-1}\,(1-u)^{\zeta-1}\ \exp\left[-\frac{\kappa}{u(1-u)}\right]\mathrm{d}u\,, \Re(\eta) > 0,\ \Re(\zeta) > 0,\ \Re(\kappa) > 0\,. \tag{1.2}$$

It is clear that for $\kappa = 0$ in (1.2) yields the classical beta function, i.e. $\mathrm{B}_0(\eta,\zeta) = \mathrm{B}(\eta,\zeta)$. Further, replacing u by $(1-u)$ in (1.2), it is also observed that $\mathrm{B}_\kappa(\eta,\zeta) = \mathrm{B}_\kappa(\zeta,\eta)$. The reasoning and defense for presenting this function are given in (*cf.* [7]), where a few properties and a statistical analysis have additionally been examined. Miller *et al.* (*cf.* [15]) further studied this function and derived some more properties.

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In the same paper, Chaudhry *et al.* (*cf.* [7]) presented an extended beta-type 1 distribution, using (1.2) as follows:

$$\mathrm{f}_{EB1}(u;\eta,\zeta,\kappa)=\frac{u^{\eta-1}\,(1-u)^{\zeta-1}}{\mathrm{B}_{\kappa}(\eta,\zeta)}\exp\left[-\frac{\kappa}{u(1-u)}\right],\quad 0<u<1\,. \tag{1.3}$$

Choi *et al.* (*cf.* [9]) introduced an additional parameter τ in (1.2) as follows:

$$\mathrm{B}_{\kappa,\tau}(\eta,\zeta)=\int_0^1 u^{\eta-1}\,(1-u)^{\zeta-1}\,\exp\left[-\frac{\kappa}{u}-\frac{\tau}{(1-u)}\right]\mathrm{d}u,\ \ \Re(\kappa)>0,\ \Re(\tau)>0\,. \tag{1.4}$$

By putting $\tau=\kappa$ in (1.4), one obtains the extended beta function given in (1.2).

Using (1.4), Choi *et al.* (*cf.* [9]) presented a further extension of the extended beta-type 1 distribution (1.3) as follows:

$$\mathrm{f}_{EEB1}(u;\eta,\zeta,\kappa,\tau)=\frac{u^{\eta-1}\,(1-u)^{\zeta-1}}{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)}\exp\left[-\frac{\kappa}{u}-\frac{\tau}{(1-u)}\right],\quad 0<u<1\,. \tag{1.5}$$

It is employed in various applications; for an overview of this concept and its relationship to other discrete statistical models, as well as their application in Bayesian analysis (*cf.* [18]). Moreover, this concept is extensively utilized in the realme of population genomics research (*cf.* [5]).

If $\mho$ alone a set with a component θ, and let A be a subset of $\mho$. Then, at that point the indicator function of A, indicated by $\mathbf{1}_A(\cdot)$, and characterized by

$$\mathbf{1}_A=\begin{cases}1, & \theta\in A\\ 0, & \theta\notin A.\end{cases}$$

The Psi function (or digamma function) is the logarithmic differential coefficient of the gamma function defined as follows:

$$\psi(z)=\frac{\mathrm{d}}{\mathrm{d}z}\big\{\ell n\,\Gamma(z)\big\}=\frac{\Gamma^{'}(z)}{\Gamma(z)}. \tag{1.6}$$

The trigamma function is the differential coefficient of the Psi function defined as follows:

$$\psi_1(z)=\frac{\mathrm{d}}{\mathrm{d}z}\psi(z)=\frac{\mathrm{d}^2}{\mathrm{d}z^2}\big\{\ell n\,\Gamma(z)\big\}. \tag{1.7}$$

The n^{th} differential coefficient of the Psi function is defined by the polygamma function of order n as follows:

$$\psi^n(z)=\frac{\mathrm{d}^n}{\mathrm{d}z^n}\psi(z)=\frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}}\big\{\ell n\,\Gamma(z)\big\}. \tag{1.8}$$

In the case when $n=0$, we have the psi function (or digamma function), while for $n=1$ we have the trigamma function. The display (1.8) is also mentioned in Askey *et al.* (*cf.* [4]).

The Laguerre polynomial (*cf.* [24, p. 84, Eq. (15)]) of degree s is defined as follows:

$$L_s(u)=\sum_{\ell=0}^{s}\frac{(-1)^{\ell}}{\ell!}\binom{s}{\ell}u^{\ell}\,, \tag{1.9}$$

where $\binom{s}{\ell}$ is the binomial coefficient. The n^{th} differential coefficient of the Laguerre polynomial $L_s(u)$ is described by:

$$\frac{\mathrm{d}^n}{\mathrm{d}u^n}L_s(u)=(-1)^nL_{s-n}^{(r)}(u)\,, \tag{1.10}$$

where $L_s^{(\alpha)}(u)$ are the generalized Laguerre polynomial of degree s described by the sum:

$$L_s^{(\alpha)}(u)=\sum_{\ell=0}^{s}\frac{(-1)^{\ell}}{\ell!}\binom{s+\alpha}{s-\ell}u^{\ell}\,. \tag{1.11}$$

The following series generate the Laguerre polynomials (*cf.* [24, p. 84, Eq. (15)]):

$$\frac{1}{(1-t)}\exp\left(-\frac{\kappa t}{1-t}\right)=\sum_{s=0}^{\infty}L_s(\kappa)\,t^s,\qquad (|\,t\,|<1)\,. \tag{1.12}$$

