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Ramanujan-type congruences for partitions functions modulo prime powers

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Abstract

In this paper, we explore the generalized partition functions $p_{[1^{\lambda_0} l^{k\lambda_1}]}(n)$ as defined by their generating functions

$$\sum_{n=0}^{\infty} p_{[1^{\lambda_0} l^{k\lambda_1}]}(n) q^n = \frac{q^{\frac{\lambda_0 + l^k \lambda_1}{24}}}{\eta(\tau)^{\lambda_0} \eta(l^k \tau)^{\lambda_1}},$$

where $\eta(\tau)$ is the Dedekind eta function, l prime number, k positive integer and λ_0, λ_1 arbitrary integers. We establish Ramanujan-type congruences for these partition functions $p_{[1^{\lambda_0} l^{k\lambda_1}]}(n)$ modulo powers of primes $l = 2, 3, 5, 7, 11, 13, 17$ for any integers λ_0, and λ_1. Our results not only confirm but also extend the previous work of Atkin, Gordon, Wang, and others concerning congruences for various partition functions.

Keywords: Modular forms, partition functions, Ramanujan's congruences, eta-quotients

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1. Introduction

A partition of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$. The λ_i are called the parts of the partition. For example, partitions of 5 are 5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1.

In 1753, Euler [7] established the generating function of $p(n)$, given by the following formula:

$$\prod_{m=1}^{\infty} \frac{1}{1 - q^m} = \sum_{n=0}^{\infty} p(n) q^n, \ |q| < 1.$$

Furthermore, Euler [7] proved the formula for the generating function of partitions into distinct parts as follows:

$$\sum_{n=0}^{\infty} p_d(n) q^n = \prod_{m=1}^{\infty} (1 + q^m), \ |q| < 1, \ p_d(0) = 1.$$

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He also established a generating formula for the partition function into odd parts.

$$\sum_{n=0}^{\infty} p_o(n)q^n = \prod_{m=1}^{\infty} \frac{1}{1-q^{2m-1}},\ |q| < 1,\ p_o(0) = 1.$$

Throughout this paper, we use the following q-series notation

$$(a;q)_\infty = \prod_{k=1}^{\infty} \left(1 - aq^{k-1}\right).$$

The partition function $p(n)$ of positive integers, satisfies the following recurrence relation discovered by Euler (*cf.* [7]):

$$p(n) = \sum_{j=1}^{n} \left(p\left(n - \frac{j(3j-1)}{2}\right) + p\left(n - \frac{j(3j+1)}{2}\right)\right). \tag{1.1}$$

Here, the terms involve the generalized pentagonal numbers, given by $\frac{j(3j-1)}{2}$ and $\frac{j(3j+1)}{2}$ for $j \in \mathbb{N}\backslash\{0\}$. For small values of n, the recurrence can be computed explicitly:

$$\begin{aligned}
&p(1) = p(0) = 1,\\
&p(2) = p(1) + p(0) = 2,\\
&p(3) = p(2) + p(1) = 3,\\
&p(4) = p(3) + p(2) = 5,\\
&p(5) = p(4) + p(3) - p(0) = 7
\end{aligned}$$

and so on.

In 1916, MacMahon [20] utilized (1.1) to compute the values of $p(n)$ for $1 \le n \le 200$.

In 1918, Hardy and Ramanujan [13] proved the following asymptotic formula

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi \sqrt{\frac{2n}{3}}} \tag{1.2}$$

which gives $p(200) \sim 4.10025 \times 10^{12}$.

The formula (1.2) was improved in 1936 [21] to a convergent one by Rademacher [21] leading to the Hardy-Ramanujan-Rademacher formula.

$$p(n) \sim \sum_{k=1}^{N(n)} \sum_{\substack{h=1\\(h,k)=1}}^{k-1} e^{-2\pi i n h/k} \times \frac{2\pi}{k}\left(\frac{\pi}{6k}\right)^{3/2} e^{\pi i s(h,k)} \times d_k(n)^{-3/2} l_{3/2}(d_k(n)) + O(1), \tag{1.3}$$

where $s(h,k) = \sum_{m=1}^{k-1} \left(\frac{m}{k}\right)\left(\frac{mh}{k}\right)$ is the Dedekind sum and $d_k(n) = \frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}$ and $N(n) \sim n^{1/2}$.

In [22]-[24] Ramanujan find congruence relations for $p(n)$.

For $l \in \{5,\ 7,\ 11\}$, for all positive integers j, let $\delta_{l,j}$ be the reciprocal of 24 modulo l^j, i.e., $24\delta_{l,j} \equiv 1 \pmod{l^j}$. For $n \ge 0$, it is known that

$$p\left(5^j n + \delta_{5,j}\right) \equiv 0 \pmod{5^j} \tag{1.4}$$

$$p\left(7^j n + \delta_{7,j}\right) \equiv 0 \pmod{7^{[j/2]+1}} \tag{1.5}$$

$$p\left(11^j n + \delta_{11,j}\right) \equiv 0 \pmod{11^j}. \tag{1.6}$$

In 1919, Ramanujan [22] proved the first two congruences for $j = 1$, using the Jacobi triple product, and later in [23] using modular forms theory on $SL_2(\mathbb{Z})$. In 1938, Watson [27] proved the congruences (1.4) and (1.5) for arbitrary

j, using modular equations of degrees 5 and 7. In 1967, Atkin [1] proved the congruence (1.6), using the modular equation of degree 11. In 1981, Hirschhorn and Hunt [17], and in 1982, Garvan [9], provided simple proofs for (1.4) and (1.5). Their demonstrations primarily rely on the Jacobi triple product and Euler's theorem.

To study a large class of restricted partitions, in this paper we are interested in the partition functions $p_{[1^{\lambda_0} l^{k\lambda_1}]}(n)$ defined by

$$\sum_{n=0}^{\infty} p_{[1^{\lambda_0} l^{k\lambda_1}]}(n) q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{\lambda_0}(1-q^{l^k n})^{\lambda_1}} = \frac{q^{\frac{\lambda_0 + l^k \lambda_1}{24}}}{\eta(\tau)^{\lambda_0} \eta(l^k \tau)^{\lambda_1}},$$

where

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n), \qquad q = e^{2\pi i \tau},\ \operatorname{Im}(\tau) > 0$$

is the Dedekind eta function.

