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Stability analysis of SIR epidemic model under vaccination coverage on vertically transmitted newborns

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Abstract

In this paper, a three-compartment SIR epidemic model under vaccination coverage in both the vertically transmitted and non-transmitted newborns is considered and discussed local stability at both the equilibrium points and global stability at endemic equilibrium point. Also, analytical and numerical approaches were made in support of the results. It is observed that with the increase in the proportion of vaccination in the newborn, the infective populations almost vanish. Also observed that with the increase in the rate of infected newborns there is an increase in the infected individuals and with the increase in the vaccination to the infected newborn, there is an increase in the recovered individuals.

Keywords: Vertical transmission, vaccination coverage rate, infected newborns, disease free equilibrium point, endemic equilibrium point, local and global stability

2020 MSC: 92D25, 92D30, 93Dxx, 93D05

1. Introduction

The numerical representation of a system or an event using the mathematical language is called mathematical model. The technique of developing a mathematical model is called mathematical modeling. It is the process of translating real-world problems into mathematical equations to analyze, understand, and predict behaviour within a system, with applications across various fields like engineering, biology, economics and medicine, allowing researchers to study complex phenomena without manipulating the real system. Essentially, it involves creating a simplified representation of a real-world situation to gain insights and make predictions based on mathematical calculations. Mathematical modeling is applied in various fields like engineering to study behaviour of the system such as Control

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systems, in medicine to study epidemic models to know the causes of transmission of the disease etc. In recent years there is a need for studying epidemic models due to raise of pandemic diseases. Many researchers started working on epidemic models to study and to analyze the transmission dynamics of the disease, ways to contain the disease, etc. In recent times, many researchers studied the behaviour, causes and ways to eradicate the pandemic Covid-19 using the mathematical modeling. An attempt is made in this paper to analyse the SIR epidemic model under specific conditions.

1.1. Background and motivation

Infectious illness dynamics are now the subject of much interdisciplinary research. Various fields such as mathematics, physics, biology, computer science in epidemiology plays significant role to effectively respond to the advancement and enhancement of public health. Within this framework, mathematical modeling seems to have enormous potential for elucidating the intricacy of infectious disease processes. Mathematical models predict how a disease will spread over time by accounting for key elements that influence disease progression, such as rates of transmission and recovery. In the analysis of infectious disease transmission and management, mathematical models have emerged as key instruments. Conventional epidemiological models separate the population into three groups: those who are susceptible, those who are infectious, and those who have recovered. It is commonly known that immunizing more number of people helps stop the spread of numerous infectious diseases.

Vaccination is the practice of immunizing the body against a certain type of virus. Upon encountering the virus and its type, our immune system attempts to fight with them. In order to eradicate an infectious disease, high vaccination rates are required if the basic reproductive rate is large. The word vertical transmission refers to the transfer of a disease or disease causing elements from the infected mother to the child directly either in the womb or during the early stages of birth. HIV, hepatitis-B, syphilis are such diseases that transmit from mothers to their infants.

1.2. Literature review

Many researchers started working on mathematical modeling in epidemiology and studied various epidemic models to analyze the strategies to contain epidemic since decades. Recent studies include analyzing the stability of SIR epidemic model along with the models incorporating delay in various compartments, included incidence rates etc. Dokala et al. [4, 8] studied on stability of various epidemic models at its equilibrium points. Gummala et al. [12, 13] studied stability of SIR epidemic model under vaccination coverage to the newborns and delay in the interaction of susceptible and infected individuals. Dokala et al. [5]-[7] started to work on SIRI, SIRS epidemic models to study stability in them. Emandi et al. [9, 10] discussed stability on SIRI epidemic model with reintroduced susceptible i.e., $\mathrm{SIR}^{\nearrow \mathrm{I}}_{\searrow \mathrm{S}}$. Naji and Hussien [21] studied the "Dynamics of epidemic model with two types of infectious diseases and vertical transmission", and several researchers are still working on vertical transmission and its consequences (*cf.* [17, 22]).

1.3. Objectives of the study

In this paper, a three-compartment SIR model (*cf.* [19]) is considered and incorporated the vaccination to the newborns that include vertical transmission (*cf.* [21]). Here, the fraction of newborns that are vaccinated immediately after birth is denoted by 'p'. Then the recruitment of susceptible population changes from v to $(1-p)v$ into the susceptible compartment while pv is the population entered into the removable compartment. The infected newborn individuals enter into the infected compartment with the rate q and fraction of infected newborns that are vaccinated immediately after birth is denoted by 'k' The reproduction rate R_0, disease free equilibrium point, endemic equilibrium point are identified, and discussed the local stability at the two equilibrium points. Also, a suitable Lyapunov function is assumed to describe global stability. Finally, numerical simulation is done to support the analysis and to identify the parameters that are noteworthy in the spread of the disease/dynamics.