Replacing t by $(1-t)$ in (1.12), we acquire

$$\exp\left(-\frac{\kappa}{t}\right)=\exp(-\kappa)\,t\sum_{s=0}^{\infty}L_s(\kappa)\,(1-t)^s\,. \tag{1.13}$$

Again replacing t by $(1-t)$ and s by r in (1.13), we find

$$\exp\left(-\frac{\kappa}{1-t}\right)=\exp(-\kappa)\,(1-t)\sum_{r=0}^{\infty}L_r(\kappa)\,t^r\,. \tag{1.14}$$

Miller *et al.* (*cf.* [15]) discussed an alternative representation for $\mathrm{B}_\kappa(\eta,\zeta)$ by replacing $\exp\left(-\frac{\kappa}{t}\right)$ and $\exp\left(-\frac{\kappa}{1-t}\right)$ by their respective series expansions involving the Laguerre polynomial in (1.2) as follows:

$$\mathrm{B}_\kappa(\eta,\zeta)=\exp(-2k)\sum_{r,s=0}^{\infty}L_r(\kappa)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\,.\quad \eta>-1,\ \zeta>-1\,. \tag{1.15}$$

Choi *et al.* (*cf.* [9]) also discussed an alternative representation for $\mathrm{B}_{\kappa,\tau}(\eta,\zeta)$ using (1.4) is as follows:

$$\mathrm{B}_{\kappa,\tau}(\eta,\zeta)=\exp(-\tau-\kappa)\sum_{r,s=0}^{\infty}L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\,,\quad \eta>-1,\ \zeta>-1. \tag{1.16}$$

If one puts $\tau=\kappa$ in (1.16), it reduces to (1.15).

2. Some new properties of the extended beta-type 1 function

In this section, we discuss the first and second order partial derivatives of the extended beta-type 1 function $\mathrm{B}_{\kappa,\tau}(\eta,\zeta)$ given by Choi *et al.* (*cf.* [9]) in terms of the psi function and Laguerre polynomials.

$$\frac{\partial}{\partial\eta}\Big\{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\Big\}=\exp(-\tau-\kappa)\sum_{r,s=0}^{\infty}L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\Big[\psi(\eta+r+1)-\psi(\eta+\zeta+r+s+2\Big],$$

$$\frac{\partial}{\partial\zeta}\Big\{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\Big\}=\exp(-\tau-\kappa)\sum_{r,s=0}^{\infty}L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\Big[\psi(\zeta+s+1)-\psi(\eta+\zeta+r+s+2\Big],$$

$$\frac{\partial}{\partial\kappa}\Big\{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\Big\}=-\exp(-\tau-\kappa)\sum_{r,s=0}^{\infty}L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\Big[1+\frac{L_{s-1}^{(1)}(\kappa)}{L_s(\kappa)}\mathbf{1}_{\mathbb{N}_1}(s)\Big],$$

$$\frac{\partial}{\partial\tau}\Big\{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\Big\}=-\exp(-\tau-\kappa)\sum_{r,s=0}^{\infty}L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\Big[1+\frac{L_{r-1}^{(1)}(\tau)}{L_r(\tau)}\mathbf{1}_{\mathbb{N}_1}(r)\Big],$$

$$\begin{aligned}\frac{\partial^2}{\partial\eta^2}\Big\{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\Big\}\quad=\quad&\exp(-\tau-\kappa)\sum_{r,s=0}^{\infty}L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\Big[\Big(\psi(\eta+r+1)-\psi(\eta+\zeta+r+s+2)\Big)^2\\&+\psi_1(\eta+r+1)-\psi_1(\eta+\zeta+r+s+2)\Big],\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial\zeta^2}\Big\{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\Big\} &= \exp(-\tau-\kappa)\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\Big[\Big(\psi(\zeta+s+1)-\psi(\eta+\zeta+r+s+2)\Big)^2\\ &\quad +\psi_1(\zeta+s+1)-\psi_1(\eta+\zeta+r+s+2)\Big],\end{aligned}$$

$$\frac{\partial^2}{\partial\kappa^2}\Big\{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\Big\} = \exp(-\tau-\kappa)\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\Big[1+2\,\frac{L_{s-1}^{(1)}(\kappa)}{L_s(\kappa)}\mathbf{1}_{\mathbb{N}_1}(s)+\frac{L_{s-2}^{(2)}(\kappa)}{L_s(\kappa)}\mathbf{1}_{\mathbb{N}_1}(s)\,\mathbf{1}_{\mathbb{N}_2}(s)\Big],$$

$$\frac{\partial^2}{\partial\tau^2}\Big\{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\Big\} = \exp(-\tau-\kappa)\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\Big[1+2\,\frac{L_{r-1}^{(1)}(\tau)}{L_r(\tau)}\mathbf{1}_{\mathbb{N}_1}(r)+\frac{L_{r-2}^{(2)}(\tau)}{L_r(\tau)}\mathbf{1}_{\mathbb{N}_1}(r)\,\mathbf{1}_{\mathbb{N}_2}(r)\Big],$$