In the paper Wang [26] proved the following congruences.

Theorem 1.1 (*cf.* [26])**.** *For any integers $n \geq 1$ and $r \geq 1$, we have*

$$p_{[1^1 11^{-11}]}(11^r n + 11^r - 5) \equiv 0 \pmod{11^r},$$

$$p_{[1^1 11^{-11}]}\left(11^{2r-1} n + \frac{7 \cdot 11^{2r-1} - 5}{12}\right) \equiv 0 \pmod{11^r},$$

$$p_{[1^1 11^1]}\left(11^r n + \frac{11^r + 1}{2}\right) \equiv 0 \pmod{11^r}.$$

Mestrige [22] proved the following congruences by using Gordon's approach.

Theorem 1.2 (*cf.* [22])**.** *For any integers c, d and for any positive integer r, we have*

$$p_{[1^c 11^d]}(11^r m + n_r) \equiv 0 \left(\bmod 11^{A_r}\right),$$

where $24 n_r \equiv (c + 11d) \pmod{11^r}$ and A_r depends only on c and d.

2. Main results and corollaries

In this section. Let $\lambda_0, \lambda_1 \in \mathbb{Z}$ and k be a positive integer. We define a sequence $(\lambda'_r)_r$ of integers as follows. For $r < k$, we have

$$\lambda'_r = \begin{cases} \lambda_0 & \text{if } r \text{ is even}, \\ 0 & \text{if } r \text{ is odd}, \end{cases}$$

$$\lambda'_k = \begin{cases} \lambda_0 + \lambda_1 & \text{if } k \text{ is even}, \\ \lambda_1 & \text{if } k \text{ is odd} \end{cases}$$

and for $r \geq k$, we set

$$\lambda'_r = \begin{cases} \lambda'_k & \text{if } r \equiv k \pmod{2}, \\ \lambda'_{k-1} & \text{if } r \not\equiv k \pmod{2}. \end{cases}$$

Theorem 2.1. *For $\ell = 5, 7, 11, 13$ and 17, $\lambda_0, \lambda_1 \in \mathbb{Z}$ and for any positive integers r, k, we have*

$$p_{[1^{\lambda_0}\, \ell^{k\lambda_1}]}(\ell^r m + n_{r,l}) \equiv 0 \pmod{\ell^{A_{r,\ell}}}, \quad \forall\, m \in \mathbb{N} \tag{2.1}$$

with $n_{r,\ell} = -\frac{(\ell^2-1)}{24} \sum_{i=0}^{r-1} \lambda'_i \ell^i$ and $A_{r,\ell}$ depend only on the integers λ_0, λ_1, r, ℓ and k.

Corollary 2.2. *For $l = 5,\ 7,\ 13$ and any positive integers $r,\ k$, we obtain*

$$p_{[1^{\lambda_0}\, l^{k\lambda_1}]}(l^r m + n_{r,l}) \equiv 0 \pmod{l^{\frac{1}{2}\alpha_l r+\epsilon}}, \quad \forall\, m \in \mathbb{N}, \tag{2.2}$$

where $24n_{r,l} \equiv (\lambda_0 + l^k\lambda_1) \pmod{l^r}$, $\epsilon = O(log|\lambda_0 + l^k\lambda_1|)$. Moreover, when $\lambda_0 + l^k\lambda_1 \geq 0$, α_l depends on the residue of $\lambda_0 + l^k\lambda_1 \pmod{24}$ which is shown in the following table, $\lambda_0 + l^k\lambda_1 < 0$ the entries in the last column need to changed to 1 for $l = 5, 7$ (Table 1).

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$l = 5$	2	1	1	1	2	2	1	1	1	1	1	0	0	0	1	1	0	0	0	1	1	0	0	0
$l = 7$	1	1	1	2	1	1	1	0	0	0	1	0	0	1	0	0	0	1	0	0	1	0	0	0
$l = 13$	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 1. Values of α_l

From Corollary 2.2, we obtain Atkin's result (*cf.* [2, 3, Theorem 1]) for $l = 5, 7, 13$. For $l = 11$, we obtain the following corollary:

Corollary 2.3. *For $l = 11$ and for any positive integers $r,\ k$, we have*

$$p_{[1^{\lambda_0}\, 11^{k\lambda_1}]}(11^r m + n_{r,11}) \equiv 0 \ \left(\mathrm{mod} 11^{\frac{1}{2}\alpha_{11} r+\epsilon}\right), \quad \forall\, m \in \mathbb{N}, \tag{2.3}$$

where $24n_{r,l} \equiv (\lambda_0 + 11^k\lambda_1) \pmod{11^r}$, $\epsilon = O(log|\lambda_0 + 11^k\lambda_1|)$. Moreover when $\lambda_0 + 11^k\lambda_1 \geq 0$, α_{11} depends on the residue of $\lambda_0 + 11^k\lambda_1 \pmod{120}$ which is shown in Table 2, for $\lambda_0 + 11^k\lambda_1 < 0$, the entries in the last column need to changed to 2, 2, 2, 0, 2.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
0	2	1	2	1	1	1	2	2	1	1	2	2	1	2	1	0	0	1	1	0	0	1	1	0
24	1	1	1	1	2	2	1	1	2	2	1	0	0	0	0	1	1	0	0	1	1	1	0	0
48	1	1	2	2	1	1	1	0	1	0	1	0	0	1	1	0	0	1	0	1	0	1	0	0
72	2	1	1	1	2	1	2	1	2	1	2	2	1	1	1	2	1	2	1	2	1	1	1	0
96	0	0	1	0	1	0	1	0	1	1	0	0	0	1	0	1	0	1	0	1	1	0	0	0

Table 2. Values of α_{11}

Here in Table 2, the entry in the row labeled $24i$ and the column labeled j is $\alpha_{11}(24i + j)$. Our Corollary 2.3 generalizes Gordon's result [15, Theorem 1.2].

Moreover, for $l = 17$ we can obtain the following result.