In Section 2, we formulated the governing equations for the SIR epidemic model under vaccination coverage to vertically transmitted and non-transmitted newborns by considering the required assumptions. Also described the flowchart, the parameters and the variables used in formulating the equations, model equations. Existence and uniqueness of solutions of the governing equations is also detailed. In Section 3, equilibrium points were identified

and stability analysis at the two equilibrium points is studied. The behaviour of the system and its stability is studied by numerical simulations for various parametric values using MATLAB in Section 4. Section 5 provides the results and conclusions drawn from the paper.

2. Mathematical formulation of SIR model under vaccination coverage to the vertically transmitted and non-transmitted newborns

The SIR model is a fundamental model in epidemiology that is often used to predict and understand the dynamics of the transmission of infectious diseases within a population. The model which was developed by Kermack and McKendrick in 1927, divides the population into three classes: Susceptible (S) – those who are at risk of getting the disease. Infective (I) – those who caught the disease and are capable of spreading the disease, and Recovered (R) – those who have recovered from infection/disease due to immunity caused by either treatment or vaccination. (*cf.* [16]).

In this section, various steps which are followed to build mathematical equations of SIR epidemic model under vaccination coverage on vertically transmitted and non-transmitted newborns were detailed. These steps include model assumptions, model diagram or flowchart, description of model variables and parameters, model equations and existence and uniqueness of solution.

2.1. Model assumptions

Here, the following assumptions were made to develop the model with vaccination on vertically transmitted and non-transmitted newborns.

- **(a)** The total population is divided into three compartments susceptible (S), infectious (I), and removable/ recovered (R).
- **(b)** The total population N is constant with respect to births and deaths over time, or to say, the sum of all the individuals in the three compartments do not change.
- **(c)** Humans are recruited into susceptible class with birth rate.
- **(d)** Humans of all classes will die with death rate due to unrelated background mortality.
- **(e)** Susceptible humans if interacted with infected humans will become infected and will go to infected class.
- **(f)** Newborn susceptible humans if vaccinated will go to recovered/removable class.
- **(g)** Some exposed humans having sufficient natural immunity will recover from the infection naturally and will go to recovered class.
- **(h)** Humans of infected class will die with disease induced death rate.
- **(i)** Infected newborn due to vertical transmission will go to infected class due to infection.
- **(j)** Infected newborn if vaccinated will go to recovered class.

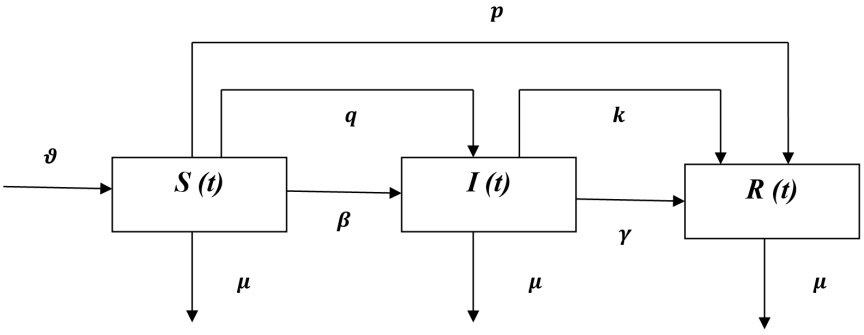

Figure 1. Flowchart showing flow of humans among model compartments

2.2. Model diagram

Based on the model assumptions listed in Section 2.1, the model diagram or model flowchart is drawn as shown in Figure 1. This flow chart describes the flow of humans among the model compartments.

The model variables and parameters description is detailed in the Table 1 and Table 2.

Variable	Description
$S(t)$	Size of susceptible population at time t
$I(t)$	Size of infected population at time t
$R(t)$	Size of recovered population at time t

Table 1. Description of model variables

Parameter	Description
v	Rate of individuals joining susceptible class or birth rate
β	Rate of transmission of infection or infection rate
γ	Rate at which infected individuals recover or recovery rate
μ	Rate of death which is the loss of the individuals or death rate
p	Proportion of vaccination on newborn or vaccination rate
q	Rate of vertically transmitted individuals (infected newborn)
k	Proportion of vaccination given to vertically transmitted individuals