$$\begin{aligned}\frac{\partial^2}{\partial\zeta\partial\eta}\Big\{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\Big\} &= \exp(-\tau-\kappa)\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\\ &\quad\times\Big[\Big(\psi(\eta+r+1)-\psi(\eta+\zeta+r+s+2)\Big)\Big(\psi(\zeta+s+1)\\ &\quad-\psi(\eta+\zeta+r+s+2)\Big)-\psi_1(\eta+\zeta+r+s+2)\Big],\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial\kappa\partial\eta}\Big\{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\Big\} &= -\exp(-\tau-\kappa)\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\Big[\psi(\eta+r+1)-\psi(\eta+\zeta+r+s+2)\Big]\\ &\quad\times\Big[1+\frac{L_{s-1}^{(1)}(\kappa)}{L_s(\kappa)}\mathbf{1}_{\mathbb{N}_1}(s)\Big],\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial\kappa\partial\zeta}\Big\{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\Big\} &= -\exp(-\tau-\kappa)\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\Big[\psi(\zeta+s+1)-\psi(\eta+\zeta+r+s+2)\Big]\\ &\quad\times\Big[1+\frac{L_{s-1}^{(1)}(\kappa)}{L_s(\kappa)}\mathbf{1}_{\mathbb{N}_1}(s)\Big],\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial\tau\partial\eta}\Big\{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\Big\} &= -\exp(-\tau-\kappa)\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\Big[\psi(\eta+r+1)-\psi(\eta+\zeta+r+s+2)\Big]\\ &\quad\times\Big[1+\frac{L_{r-1}^{(1)}(\tau)}{L_r(\tau)}\mathbf{1}_{\mathbb{N}_1}(r)\Big],\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial\tau\partial\zeta}\Big\{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\Big\} &= -\exp(-\tau-\kappa)\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\Big[\psi(\zeta+s+1)-\psi(\eta+\zeta+r+s+2)\Big]\\ &\quad\times\Big[1+\frac{L_{r-1}^{(1)}(\tau)}{L_r(\tau)}\mathbf{1}_{\mathbb{N}_1}(r)\Big],\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial\tau\partial\kappa}\Big\{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\Big\} &= \exp(-\tau-\kappa)\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\Big[1+\frac{L_{r-1}^{(1)}(\tau)}{L_r(\tau)}\mathbf{1}_{\mathbb{N}_1}(r)+\frac{L_{s-1}^{(1)}(\kappa)}{L_s(\kappa)}\mathbf{1}_{\mathbb{N}_1}(s)\\ &\quad+\frac{L_{s-1}^{(1)}(\kappa)}{L_s(\kappa)}\,\frac{L_{r-1}^{(1)}(\tau)}{L_r(\tau)}\,\mathbf{1}_{\mathbb{N}_1}(s)\,\mathbf{1}_{\mathbb{N}_1}(r)\Big],\end{aligned}$$

where $\mathbb{N}_i=\Big\{i,i+1,\cdots\Big\}$.

3. Entropies

In this section, we discuss the exact form of Rényi and Shannon entropies for the extended beta-type 1 distribution $U \sim EEB1(\eta, \zeta; \kappa, \tau)$.

Entropy is a fundamental concept in various disciplines related to communications and coding theory, playing a crucial role in the efficient representation of information, encompassing both audio and video (*cf.* [10]).

Let $(\mathfrak{X}, \mathfrak{B}, P)$ be a probability space and consider a probability density function f associated with P, dominated by $\sigma-$finite measure μ on $\mathfrak{X}$. Shannon (*cf.* [22]) presented the well-known Shannon entropy is defined by:

$$\mathcal{H}_{Sh}(f) = -\int_{\mathfrak{X}} f(x)\ \log\ f(x)\, d\mu\,. \tag{3.1}$$

It has good extensively applications in various areas, including hydrology, ecology, and water resources. In ecology, entropy helps measure how many different species are present in an ecosystem. The entropy is used for earthquake forecasting (*cf.* [13]).

Rényi (*cf.* [21]) derived further extensions of (3.1) and gave the generalized entropy measure as follows:

$$\mathcal{H}_R(\vartheta) = \mathcal{H}_R(\vartheta, f) = \frac{\ell n\, G(\vartheta)}{1-\vartheta}\,, \tag{3.2}$$

where $\vartheta > 0$, $\vartheta \neq 1$, and $G(\vartheta) = \int_{\mathfrak{X}} f^{\vartheta}\ \mathrm{d}\mu$.

Like Shannon's entropy, the Rényi entropy has also been used by several authors since its introduction in the literature. There are numerous uses for Rényi entropy in dynamical systems to quantify uncertainty and it has also proved to be a useful criterion for optimization problems (*cf.* [6, 14]).

The involved parameter ϑ is utilized to describe complex behavior in probability models and the related process under investigation (*cf.* [16, 26]). The limiting case of $\mathcal{H}_R(\vartheta)$ as $\vartheta \uparrow 1$ is Shannon entropy $\mathcal{H}_{Sh}$. Rényi entropy $\mathcal{H}_R(\vartheta)$ is monotonically decreasing in ϑ, while the Shannon entropy (3.1) is acquired from (3.2) for $\vartheta \uparrow 1$.