Corollary 2.4. *For $l = 17$ and for any positive integers $r,\ k$, we have*

$$p_{[1^{\lambda_0}\, 17^{k\lambda_1}]}(17^r m + n_{r,17}) \equiv 0 \ \left(\mathrm{mod} 17^{\frac{1}{2}\alpha_{17} r+\epsilon}\right), \quad \forall\, m \in \mathbb{N}, \tag{2.4}$$

where $24n_r \equiv (\lambda_0 + 17^k\lambda_1) \pmod{17^r}$, $\epsilon = O(log|\lambda_0 + 17^k\lambda_1|)$. Moreover when $\lambda_0 + 17^k\lambda_1 \geq 0$, α_{17} depends on the residue of $\lambda_0 + 17^k\lambda_1 \pmod{96}$, which is shown in Table 3, for $\lambda_0 + 17\lambda_1 < 0$, the entries in the last column need to changed to 0, 2, 0, 0.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
24	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
48	1	1	1	1	1	2	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
72	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 3. Values of α_{17}

Note that Corollary 2.4 generalizes Hughes result (*cf.* [14, Theorem 3]).

Let us consider the following modular forms.

$$q^{-\frac{(\ell-1)t}{24}}\left(\frac{\eta(\ell\tau)}{\eta(\tau)}\right)^t, \quad q^{-\frac{(\ell^2-1)t}{24}}\left(\frac{\eta(\ell^2\tau)}{\eta(\tau)}\right)^t \tag{2.5}$$

with t be a positive integer.

From Theorem 2.1, we obtain the following corollaries.

Corollary 2.5. *For $\ell = 7$, $k = 2$ and $t = 1$, we have*

$$p_{[1^1\, 49^{-1}]}(7^r n - 2) \equiv 0 \pmod{7^r}, \quad \forall\, r \geq 1, \forall\, n \geq 1. \tag{2.6}$$

Corollary 2.6. *Let $\ell = 5, 7$ and $t = \frac{24}{\gcd(\ell-1,24)}t'$, with $t' = \sum_{i=0}^{\infty} t'_i \ell^i$, such that $t'_0 = 1, 2$ and $t'_i = 0, 1$, $\forall i \geq 1$. We have*

$$p_{[1^t\, \ell^{-t}]}(\ell^r n - t') \equiv 0 \pmod{\ell^r}, \quad \forall\, r \geq 1, \forall\, n \in \mathbb{N}.$$

Corollary 2.7. *For $t = \frac{2}{3}(13^{m+1} + 2)$ with m be a positive integer, we obtain*

$$p_{[1^t\, 13^{-t}]}\left(13^r n - \frac{t}{2}\right) \equiv 0 \pmod{13^{\min(m,r)}}, \quad \forall\, r \geq 1, \forall\, n, m \in \mathbb{N}. \tag{2.7}$$

Corollary 2.8. *For $t = \sum_{i=0}^{\infty} t_i 5^i$, such that $t_0 = 1, 2$ and $t_i = 0, 1$, $\forall\, i \geq 1$. We have*

$$p_{[1^t\, 25^{-t}]}(5^r n - t) \equiv 0 \pmod{5^r}, \quad \forall\, r \geq 1, \forall\, n \in \mathbb{N}. \tag{2.8}$$

3. Further results on modular forms and Hecke operators

Modular forms have been used extensively to prove partition congruences. See previous studies [18, 19] for basic definitions, theoretical properties, and applications. This is due to the generating functions of partitions are related to modular forms.

Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\}$ the Poincaré half-plane and the modular group

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1\right\}$$

act on $\mathbb{H}$ by

$$\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \forall \tau \in \mathbb{H} : \gamma.\tau = \frac{a\,\tau + b}{c\,\tau + d}\,.$$

In particular, the congruence subgroups $\Gamma_0(\ell)$, defined by

$$\Gamma_0(\ell) := \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \ell \mid c\right\}, \text{ where } \ell \in \mathbb{N}^*$$

operate through the same action on $\mathbb{H}$. A modular function f for $\Gamma_0(\ell)$, is a function $f : \mathbb{H} \longrightarrow \mathbb{C}$ that satisfies the following conditions:

(i) f is meromorphic on $\mathbb{H}$,

(ii) $f\left(\frac{a\tau+b}{c\tau+d}\right) = f(\tau)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(l)$ and for any $\tau \in \mathbb{H}$,

(iii) f is meromorphic at cusps.

The Dedekind eta function is non-vanishing on the upper half plane as a function. Precisely we have

Definition 3.1. Let $\tau \in \mathbb{H}$ and $q = e^{2\pi i\tau}$. The Dedekind Eta function, $\eta(\tau)$ is defined as:

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n). \tag{3.1}$$

The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} = \frac{q^{1/24}}{\eta(\tau)}.$$

Now we introduce Atkin's operators, is a class of linear operators acting on modular forms, which play a crucial role in the theory of modular forms and their applications.

Definition 3.2. For a Laurent series $f(\tau) = \sum\limits_{n\geq N} a_n\, q^n$, we define the Atkin U_ℓ-operator by

$$U_\ell\,(f(\tau)) = \sum_{\ell\, n \geq N} a(\ell n)q^n, \quad (q = e^{2\pi i\tau}). \tag{3.2}$$

Let $g(\tau) = \sum\limits_{n\geq N} b(n)q^n$ be an another Laurent series.

$$U_\ell\,(f(\tau)g(\ell\,\tau)) = g(\tau)\,U_\ell(f(\tau)).$$

Theorem 3.3 (Atkin-Lehner)**.** *Let ℓ be an integer, if $f(\tau)$ is a modular function for $\Gamma_0(N)$ and if $\ell^2|N$, then $U_\ell\,(f(\tau))$ is a modular function for $\Gamma_0(N/\ell)$.*

Next we give the relation between Atkin and Hecke's operator.

Lemma 3.4. *Let $f(\tau) = \sum\limits_{n\geq n_0} a_n q^n$ be Laurent series and ℓ an integer. We have*

$$U_\ell\,(f(\tau)) = \frac{1}{\ell}\sum_{h=0}^{\ell-1} f\left(\frac{\tau+h}{\ell}\right).$$

Proof. See [4, Lemma 3.3]. □

3.1. Modular equations and divisibility

Let $z_\ell(\tau) = \left(\frac{\eta(\ell\tau)}{\eta(\tau)}\right)^r$ and $\phi_\ell(\tau) = \frac{\eta(\ell^2\tau)}{\eta(\tau)}$, where r is minimum positive integers which satisfies $r(\ell-1) \equiv 0 \pmod{24}$.