Table 2. Description of model parameters

2.3. Model equations

The basic governing equations of the model are taken from the Chapter "Mathematical models in Infectious disease Epidemiology", taken from the textbook "Modern Infectious disease Epidemiology", by Mirjam Kretzschmar, Jacco Wallinga (*cf.* [18, 19]) in which the governing equations of a three compartment SIR epidemic model under vaccination coverage is given but the stability of the system is not studied. We made an attempt to study the stability of the system for the model (*cf.* [13]) and in this paper we introduced vertical transmission and vaccination is given to both vertically transmitted and non-transmitted newborns to study stability analysis. Based on the model assumptions, model flow chart and description of model variables and parameters, the system of equations of the model are given in (2.1). The system is a group of three nonlinear ordinary differential equations. The system of equations of the SIR model after inducing vaccination to vertically transmitted and non-transmitted newborns is,

$$\begin{aligned} \frac{dS}{dt} &= v - pv - \beta\frac{SI}{N} - \mu S - qI, \\ \frac{dI}{dt} &= \beta\frac{SI}{N} - \gamma I - \mu I + qI - kI, \\ \frac{dR}{dt} &= \gamma I - \mu R + vp + kI \quad for\ 0 \le p \le 1. \end{aligned} \tag{2.1}$$

The equation representing the rate of change in the susceptible population is obtained as v represents the birth rate, introducing new individuals into the susceptible population, the term pv represents the fraction of newborns that are vaccinated soon after birth and thus move directly to the recovered compartment, β being the transmission rate, the term $\beta\frac{SI}{N}$ models the rate at which susceptible individuals become infected through contact with infective individuals, μS represents the natural death of susceptible individuals and qI indicates the newborn individuals who are infected by birth move directly to the infected compartment.

The equation representing the rate of change in the infected population is due to $\beta\frac{SI}{N}$, the rate at which susceptible individuals become infected, γI, the infective individuals recover and move to the recovered compartment and μI, the natural death of infective individuals, qI, being the newborn infected individuals and kI, is the vaccinated infected newborn individuals moving to removable compartment.

The equation representing the rate of change in the removed class is because of γI, the individuals recover from infection, μR, the natural death of recovered individuals and pv, the fraction of newborns that are vaccinated and thus enter directly into the recovered compartment, kI, vaccinated infected newborn individuals moving to the removable compartment.

2.4. Existence and uniqueness of solutions

Theorem 2.1. *Let F be a continuously differentiable function satisfying the Lipschitz condition. Then there exists a unique solution* $(S(t), I(t), R(t))$ *for the system of equations* (2.1).

Proof. Let $F(S, I, R) = [f_1, f_2, f_3]$ represents the system of equations (2.1) where,

$$f_1 = v - pv - \frac{\beta S I}{N} - \mu S - qI,$$

$$f_2 = \frac{\beta S I}{N} - \gamma I - \mu I + qI - kI,$$

$$f_3 = \gamma I - \mu R + vp + kI.$$

The Jacobian J of F is given by,

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial S} & \frac{\partial f_1}{\partial I} & \frac{\partial f_1}{\partial R} \\ \frac{\partial f_2}{\partial S} & \frac{\partial f_2}{\partial I} & \frac{\partial f_2}{\partial R} \\ \frac{\partial f_3}{\partial S} & \frac{\partial f_3}{\partial I} & \frac{\partial f_3}{\partial R} \end{bmatrix},$$

$$J = \begin{bmatrix} \beta\frac{I}{N} - \mu & -\beta\frac{S}{N} - q & 0 \\ \beta\frac{I}{N} & \beta\frac{S^*}{N} - (\mu + \gamma + k - q) & 0 \\ 0 & \gamma + k & -\mu \end{bmatrix}.$$

Since all the partial derivatives exists and they are finite, J is continuous and bounded in the domain. And hence the Lipshcitz condition is satisfied.

Therefore, the solution of the system exists and is unique. □

3. Equilibrium points and stability analysis

In this section, equilibrium points are derived from the system of equations, where the disease-free equilibrium point (DFE) occurs when there are no infective individuals in the population. Existence of disease-free equilibrium (DFE) and endemic equilibrium (EE) points corroborate the state of eradication of the disease and the state where the disease persists at a constant level respectively. Hence the two points are determined to study the stability of the system which helps to predict whether the disease will spread rapidly or remain under control. By analyzing the stability of the system at equilibrium points one can evaluate the effectiveness of control measures such as vaccination, quarantine etc, and basic reproduction rate is the number of new infections that an each infected individual generates on an average which is denoted by R_0.