Lemma 3.1. *Let* $g(\eta, \zeta, \kappa, \tau) = \lim\limits_{\vartheta \to 1} h(\vartheta)$, *where* $h(\vartheta) = \frac{d}{d\vartheta}\Big[\mathrm{B}_{\vartheta\kappa,\vartheta\tau}\Big(\vartheta(\eta-1)+1, \vartheta(\zeta-1)+1\Big)\Big]$. *Then*

$$\begin{aligned} g(\eta, \zeta, \kappa, \tau) \quad &= \quad \exp(-\tau-\kappa) \sum_{r,s=0}^{\infty} L_r(\tau) L_s(\kappa)\, \mathrm{B}(\eta+r+1, \zeta+s+1) \\ &\quad \times\Big[-\tau\, \frac{L_{r-1}^{(1)}(\tau)}{L_r(\tau)} \boldsymbol{I}_{\mathbb{N}_1}(r) - \kappa\, \frac{L_{s-1}^{(1)}(\kappa)}{L_s(\kappa)} \boldsymbol{I}_{\mathbb{N}_1}(s) - \kappa - \tau + (\eta-1)\, \psi(\eta+r+1) \\ &\quad +(\zeta-1)\, \psi(\zeta+s+1) - (\eta+\zeta-2)\, \psi(\eta+\zeta+r+s+2)\Big], \end{aligned}$$

where $\psi(.)$ *is the Psi function (or digamma function).*

Proof. Using (1.16), we expand $\Big[\mathrm{B}_{\vartheta\kappa,\vartheta\tau}\Big(\vartheta(\eta-1)+1, \vartheta(\zeta-1)+1\Big)\Big]$ in a series involving Laguerre polynomials as

$$h(\vartheta) = \frac{d}{d\vartheta} \sum_{r,s=0}^{\infty} \Delta_{rs}(\vartheta) = \sum_{r,s=0}^{\infty} \Big[\frac{d}{d\vartheta} \Delta_{rs}(\vartheta)\Big], \tag{3.3}$$

where

$$\Delta_{rs}(\vartheta) = \mathrm{e}^{-\vartheta(\tau+\kappa)} \sum_{r,s=0}^{\infty} L_r(\vartheta\tau) L_s(\vartheta\kappa)\, \frac{\Gamma(\vartheta(\eta-1)+r+2)\, \Gamma(\vartheta(\zeta-1)+s+2)}{\Gamma(\vartheta(\eta+\zeta-2)+r+s+4)}.$$

Now, the derivative of the logarithms of $\Delta_{rs}(\vartheta)$ with regard to ϑ can be calculated as

$$\begin{aligned} \frac{d}{d\vartheta} \Delta_{rs}(\vartheta) \quad &= \quad \Big[-\tau\, \frac{L_{r-1}^{(1)}(\tau)}{L_r(\tau)} \mathbf{1}_{\mathbb{N}_1}(r) - \kappa\, \frac{L_{s-1}^{(1)}(\kappa)}{L_s(\kappa)} \mathbf{1}_{\mathbb{N}_1}(s) - \kappa - \tau \\ &\quad +(\eta-1)\, \psi(\vartheta(\eta-1)+r+2) + (\zeta-1)\, \psi(\vartheta(\zeta-1)+s+2) \\ &\quad -(\eta+\zeta-2)\, \psi(\vartheta(\eta+\zeta-2)+r+s+4)\Big]. \end{aligned} \tag{3.4}$$

Finally, putting (3.4) in (3.3) and taking $\vartheta \to 1$, we obtain the stated result. □

Theorem 3.2. *Rényi and Shannon entropies for the extended beta-type* 1 *distribution defined by the p.d.f.* (1.5) *are given by*

$$\mathcal{H}_R(\vartheta, f) = \frac{1}{1-\vartheta}\Big[\ell n\ \mathrm{B}_{\vartheta\kappa,\vartheta\tau}\Big(\vartheta(\eta-1)+1, \vartheta(\zeta-1)+1\Big) - \vartheta\ \ell n\ \mathrm{B}_{\kappa,\tau}(\eta,\zeta)\Big], \quad \vartheta > 0 \text{ and } \vartheta \neq 1 \tag{3.5}$$

and

$$\mathcal{H}_{Sh}(f) = -\frac{g(\eta,\zeta,\kappa,\tau)}{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)} + \ell n\ \mathrm{B}_{\kappa,\tau}(\eta,\zeta), \tag{3.6}$$

where $g(\eta,\zeta,\kappa,\tau)$ is given by Lemma 3.1.

Proof. Using the p.d.f. (1.5) and equation (1.4), we have obtained

$$\begin{aligned} G(\vartheta) &= \int_0^1 \Big[\mathrm{f}_{EEB1}(u;\eta,\zeta,\kappa,\tau)\Big]^{\vartheta} \mathrm{d}u \\ &= \frac{1}{[\mathrm{B}_{\kappa,\tau}(\eta,\zeta)]^{\vartheta}} \int_0^1 u^{\vartheta(\eta-1)}\,(1-u)^{\vartheta(\zeta-1)}\ \exp\left[-\frac{\vartheta\kappa}{u} - \frac{\vartheta\tau}{(1-u)}\right] \mathrm{d}u \\ &= \frac{\mathrm{B}_{\vartheta\kappa,\vartheta\tau}\Big(\vartheta(\eta-1)+1, \vartheta(\zeta-1)+1\Big)}{[\mathrm{B}_{\kappa,\tau}(\eta,\zeta)]^{\vartheta}}. \end{aligned}$$

Further, on taking the logarithm of $G(\vartheta)$ and utilizing (3.2), we acquire (3.5).