Definition 3.5. The modular equation for prime ℓ, we mean the equation connecting two modular $z_\ell(\tau)$ and $\phi_\ell(\tau)$.

ℓ	$z_\ell(\tau)$	$\phi_\ell(\tau)$
5	$\left(\frac{\eta(5\tau)}{\eta(\tau)}\right)^6$	$\frac{\eta(25\tau)}{\eta(\tau)}$
7	$\left(\frac{\eta(7\tau)}{\eta(\tau)}\right)^4$	$\frac{\eta(49\tau)}{\eta(\tau)}$
13	$\left(\frac{\eta(13\tau)}{\eta(\tau)}\right)^2$	$\frac{\eta(169\tau)}{\eta(\tau)}$

Table 4. Modular forms corresponding to $\ell = 5, 7, 13$

Theorem 3.6 (*cf.* [3, 8])**.** *For $l = 5$, 7 and 13. We have the following modular equations*

$$\phi_5^5(\tau) = z_5(\tau)\left(5^2\phi_5^4(\tau) + 5^2\phi_5^3(\tau) + 3.5\phi_5^2(\tau) + 5\phi_5(\tau) + 1\right),$$

$$\phi_7^7(\tau) = z_7(\tau)(7^2\phi_7^6(\tau) + 5.7\phi_7^5(\tau) + 7\phi_7^4(\tau)) + z_7^2(\tau)(7^3\phi_7^6(\tau) + 7^3\phi_7^5(\tau) + 3.7^2\phi_7^4(\tau) + 7^2\phi_7^3(\tau) + 3.7\phi_7^2(\tau) + 7\phi_7(\tau) + 1)$$

and

$$\phi_{13}^{13}(\tau) + \sum_{r=1}^{13}\sum_{\rho=\left[\frac{r+2}{2}\right]}^{7} m_{r\rho}\, z_{13}^{\rho}(\tau)\, \phi_{13}^{13-r}(\tau) = 0,$$

the explicit values of $m_{r\rho}$ are given in Table 5.

r \ ρ	1	2	3	4	5	6	7
1	-11.13	-36.13^2	-38.13^3	-20.13^4	-6.13^5	-13^6	-13^6
2		204.13	346.13^2	222.13^3	74.13^4	13^6	13^6
3		-36.13	-126.13^2	-102.13^3	-38.13^4	-7.13^5	-7.13^5
4			346.13	422.13^2	184.13^3	37.13^4	3.13^5
5			-38.13	-102.13^2	-56.13^3	-13^5	-15.13^4
6				222.13	184.13^2	51.13^3	5.13^4
6				222.13	184.13^2	51.13^3	5.13^4
7				-20.13	-38.13^2	-13^4	-19.13^3
8					74.13	37.13^2	5.13^3
9					-6.13	-7.13^2	-15.13^2
10						13^2	3.13^2
11						-13	-7.13
12							13
13							-1

Table 5. Table of $m_{r,\rho}$

Definition 3.7. For any integer r, we define $S_{r,\ell}(\tau) = \ell\, U_\ell\left(\phi_\ell^r(\tau)\right)$. This is the sum of the r^{th} power of the roots of modular equations for prime ℓ.

Proposition 3.8. *We have*

- **(i)** *For any integer r, $S_{r,\ell}$ is a modular function on $\Gamma_0(\ell)$.*
- **(ii)** *For $\ell = 5, 7, 13$, a basis for the vector space of modular functions on $\Gamma_0(\ell)$, can be given as powers of $z_\ell(\tau)$.*
- **(iii)** *Let r be any integer and $\ell = 5, 7, 13$. Then*

$$S_{r,\ell}(\tau) = \sum_{\rho=-\infty}^{+\infty} a_{r,\rho}^{\ell} z_\ell^{\rho}.$$

Proof. See [2, 8, 15]. ☐

Let $\pi_\ell(a)$ be the ℓ-adic order of the number a.

Lemma 3.9. *Let $\ell = 5, 7$, for r and ρ two integers, we have*

$$\pi_5(a_{r\rho}^5) \geq \left[\frac{5\rho - r + 1}{2}\right], \quad \pi_7(a_{r\rho}^7) \geq \left[\frac{7\rho - 2r + 3}{4}\right]$$

where

(i) $a^5_{r\rho} = 0$ *unless* $[(r+4)/5] \leq \rho \leq r$,

(ii) $a^7_{r\rho} = 0$ *unless* $[(2r+6)/7] \leq \rho \leq 2r$.

Lemma 3.10. *Let $\ell = 13$, for r and ρ two integers, we have*

$$\pi_{13}(a^{13}_{r\rho}) \geq \left[\frac{13\rho - 7r + 13}{14}\right],$$

where $a^{13}_{r\rho} = 0$ *unless* $\left[\frac{7r+12}{13}\right] \leq \rho \leq 7r$.

Theorem 3.11 (*cf.* [8, 15])**.** *For $\ell = 11$, let $t_{11} = \phi_{11}\left(q^{1/11}\right)$, t_{11} satisfies the following modular equation of degree* 11 *with coefficients in V_{11}*

$$t^{11}_{11} + \sum_{k=1}^{11} (-1)^k \sigma_k t^{11-k} = 0, \tag{3.3}$$

where the σ_k are given in [15].

Theorem 3.12 (*cf.* [14])**.** *For $\ell = 17$, let $t_{17} = \phi_{17}(q^{1/17})$, t_{17} satisfies following modular equation of degree* 17 *with coefficients in V_{17}*

$$t^{17}_{17} + \sum_{k=1}^{17} (-1)^k \sigma_k t^{17-k} = 0, \tag{3.4}$$

where the σ_k are given in [14].

4. Proofs of Theorem 2.1, Corollaries 2.2, 2.3 and 2.4

4.1. Modular functions associated to $p_{[1^{\lambda_0}\ \ell^{k\lambda_1}]}(n)$

We construct a sequence of modular functions on $\Gamma_0(\ell)$ that are the generating functions for the partition $p_{[1^{\lambda_0}\ \ell^{k\lambda_1}]}(n)$ restricted to certain arithmetic progressions. Let

$$\begin{aligned}
L_{0,\ell}(\tau) &= 1 \\
L_{1,\ell}(\tau) &= U_\ell\left(\phi_\ell(\tau)^{\lambda_0} \prod_{n=1}^{\infty} \frac{\left(1-q^{\ell^k n}\right)^{\lambda_1}}{\left(1-q^{\ell^k n}\right)^{\lambda_1}}\right) \\
&= U_\ell\left(q^{\delta_\ell.\lambda_0} \prod_{n=1}^{\infty} \frac{(1-q^{\ell^2 n})^{\lambda_0}\left(1-q^{\ell^k n}\right)^{\lambda_1}}{(1-q^n)^{\lambda_0}\left(1-q^{\ell^k n}\right)^{\lambda_1}}\right) \\
&= \prod_{n=1}^{\infty}\left(1-q^{\ell n}\right)^{\lambda_0} . \left(1-q^{\ell^{k-1} n}\right)^{\lambda_1} . \sum_{n \geq \mu_{1,\ell}} p_{[1^{\lambda_0}\ \ell^{k\lambda_1}]}(\ell n + n_{1,\ell}) q^n,
\end{aligned}$$

where $\delta_\ell = \frac{\ell^2-1}{24}$.

$$L_{2,\ell}(\tau) = U_\ell(\phi_\ell(\tau)^0 L_{1,\ell}(\tau)).$$

Define

$$L_{r,\ell}(\tau) := U_l\left(\phi_\ell^{\lambda'_{r-1}}(\tau) L_{r-1,\ell}(\tau)\right), \quad \text{if } r \geq 1,$$

where the integers λ'_r are defined for $r < k$ as

$$\lambda'_r = \begin{cases} \lambda_0 & \text{if } r \text{ is even}, \\ 0 & \text{if } r \text{ is odd}, \end{cases} \tag{4.1}$$

$$\lambda'_k = \begin{cases} \lambda_0 + \lambda_1 & \text{if } k \text{ is even,} \\ \lambda_1 & \text{if } k \text{ is odd,} \end{cases} \tag{4.2}$$

$$L_{2r,\ell}(\tau) = \prod_{n=1}^{\infty} (1-q^n)^{\lambda_0} \left(1-q^{\ell^{k-2r}n}\right)^{\lambda_1} \sum_{n \geq \mu_{2r,\ell}} p_{[1^{\lambda_0}\,\ell^{k\lambda_1}]}\left(l^{2r}n + n_{2r,\ell}\right) q^n,$$

$$L_{2r-1,\ell}(\tau) = \prod_{n=1}^{\infty} \left(1-q^{\ell n}\right)^{\lambda_0} \left(1-q^{\ell^{k-2r+1}n}\right)^{\lambda_1} \sum_{n \geq \mu_{r-1,\ell}} p_{[1^{\lambda_0}\,\ell^{k\lambda_1}]}\left(\ell^{2r-1}n + n_{2r-1,\ell}\right) q^n.$$

From the recurrence relation between $L_{2r,\ell}(\tau)$ and $L_{2r-1,\ell}(\tau)$,

$$n_{r,\ell} = -\lambda_0 \frac{\ell^{2\,[(r+1)/2]}-1}{24}, \quad \mu_{r,\ell} = \left\lceil \frac{\lambda_0\left(\ell^{2\left[\frac{r+1}{2}\right]}-1\right)}{24\,\ell^r} \right\rceil, \quad 0 \leq r \leq k.$$

For $r \geq k$, we set:

$$\lambda'_r = \begin{cases} \lambda'_k & \text{if } r \equiv k \pmod 2, \\ \lambda'_{k-1} & \text{if } r \not\equiv k \pmod 2. \end{cases} \tag{4.3}$$

Theorem 4.1. *For $r \geq k$ be a positive integers, we have*

$$L_{r,\ell}(\tau) = \prod_{n=1}^{\infty} (1-q^n)^{\lambda'_r} \left(1-q^{\ell n}\right)^{\lambda'_{r-1}} \sum_{n \geq \mu_{r,\ell}} p_{[1^{\lambda_0}\,\ell^{k\lambda_1}]}\left(\ell^r\, n + n_{r,\ell}\right),$$

where

$$n_{r,\ell} = -\frac{(\ell^2-1)}{24} \sum_{i=0}^{r-1} \lambda'_i\, \ell^i, \quad \mu_{r,\ell} = \left\lceil \frac{(\ell^2-1)}{24\,\ell^r} \sum_{i=0}^{r-1} \lambda'_i\, \ell^i \right\rceil.$$

We write the proof for k even. We have

$$\begin{aligned}
L_{k,\ell}(\tau) &= \prod_{n=1}^{\infty} (1-q^n)^{\lambda_0+\lambda_1} \sum_{n \geq \mu_{k,\ell}} p_{[1^{\lambda_0}\,\ell^{k\lambda_1}]}\left(\ell^k n + n_{k,\ell}\right) q^n, \\
L_{k+1,\ell}(\tau) &= U_\ell\left(\phi_\ell(\tau)^{\lambda_0+\lambda_1} L_{k,\ell}(\tau)\right) \\
&= \prod_{n=1}^{\infty} \left(1-q^{\ell n}\right)^{\lambda_0+\lambda_1} \sum_{n \geq \mu_{k+1,\ell}} p_{[1^{\lambda_0}\,\ell^{k\lambda_1}]}\left(\ell^{k+1}n + n_{k,\ell} - \left(\frac{\ell^2-1}{24}\right) \ell^k\,(\lambda_0+\lambda_1)\right) q^n, \\
L_{k+2,\ell}(\tau) &= U_\ell(L_{k+1,\ell}(\tau)) \\
&= \prod_{n=1}^{\infty} (1-q^n)^{\lambda_0+\lambda_1} \sum_{n \geq \mu_{k+2,\ell}} p_{[1^{\lambda_0}\,\ell^{k\lambda_1}]}\left(l^{k+2}n + n_{k,\ell} - \left(\frac{\ell^2-1}{24}\right) \ell^k(\lambda_0+\lambda_1)\right) q^n
\end{aligned} \tag{4.4}$$

and for any non-negative integer s, we have

$$\begin{cases} L_{k+(2s+1),\ell}(\tau) = U_\ell\left(\phi_\ell(\tau)^{\lambda_0+\lambda_1} L_{k+2s,\ell}(\tau)\right) \\ \quad L_{k+2s,\ell}(\tau) = U_\ell\left(L_{k+(2s-1),\ell}(\tau)\right). \end{cases} \tag{4.5}$$