Application of L'Hospital rule on the right-hand side of (3.5) results in

$$-\frac{\frac{\mathrm{d}}{\mathrm{d}\vartheta}\Big[\mathrm{B}_{\vartheta\kappa,\vartheta\tau}\Big(\vartheta(\eta-1)+1, \vartheta(\zeta-1)+1\Big)\Big]}{\mathrm{B}_{\vartheta\kappa,\vartheta\tau}\Big(\vartheta(\eta-1)+1, \vartheta(\zeta-1)+1\Big)} + \ell n\ \mathrm{B}_{\kappa,\tau}(\eta,\zeta).$$

Taking $\lim\limits_{\vartheta\to 1}$, one obtains (3.6), which is the required result. □

4. Fisher information matrix

In this segment, we evaluate the Fisher information matrix for the generalized extended beta-type 1 distribution (1.5). Fisher information plays a pivotal role throughout statistical modeling, system analysis, inference and estimation, including input design, prediction bounds for Bayesian analysis. The Fisher information matrix for a given observation u is as follows:

$$\begin{bmatrix} \mathbb{E}\left(\frac{\partial^2}{\partial\eta^2}\ \ell nL\right) & \mathbb{E}\left(\frac{\partial^2}{\partial\eta\partial\zeta}\ \ell nL\right) & \mathbb{E}\left(\frac{\partial^2}{\partial\eta\partial\kappa}\ \ell nL\right) & \mathbb{E}\left(\frac{\partial^2}{\partial\eta\partial\tau}\ \ell nL\right) \\ \mathbb{E}\left(\frac{\partial^2}{\partial\eta\partial\zeta}\ \ell nL\right) & \mathbb{E}\left(\frac{\partial^2}{\partial\zeta^2}\ \ell nL\right) & \mathbb{E}\left(\frac{\partial^2}{\partial\zeta\partial\kappa}\ \ell nL\right) & \mathbb{E}\left(\frac{\partial^2}{\partial\zeta\partial\tau}\ \ell nL\right) \\ \mathbb{E}\left(\frac{\partial^2}{\partial\eta\partial\kappa}\ \ell nL\right) & \mathbb{E}\left(\frac{\partial^2}{\partial\zeta\partial\kappa}\ \ell nL\right) & \mathbb{E}\left(\frac{\partial^2}{\partial\kappa^2}\ \ell nL\right) & \mathbb{E}\left(\frac{\partial^2}{\partial\kappa\partial\tau}\ \ell nL\right) \\ \mathbb{E}\left(\frac{\partial^2}{\partial\eta\partial\tau}\ \ell nL\right) & \mathbb{E}\left(\frac{\partial^2}{\partial\zeta\partial\tau}\ \ell nL\right) & \mathbb{E}\left(\frac{\partial^2}{\partial\kappa\partial\tau}\ \ell nL\right) & \mathbb{E}\left(\frac{\partial^2}{\partial\tau^2}\ \ell nL\right) \end{bmatrix},$$

where $L(\eta,\zeta,\kappa,\tau) = \ell n\ \mathrm{f}_{EEB1}(u;\eta,\zeta,\kappa,\tau)$.

From (1.5), the natural logarithms of $L(\eta,\zeta.\kappa,\tau)$ can be obtained as

$$\ell n\ L(\eta,\zeta,\kappa,\tau) = -\ell n\ \mathrm{B}_{\kappa,\tau}(\eta,\zeta) + (\eta-1)\ \ell nu + (\zeta-1)\ \ell n(1-u) - \frac{\kappa}{u} - \frac{\tau}{1-u}.$$

We note that, all the second-order partial derivatives of the $\ell n\ L(\eta,\zeta,\kappa,\tau)$ are constant and the expected value of a constant is the constant itself, we have