From (4.4), we obtain

$$L_{r,\ell}(\tau) = \prod_{n=1}^{\infty} (1-q^n)^{\lambda'_r} \left(1-q^{\ell n}\right)^{\lambda'_{r-1}} \sum_{n \geq \mu_{r,\ell}} p_{[1^{\lambda_0}\,\ell^{k\lambda_1}]}\left(\ell^r\, n + n_{r,\ell}\right).$$

Again, from (4.4) and (4.5), we deduce

$$n_{r,\ell} = -\lambda_0 \frac{\ell^k-1}{24} - \ell^k(\lambda_0+\lambda_1) \frac{\ell^{2\,[(r-k+1)/2]}-1}{24}, \quad \mu_{r,\ell} = \left\lceil \frac{(\lambda_0+\lambda_1)\,\ell^{2[(r-k+1)/2]}}{24\,\ell^{r-k}} - \frac{\lambda_0+\ell^k\lambda_1}{24.\ell^r} \right\rceil.$$

Similarly, for k odd we have

$$\begin{aligned}
L_{k,\ell}(\tau) &= \prod_{n=1}^{\infty}(1-q^{\ell n})^{\lambda_0}\,(1-q^n)^{\lambda_1}\sum_{n\geq\mu_{k,\ell}} p_{[1^{\lambda_0}\,\ell^{k\lambda_1}]}(\ell^k n + n_{k,\ell})q^n,\\
L_{k+1,\ell}(\tau) &= U_\ell\left(\phi_\ell(\tau)^{\lambda_1} L_{k,\ell}(\tau)\right)\\
&= \prod_{n=1}^{\infty}(1-q^n)^{\lambda_0}\left(1-q^{\ell n}\right)^{\lambda_1}\sum_{n\geq\mu_{k+1,\ell}} p_{[1^{\lambda_0}\,\ell^{k\lambda_1}]}\left(\ell^{k+1}n + n_{k,\ell} - \left(\frac{\ell^2-1}{24}\right)\lambda_1\,\ell^k\right)q^n,\\
L_{k+2,\ell}(\tau) &= U_\ell\left(\phi_\ell(\tau)^{\lambda_0} L_{k+1,\ell}(\tau)\right)\\
&= \prod_{n=1}^{\infty}(1-q^{\ell n})^{\lambda_0}\,(1-q^n)^{\lambda_1}\sum_{n\geq\mu_{k+2,\ell}} p_{[1^{\lambda_0}\,\ell^{k\lambda_1}]}(\ell^{k+2}n + n_{k,\ell} - \left(\frac{\ell^2-1}{24}\right)\lambda_1\ell^k - \left(\frac{\ell^2-1}{24}\right)\lambda_0\ell^{k+1})q^n \qquad (4.6)
\end{aligned}$$

and for any non-negative integer s, we have

$$\begin{cases} L_{k+(2s+1),\ell}(\tau) = U_\ell\left(\phi_l(\tau)^{\lambda_1} L_{k+2s,\ell}(\tau)\right)\\ \quad L_{k+2s,\ell}(\tau) = U_\ell\left(\phi_\ell(\tau)^{\lambda_0} L_{k+(2s-1),\ell}(\tau)\right). \end{cases} \qquad (4.7)$$

From (4.6), we obtain

$$L_{r,\ell}(\tau) = \prod_{n=1}^{\infty}(1-q^n)^{\lambda'_r}\left(1-q^{\ell n}\right)^{\lambda'_{r-1}}\sum_{n\geq\mu_{r,\ell}} p_{[1^{\lambda_0}\,\ell^{k\lambda_1}]}\left(\ell^r n + n_{r,\ell}\right).$$

Again, from (4.6) and (4.7), we deduce

$$\begin{aligned}
n_{r,\ell} &= -\frac{\lambda_0}{24}(\ell^{k+2[(r-k)/2]+1}-1) - \ell^k\,\lambda_1\frac{\ell^{2\,[(r-k+1)/2]}-1}{24},\\
\mu_{r,\ell} &= \left\lceil \frac{\lambda_0\,\ell^{2[(r-k)/2]+1} + \lambda_1\,\ell^{2[(r-k+1)/2]}}{24\,\ell^{r-k}} - \frac{\lambda_0+\ell^k\lambda_1}{24\,\ell^r}\right\rceil.
\end{aligned}$$

Lemma 4.2. *With the same notations as before, we have*

$$24\,n_{r,\ell} \equiv (\lambda_0 + \ell^k\,\lambda_1) \pmod{\ell^r}.$$

Proof. We have

$$\begin{aligned}
24\,n_{r,\ell} &= \left(\sum_{i=0}^{r-1}\lambda'_i\,\ell^i\right)\left(1-\ell^2\right) = \sum_{i=0}^{r-1}\lambda'_i\,\ell^i - \sum_{i=0}^{r-1}\lambda'_i\,\ell^{i+2}\\
&= \lambda'_0 + \lambda'_1\,\ell + \sum_{2\leq i\leq r-3}\left(\lambda'_i - \lambda'_{i-2}\right)\ell^i - \lambda'_{r-2}\,\ell^r - \lambda'_{r-1}\,\ell^{r+1}. \qquad (4.8)
\end{aligned}$$

From the equations (4.1), (4.2) and (4.3), we get

$$\lambda'_0 = \lambda_0,\ \lambda'_1 = 0,\ \lambda'_i - \lambda'_{i-2} = \begin{cases} \lambda_1, & \text{if } r = k\\ 0, & \text{else.}\end{cases} \qquad (4.9)$$

Therefore, from (4.8) and (4.9), we obtain

$$24\,n_{r,\ell} \equiv \left(\lambda_0 + \ell^k\,\lambda_1\right) \pmod{\ell^r}.$$

□

The constants $\mu_{r,l}$ are given in the following result.