$$\begin{aligned}\mathbb{E}\left(\frac{\partial^2}{\partial\eta^2}\ell n\ L(\eta,\zeta,\kappa,\tau)\right) &= \frac{\partial^2}{\partial\eta^2}\ell n\ L(\eta,\zeta,\kappa,\tau)\\ &= \frac{1}{[\mathrm{B}_{\kappa,\tau}(\eta,\zeta)]^2}\left[\frac{\partial}{\partial\eta}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right]^2-\frac{1}{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)}\left[\frac{\partial^2}{\partial\eta^2}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right],\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left(\frac{\partial^2}{\partial\zeta^2}\ell n\ L(\eta,\zeta,\kappa,\tau)\right) &= \frac{\partial^2}{\partial\zeta^2}\ell n\ L(\eta,\zeta,\kappa,\tau)\\ &= \frac{1}{[\mathrm{B}_{\kappa,\tau}(\eta,\zeta)]^2}\left[\frac{\partial}{\partial\zeta}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right]^2-\frac{1}{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)}\left[\frac{\partial^2}{\partial\zeta^2}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right],\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left(\frac{\partial^2}{\partial\kappa^2}\ell n\ L(\eta,\zeta,\kappa,\tau)\right) &= \frac{\partial^2}{\partial\kappa^2}\ell n\ L(\eta,\zeta,\kappa,\tau)\\ &= \frac{1}{[\mathrm{B}_{\kappa,\tau}(\eta,\zeta)]^2}\left[\frac{\partial}{\partial\kappa}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right]^2-\frac{1}{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)}\left[\frac{\partial^2}{\partial\kappa^2}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right],\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left(\frac{\partial^2}{\partial\tau^2}\ell n\ L(\eta,\zeta,\kappa,\tau)\right) &= \frac{\partial^2}{\partial\tau^2}\ell n\ L(\eta,\zeta,\kappa,\tau)\\ &= \frac{1}{[\mathrm{B}_{\kappa,\tau}(\eta,\zeta)]^2}\left[\frac{\partial}{\partial\tau}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right]^2-\frac{1}{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)}\left[\frac{\partial^2}{\partial\tau^2}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right],\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left(\frac{\partial^2}{\partial\eta\partial\zeta}\ell n\ L(\eta,\zeta,\kappa,\tau)\right) &= \frac{\partial^2}{\partial\eta\partial\zeta}\ell n\ L(\eta,\zeta,\kappa,\tau)\\ &= \frac{1}{[\mathrm{B}_{\kappa,\tau}(\eta,\zeta)]^2}\left[\frac{\partial}{\partial\eta}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right]\left[\frac{\partial}{\partial\zeta}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right]-\frac{1}{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)}\left[\frac{\partial^2}{\partial\eta\partial\zeta}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right],\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left(\frac{\partial^2}{\partial\eta\partial\kappa}\ell n\ L(\eta,\zeta,\kappa,\tau)\right) &= \frac{\partial^2}{\partial\eta\partial\kappa}\ell n\ L(\eta,\zeta,\kappa,\tau)\\ &= \frac{1}{[\mathrm{B}_{\kappa,\tau}(\eta,\zeta)]^2}\left[\frac{\partial}{\partial\eta}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right]\left[\frac{\partial}{\partial\kappa}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right]-\frac{1}{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)}\left[\frac{\partial^2}{\partial\eta\partial\kappa}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right],\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left(\frac{\partial^2}{\partial\eta\partial\tau}\ell n\ L(\eta,\zeta,\kappa,\tau)\right) &= \frac{\partial^2}{\partial\eta\partial\tau}\ell n\ L(\eta,\zeta,\kappa,\tau)\\ &= \frac{1}{[\mathrm{B}_{\kappa,\tau}(\eta,\zeta)]^2}\left[\frac{\partial}{\partial\eta}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right]\left[\frac{\partial}{\partial\tau}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right]-\frac{1}{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)}\left[\frac{\partial^2}{\partial\eta\partial\tau}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right],\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left(\frac{\partial^2}{\partial\zeta\partial\kappa}\ell n\ L(\eta,\zeta,\kappa,\tau)\right) &= \frac{\partial^2}{\partial\zeta\partial\kappa}\ell n\ L(\eta,\zeta,\kappa,\tau)\\ &= \frac{1}{[\mathrm{B}_{\kappa,\tau}(\eta,\zeta)]^2}\left[\frac{\partial}{\partial\zeta}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right]\left[\frac{\partial}{\partial\kappa}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right]-\frac{1}{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)}\left[\frac{\partial^2}{\partial\zeta\partial\kappa}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right],\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left(\frac{\partial^2}{\partial\zeta\partial\tau}\ell n\, L(\eta,\zeta,\kappa,\tau)\right) &= \frac{\partial^2}{\partial\zeta\partial\tau}\ell n\, L(\eta,\zeta,\kappa,\tau)\\ &= \frac{1}{[\mathrm{B}_{\kappa,\tau}(\eta,\zeta)]^2}\left[\frac{\partial}{\partial\zeta}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right]\left[\frac{\partial}{\partial\tau}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right]-\frac{1}{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)}\left[\frac{\partial^2}{\partial\zeta\partial\tau}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right],\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left(\frac{\partial^2}{\partial\kappa\partial\tau}\ell n\, L(\eta,\zeta,\kappa,\tau)\right) &= \frac{\partial^2}{\partial\kappa\partial\tau}\ell n\, L(\eta,\zeta,\kappa,\tau)\\ &= \frac{1}{[\mathrm{B}_{\kappa,\tau}(\eta,\zeta)]^2}\left[\frac{\partial}{\partial\kappa}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right]\left[\frac{\partial}{\partial\tau}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right]-\frac{1}{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)}\left[\frac{\partial^2}{\partial\kappa\partial\tau}\mathrm{B}_{\kappa,\tau}(\eta,\zeta)\right].\end{aligned}$$

Utilizing results of section 2, putting explicit expressions for the partial derivatives of $\mathrm{B}_{\kappa,\tau}(\eta,\zeta)$, we acquire