Theorem 4.3. *Let k and r be positive integers, we have*

$$\mu_{r,l}(k) = \left\lceil \lambda_0 \frac{l^{2[(r+1)/2]-r}}{24} \right\rceil + \omega(\lambda_0) \quad \text{if} \quad |\lambda_0| < l^r,\ 0 \le r \le k \tag{4.10}$$

with

$$\omega(\lambda_0) = \begin{cases} 1 & \text{if } \lambda_0 < 0 \text{ and } 24|\lambda_0 \\ 0 & \text{else} \end{cases}$$

and

$$\mu_{r,l} = \begin{cases} \left\lceil \dfrac{(\lambda_0 + \lambda_1) l^{2[(r-k+1)/2]-r+k}}{24} \right\rceil + \omega(\lambda_0, \lambda_1) & \text{if } |\lambda_0 + l^k \lambda_1| < l^r,\ r \ge k,\ k \text{ even} \\ \left\lceil \dfrac{\lambda_0\, l^{2[(r-k)/2]+1-r+k} + \lambda_1\, l^{2[(r-k+1)/2]-r+k}}{24} \right\rceil + \omega(\lambda_0, \lambda_1) & \text{if } |\lambda_0 + l^k \lambda_1| < l^r,\ r \ge k,\ k \text{ odd,} \end{cases} \tag{4.11}$$

where

$$\omega(\lambda_0, \lambda_1) = \begin{cases} 1 & \text{if } \lambda_0 + l^k \lambda_1 < 0 \text{ and } 24|(\lambda_0 + l^k \lambda_1) \\ 0 & \text{else.} \end{cases}$$

4.2. Computation of $\theta_\ell(\lambda, \mu)$

Let f be a modular function on $\Gamma_0(\ell)$ and consider the linear transformation,

$$T_\lambda \,:\, f(\tau) \longrightarrow U_\ell\left(\phi_\ell(\tau)^\lambda f(\tau)\right),$$

where λ is an integer. Let $\left(C^\lambda_{\mu,\nu}\right)_{\mu,\nu}$ be the matrix of the linear transformation T_λ with respect to the basis elements z_ℓ^μ.

Fourier series of $T_\lambda(z_\ell^\mu)$ has all coefficients divisible by ℓ if and only if

$$C^\lambda_{\mu,\nu} \equiv 0 \pmod{\ell} \text{ for all } \nu.$$

Now we define $\theta_\ell(\lambda, \mu)$ by

$$\theta_\ell(\lambda, \mu) = \begin{cases} 1 & \text{if } \ell | U_\ell\left(\phi_\ell^\lambda z_\ell^\mu\right) \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.4. *Let $\ell = 5$, for any integers λ and μ, we have*

$$\theta_5(\lambda, \mu) = \begin{cases} 1 & \textit{if } \mu + \lambda \equiv 1, 2 \pmod{5} \\ 0 & \textit{else.} \end{cases}$$

Lemma 4.5. *Let $\ell = 7$, for any integer λ and μ, we have*

$$\theta_7(\lambda, \mu) = \begin{cases} 1 & \textit{if } \mu + 2\lambda \equiv 1, 2 \pmod{7} \\ 0 & \textit{else.} \end{cases}$$

Lemma 4.6. *Let $\ell = 13$, for any integer λ and μ, we have*

$$\theta_{13}(\lambda, \mu) = \begin{cases} 1 & \textit{if } 2\mu + \lambda \equiv 10 \pmod{13} \\ 0 & \textit{else.} \end{cases}$$

4.2.1. The cas $\ell = 11$

Let V_{11} be the vector space of modular functions on $\Gamma_0(11)$, which are holomorphic everywhere except possibly at 0 and ∞.

In [1] Atkin constructed a basis for V_{11}. Gordon in [15] slightly modified the basis elements and defined

$$\{J_\nu,\ \nu \in \mathbb{Z}\}.$$

For more information about the construction of this basis see [1].

Lemma 4.7 (*cf.* [15]). *For all $\nu \in \mathbb{Z}$, we have*

(i) $J_{\nu+5}(\tau) = J_\nu(\tau) J_5(\tau)$

(ii) $\mathrm{ord}_\infty\, J_\nu(\tau) = \nu$

(iii) $$\mathrm{ord}_0\, J_\nu(\tau) = \begin{cases} \nu & \text{if } \nu \equiv 0 \pmod 5, \\ -\nu - 1 & \text{if } \nu \equiv 1, 2 \text{ or } 3 \pmod 5, \\ -\nu - 2 & \text{if } \nu \equiv 4 \pmod 5. \end{cases}$$

(iv) *The Fourier series of $J_\nu(\tau)$ has integer coefficients, and is of the form $J_\nu(\tau) = q^\nu + \cdots$.*

Let $\left(C^\lambda_{\mu,\nu}\right)_{\mu,\nu}$ be the matrix of the linear transformation T_λ, with respect to the basis elements J_ν

$$U_{11}\left(\phi_{11}(\tau)^\lambda J_\mu(\tau)\right) = \sum_\nu C^\lambda_{\mu,\nu} J_\nu(\tau).$$

In [15] Gordon has proven:

$$\pi_{11}\left(C^\lambda_{\mu,\nu}\right) \geq \left\lceil \frac{11\nu - \mu - 5\lambda - 1}{10} \right\rceil.$$

Lemma 4.8. *For any integers λ and μ, we have*

$$\theta_{11}(\lambda - 11, \mu) = \theta_{11}(\lambda, \mu)$$
$$\theta_{11}(\lambda + 12, \mu - 5) = \theta_{11}(\lambda, \mu), \quad 0 \leq \lambda \leq 10.$$

4.2.2. The case $\ell = 17$

Let V_{17} be the vector space of modular functions on $\Gamma_0(17)$, which are holomorphic everywhere except possibly at 0 and ∞.

Lemma 4.9 (*cf.* [14]). *For all $\nu \in \mathbb{Z}$, we have*

(i) $J_{\nu+4}(\tau) = J_\nu(\tau) J_4(\tau)$

(ii) $\{J_\nu(\tau) | -\infty < \nu < +\infty\}$ *is basis of* V_{17}

(iii) $\mathrm{ord}_\infty\, J_\nu(\tau) = \nu$

(iv) $$\mathrm{ord}_0\, J_\nu(\tau) = \begin{cases} -\nu & \text{if } \nu \equiv 0 \pmod 4, \\ -\nu - 1 & \text{if } \nu \equiv 1 \text{ or } 2 \pmod 4, \\ -\nu - 2 & \text{if } \nu \equiv 3 \pmod 4. \end{cases}$$

(v) *The Fourier series of $J_\nu(\tau)$ has integer coefficients, and is of the form $J_\nu(\tau) = q^\nu + \cdots$.*

Let $\left(C^\lambda_{\mu,\nu}\right)_{\mu,\nu}$ be the matrix of linear transformation T_λ, with respect to the basis elements $\{J_\nu(\tau)\}$.

$$U_{17}\left(\phi_{17}(\tau)^\lambda J_\mu(\tau)\right) = \sum_\nu C^\lambda_{\mu,\nu} J_\nu(\tau).$$

In [14] Hughes proved the following:

$$\pi\left(C^\lambda_{\mu,\nu}\right) \geq \left\lfloor \frac{17\nu - \mu - 12\lambda - 2}{24} \right\rfloor.$$

Lemma 4.10. *For any integers λ and μ, we have*

$$\theta_{17}(\lambda, \mu) = \theta_{17}(\lambda - 17, \mu)$$
$$\theta_{17}(\lambda, \mu) = \theta_{17}(\lambda + 6, \mu - 4), \quad 0 \leq \lambda \leq 16.$$

4.3. Evaluation of $A_{r,\ell}(\lambda_0,\lambda_1)$

For $\ell = 5,\ 7,\ 11,\ 13$ and 17, we define

$$A_{r,\ell} = \sum_{i=0}^{r-1} \theta_\ell(\lambda_i', \mu_{i,\ell}), \quad A_{0,\ell} = 0. \tag{4.12}$$

We have two cases

- For $r < k$, we obtain
$$A_{r,\ell} = \frac{1}{2}\alpha_\ell r + O(\log|\lambda_0|)$$
with
$$\alpha_\ell = \theta_\ell(0, \lceil l\lambda_0/24\rceil + \omega(\lambda_0)) + \theta_\ell(\lambda_0, \lceil \lambda_0/24\rceil + \omega(\lambda_0)).$$
- For $r \geq k$, we obtain
$$A_{r,\ell} = \frac{1}{2}\alpha_\ell r + O\Big(\log|\lambda_0 + \ell^k\lambda_1|\Big)$$
with
$$\alpha_\ell = \theta_\ell\left(\lambda_1, \left\lceil \frac{l(\lambda_0+\lambda_1)}{24}\right\rceil + \omega(\lambda_0,\lambda_1)\right) + \theta_\ell\left(\lambda_0, \left\lceil \frac{\lambda_0+\lambda_1}{24}\right\rceil + \omega(\lambda_0,\lambda_1)\right) \quad k \text{ even}$$

and
$$\alpha_\ell = \theta_\ell\left(\lambda_1, \left\lceil \frac{\ell\lambda_0+\lambda_1}{24}\right\rceil + \omega(\lambda_0,\lambda_1)\right) + \theta_\ell\left(\lambda_0, \left\lceil \frac{\lambda_0+\ell\lambda_1}{24}\right\rceil + \omega(\lambda_0,\lambda_1)\right), \quad k \text{ odd}.$$

Theorem 4.11. *For each $r \geq 0$, for $l = 5,\ 7,\ 11,\ 13,\ 17$ and for any integers $\lambda_0,\ \lambda_1$, we have*

$$\pi_\ell\left(L_{r,\ell}(\tau)\right) \geq A_{r,\ell}(\lambda_0,\lambda_1).$$

Note that α_ℓ, $\ell \in \{5,\ 7,\ 13\}$ depends on $\lambda_0 + \ell^k\lambda_1$ with period 24 and k positive integer. Periodicity follows by the fact that $\alpha_\ell(\lambda_0 + \ell^k\lambda_1)$ is invariant under the maps

$$\lambda_0 \longrightarrow \lambda_0 + 24 - \ell\, d \quad \text{and} \quad \lambda_1 \longrightarrow \lambda_1 + d \quad \text{for each integer } d.$$

For $\ell = 11$, Periodicity follows by the fact that $\alpha_{11}(\lambda_0 + 11^k\lambda_1)$ is invariant under the maps

$$\lambda_0 \longrightarrow \lambda_0 + 120 - 11d \quad \text{and} \quad \lambda_1 \longrightarrow \lambda_1 + d \quad \text{for each integer } d.$$

Also note that when $\ell = 17$, $\alpha_{17}(\lambda_0 + 17^k\lambda_1)$ is invariant under the maps

$$\lambda_0 \longrightarrow \lambda_0 + 96 - 17d \quad \text{and} \quad \lambda_1 \longrightarrow \lambda_1 + d \quad \text{for each integer } d.$$

5. Conclusion and future perspectives

In this work, we study the Fourier coefficients of the following eta-quotients

$$q^{\frac{\lambda_0+l^k\lambda_1}{24}}\eta(\tau)^{-\lambda_0}\eta(l^k\tau)^{-\lambda_1}.$$

We establish Ramanujan-type congruences for the Fourier coefficients of these eta-quotients modulo powers of prime l. Our main result is Theorem 2.1. Interesting corollaries are given. From Corollary 2.2 we obtain Atkin's result [2, 3, Theorem 1]. The Corollary 2.3 generalize Gordon's result [15, Theorem 1.2] to the c-colored partitions, with $c = \lambda_0+11^k\lambda_1$. Corollary 2.4 generalizes Hughes result [14, Theorem 3] to the c-colored partitions, with $c = \lambda_0+17^k\lambda_1$.

The theme of partition theory in relation to modular forms is particularly rich and inspires both active and dynamic current research. We continue to explore this direction in order to lay the groundwork for promising future studies.

It is pertinent to examine the generalizations of the relationships (1.2) and (1.3) established by Hardy-Ramanujan and Rademacher concerning the partition function $p(n)$. More specifically, we aim to investigate the extensions of these relationships in the context of partition functions $p_{[1^{\lambda_0}l^{k\lambda_1}]}(n)$.

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