$$\begin{aligned}\mathbb{E}\left(\frac{\partial^2}{\partial\eta^2}\ell n\, L(\eta,\zeta,\kappa,\tau)\right) &= \frac{e^{-2(\kappa+\tau)}}{[\mathrm{B}_{\kappa,\tau}(\eta,\zeta)]^2}\left[\sum_{r,s=0}^{\infty}L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\Big\{\psi(\eta+r+1)-\psi(\eta+\zeta+r+s+2)\Big\}\right]^2\\ &\quad-\frac{e^{-(\kappa+\tau)}}{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)}\sum_{r,s=0}^{\infty}L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\bigg[\Big(\psi(\eta+r+1)-\psi(\eta+\zeta+r+s+2)\Big)^2\\ &\quad+\psi_1(\eta+r+1)-\psi_1(\eta+\zeta+r+s+2)\bigg],\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left(\frac{\partial^2}{\partial\zeta^2}\ell n\, L(\eta,\zeta,\kappa,\tau)\right) &= \frac{e^{-2(\kappa+\tau)}}{[\mathrm{B}_{\kappa,\tau}(\eta,\zeta)]^2}\left[\sum_{r,s=0}^{\infty}L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\Big\{\psi(\zeta+s+1)-\psi(\eta+\zeta+r+s+2)\Big\}\right]^2\\ &\quad-\frac{e^{-(\kappa+\tau)}}{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)}\sum_{r,s=0}^{\infty}L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\\ &\quad\times\bigg[\Big(\psi(\zeta+s+1)-\psi(\eta+\zeta+r+s+2)\Big)^2+\psi_1(\zeta+s+1)-\psi_1(\eta+\zeta+r+s+2)\bigg],\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left(\frac{\partial^2}{\partial\kappa^2}\ell n\, L(\eta,\zeta,\kappa,\tau)\right) &= \frac{e^{-2(\kappa+\tau)}}{[\mathrm{B}_{\kappa,\tau}(\eta,\zeta)]^2}\left[\sum_{r,s=0}^{\infty}L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\left(1+\frac{L_{s-1}^{(1)}(\kappa)}{L_s(\kappa)}\mathbf{1}_{\mathbb{N}_1}(s)\right)\right]^2\\ &\quad-\frac{e^{-(\kappa+\tau)}}{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)}\sum_{r,s=0}^{\infty}L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\\ &\quad\times\left[1+2\,\frac{L_{s-1}^{(1)}(\kappa)}{L_s(\kappa)}\mathbf{1}_{\mathbb{N}_1}(s)+\frac{L_{s-2}^{(2)}(\kappa)}{L_s(\kappa)}\mathbf{1}_{\mathbb{N}_1}(s)\,\mathbf{1}_{\mathbb{N}_2}(s)\right],\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left(\frac{\partial^2}{\partial\tau^2}\ell n\, L(\eta,\zeta,\kappa,\tau)\right) &= \frac{e^{-2(\kappa+\tau)}}{[\mathrm{B}_{\kappa,\tau}(\eta,\zeta)]^2}\left[\sum_{r,s=0}^{\infty}L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\left(1+\frac{L_{r-1}^{(1)}(\tau)}{L_r(\tau)}\mathbf{1}_{\mathbb{N}_1}(r)\right)\right]^2\\ &\quad-\frac{e^{-(\kappa+\tau)}}{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)}\sum_{r,s=0}^{\infty}L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\\ &\quad\times\left[1+2\,\frac{L_{r-1}^{(1)}(\tau)}{L_r(\tau)}\mathbf{1}_{\mathbb{N}_1}(r)+\frac{L_{r-2}^{(2)}(\tau)}{L_r(\tau)}\mathbf{1}_{\mathbb{N}_1}(r)\,\mathbf{1}_{\mathbb{N}_2}(r)\right],\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}\left(\frac{\partial^2}{\partial\eta\partial\zeta}\ell n\, L(\eta,\zeta,\kappa,\tau)\right) &= \frac{\mathrm{e}^{-2(\kappa+\tau)}}{[\mathrm{B}_{\kappa,\tau}(\eta,\zeta)]^2}\Bigg[\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\\
&\quad\times\Big(\psi(\eta+r+1)-\psi(\eta+\zeta+r+s+2)\Big)\Bigg]\\
&\quad\times\Bigg[\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\mathrm{B}(\eta+r+1,\zeta+s+1)\Big(\psi(\zeta+s+1)-\psi(\eta+\zeta+r+s+2)\Big)\Bigg]\\
&\quad-\frac{\mathrm{e}^{-(\kappa+\tau)}}{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)}\Bigg[\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\\
&\quad\times\Big\{\Big(\psi(\eta+r+1)-\psi(\eta+\zeta+r+s+2)\Big)\Big(\psi(\zeta+s+1)\\
&\quad-\psi(\eta+\zeta+r+s+2)\Big)-\psi_1(\eta+\zeta+r+s+2)\Big\}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left(\frac{\partial^2}{\partial\eta\partial\kappa}\ell n\, L(\eta,\zeta,\kappa,\tau)\right) &= -\frac{\mathrm{e}^{-2(\kappa+\tau)}}{[\mathrm{B}_{\kappa,\tau}(\eta,\zeta)]^2}\Bigg[\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\\
&\quad\times\Big(\psi(\eta+r+1)-\psi(\eta+\zeta+r+s+2)\Big)\Bigg]\\
&\quad\times\Bigg[\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\mathrm{B}(\eta+r+1,\zeta+s+1)\Big(1+\frac{L_{s-1}^{(1)}(\kappa)}{L_s(\kappa)}\mathbf{1}_{\mathbb{N}_1}(s)\Big)\Bigg]\\
&\quad+\frac{\mathrm{e}^{-(\kappa+\tau)}}{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)}\Bigg[\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\\
&\quad\times\Big\{\Big(\psi(\eta+r+1)-\psi(\eta+\zeta+r+s+2)\Big)\Big(1+\frac{L_{s-1}^{(1)}(\kappa)}{L_s(\kappa)}\mathbf{1}_{\mathbb{N}_1}(s)\Big)\Big\}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left(\frac{\partial^2}{\partial\eta\partial\tau}\ell n\, L(\eta,\zeta,\kappa,\tau)\right) &= -\frac{\mathrm{e}^{-2(\kappa+\tau)}}{[\mathrm{B}_{\kappa,\tau}(\eta,\zeta)]^2}\Bigg[\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\\
&\quad\times\Big(\psi(\eta+r+1)-\psi(\eta+\zeta+r+s+2)\Big)\Bigg]\\
&\quad\times\Bigg[\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\mathrm{B}(\eta+r+1,\zeta+s+1)\Big(1+\frac{L_{r-1}^{(1)}(\tau)}{L_r(\tau)}\mathbf{1}_{\mathbb{N}_1}(r)\Big)\Bigg]\\
&\quad+\frac{\mathrm{e}^{-(\kappa+\tau)}}{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)}\Bigg[\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\\
&\quad\times\Big\{\Big(\psi(\eta+r+1)-\psi(\eta+\zeta+r+s+2)\Big)\Big(1+\frac{L_{r-1}^{(1)}(\tau)}{L_s(\tau)}\mathbf{1}_{\mathbb{N}_1}(r)\Big)\Big\}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left(\frac{\partial^2}{\partial\zeta\partial\kappa}\ell n\, L(\eta,\zeta,\kappa,\tau)\right) &= -\frac{\mathrm{e}^{-2(\kappa+\tau)}}{[\mathrm{B}_{\kappa,\tau}(\eta,\zeta)]^2}\Bigg[\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\\
&\quad\times\Big(\psi(\zeta+s+1)-\psi(\eta+\zeta+r+s+2)\Big)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\times\Bigg[\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\mathrm{B}(\eta+r+1,\zeta+s+1)\bigg(1+\frac{L_{s-1}^{(1)}(\kappa)}{L_s(\kappa)}\mathbf{1}_{\mathbb{N}_1}(s)\bigg)\Bigg]\\
&+\frac{\mathrm{e}^{-(\kappa+\tau)}}{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)}\Bigg[\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\\
&\times\bigg\{\Big(\psi(\zeta+s+1)-\psi(\eta+\zeta+r+s+2)\Big)\bigg(1+\frac{L_{s-1}^{(1)}(\kappa)}{L_s(\kappa)}\mathbf{1}_{\mathbb{N}_1}(s)\bigg)\bigg\}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\bigg(\frac{\partial^2}{\partial\zeta\partial\tau}\ell n\; L(\eta,\zeta,\kappa,\tau)\bigg) \;=\;& -\frac{\mathrm{e}^{-2(\kappa+\tau)}}{[\mathrm{B}_{\kappa,\tau}(\eta,\zeta)]^2}\Bigg[\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\\
&\times\Big(\psi(\zeta+s+1)-\psi(\eta+\zeta+r+s+2)\Big)\Bigg]\\
&\times\Bigg[\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\mathrm{B}(\eta+r+1,\zeta+s+1)\bigg(1+\frac{L_{r-1}^{(1)}(\tau)}{L_r(\tau)}\mathbf{1}_{\mathbb{N}_1}(r)\bigg)\Bigg]\\
&+\frac{\mathrm{e}^{-(\kappa+\tau)}}{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)}\Bigg[\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\\
&\times\bigg\{\Big(\psi(\zeta+s+1)-\psi(\eta+\zeta+r+s+2)\Big)\bigg(1+\frac{L_{r-1}^{(1)}(\tau)}{L_s(\tau)}\mathbf{1}_{\mathbb{N}_1}(r)\bigg)\bigg\}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\bigg(\frac{\partial^2}{\partial\kappa\partial\tau}\ell n\; L(\eta,\zeta,\kappa,\tau)\bigg) \;=\;& \frac{\mathrm{e}^{-2(\kappa+\tau)}}{[\mathrm{B}_{\kappa,\tau}(\eta,\zeta)]^2}\Bigg[\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\bigg(1+\frac{L_{s-1}^{(1)}(\kappa)}{L_s(\kappa)}\mathbf{1}_{\mathbb{N}_1}(s)\bigg)\Bigg]\\
&\times\Bigg[\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\mathrm{B}(\eta+r+1,\zeta+s+1)\bigg(1+\frac{L_{r-1}^{(1)}(\tau)}{L_r(\tau)}\mathbf{1}_{\mathbb{N}_1}(r)\bigg)\Bigg]\\
&-\frac{\mathrm{e}^{-(\kappa+\tau)}}{\mathrm{B}_{\kappa,\tau}(\eta,\zeta)}\Bigg[\sum_{r,s=0}^{\infty} L_r(\tau)L_s(\kappa)\,\mathrm{B}(\eta+r+1,\zeta+s+1)\\
&\times\bigg\{\bigg(1+\frac{L_{s-1}^{(1)}(\kappa)}{L_s(\kappa)}\mathbf{1}_{\mathbb{N}_1}(s)\bigg)\bigg(1+\frac{L_{r-1}^{(1)}(\tau)}{L_r(\tau)}\mathbf{1}_{\mathbb{N}_1}(r)\bigg)\bigg\}\Bigg].
\end{aligned}
$$

5. Concluding remarks

In this present article, we have determined some more properties of the extended beta type 1 distribution presented by Choi *et al.* (*cf.* [9]) which is a versatile tool used in various fields like statistics, quality control and enabling better decision-making in real world scenarios. Additionally, Fisher information matrix, such as those of Rényi and Shannon for above mentioned distribution.

Moreover, for future work, we can easily derive Rényi and Shannon's entropies for distinct extended beta type 1 distribution in a same manner given in the preceding sections. Such results are still an open problem for further research.

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