3.1. Steady state/Equilibrium points

Here, disease-free equilibrium point, and endemic equilibrium point are identified by solving the equations in (2.1) after equating them to zero individually that is,

$$\frac{dS}{dt} = 0, \frac{dI}{dt} = 0, \frac{dR}{dt} = 0.$$

The equilibrium points thus obtained are,
E_1 : Disease-free equilibrium point:

$$E_1(S^*, I^*, R^*) = \left(\frac{v - pv}{\mu}, 0, \frac{vp}{\mu}\right). \tag{3.1}$$

E_2 : Endemic equilibrium point:

$$E_2(S^*, I^*, R^*) = \left(\frac{(\mu + \gamma + k - q)N}{\beta}, \frac{N(v - pv - \mu S^*)}{(\beta S^* + Nq)}, \frac{(k + \gamma)I^* + vp}{\mu}\right). \tag{3.2}$$

Here the basic reproduction rate is,

$$R_0 = \frac{\beta}{\mu + \gamma}$$

and the growth of population is,

$$N^* = \frac{v}{\mu}.$$

It is known that, if

$$R_0 = \frac{\beta}{\mu + \gamma} < 1,$$

the disease cannot spread in the population and therefore only the susceptible (S) population remains, and if

$$R_0 = \frac{\beta}{\mu + \gamma} > 1,$$

the infection increases and an endemic equilibrium point is resulted. With the increased basic reproductive number, high levels of vaccination are to be provided to eradicate epidemic.

3.2. Local stability at disease free equilibrium (DFE) point

Theorem 3.1. *At DFE point $E_1(S^*, I^*, R^*)$, the system* (2.1) *becomes locally asymptotically stable provided,*

$$S^* < (\mu + \gamma + k - q)\frac{N}{\beta}.$$

Proof. Let, the system of equations (2.1) be represented as

$$f_1 = v - pv - \frac{\beta S I}{N} - \mu S - qI,$$

$$f_2 = \frac{\beta S I}{N} - \gamma I - \mu I + qI - kI,$$

$$f_3 = \gamma I - \mu R + vp + kI.$$

The Jacobian matrix of the system of equations is given by,

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial S} & \frac{\partial f_1}{\partial I} & \frac{\partial f_1}{\partial R} \\ \frac{\partial f_2}{\partial S} & \frac{\partial f_2}{\partial I} & \frac{\partial f_2}{\partial R} \\ \frac{\partial f_3}{\partial S} & \frac{\partial f_3}{\partial I} & \frac{\partial f_3}{\partial R} \end{bmatrix},$$

$$J = \begin{bmatrix} \beta\frac{I}{N} - \mu & -\beta\frac{S}{N} - q & 0 \\ \beta\frac{I}{N} & \beta\frac{S^*}{N} - (\mu + \gamma + k - q) & 0 \\ 0 & \gamma + k & -\mu \end{bmatrix}.$$

At the DFE point $E_1(S^*, I^*, R^*)$ Jacobian J is given by,

$$J = \begin{bmatrix} -\mu & -\beta\frac{S^*}{N} - q & 0 \\ 0 & \beta\frac{S^*}{N} - (\mu+\gamma+k-q) & 0 \\ 0 & \gamma+k & -\mu \end{bmatrix}. \tag{3.3}$$

The characteristic equation of (3.3) is given by

$$|J - \lambda I| = 0,$$

where λ is a parameter.

$$(\mu+\lambda)^2\left[\beta\frac{S^*}{N} - (\mu+\gamma+k-q) - \lambda\right] = 0.$$

The roots of the characteristic equation are

$$\lambda_1 = -\mu, \lambda_2 = -\mu, \lambda_3 = \beta\frac{S^*}{N} - (\mu+\gamma+k-q).$$

Here the roots λ_1, λ_2 are negative and the root λ_3 is negative if,

$$S^* < (\mu+\gamma+k-q)\frac{N}{\beta}.$$

Thus at DFE point $E_1(S^*, I^*, R^*)$, (2.1) locally asymptotically stable if

$$S^* < (\mu+\gamma+k-q)\frac{N}{\beta}.$$

□

3.3. Local stability at endemic equilibrium (EE) point

Theorem 3.2. *At EE point $E_2(S^*, I^*, R^*)$, the system* (2.1) *is locally asymptotically stable provided $I^* > S^*$.*

Proof. Let, the system of equations (2.1) be represented as

$$f_1 = v - pv - \frac{\beta S I}{N} - \mu S - qI,$$

$$f_2 = \frac{\beta S I}{N} - \gamma I - \mu I + qI - kI,$$

$$f_3 = \gamma I - \mu R + vp + kI.$$

The Jacobian matrix of the system of equations is given by,

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial S} & \frac{\partial f_1}{\partial I} & \frac{\partial f_1}{\partial R} \\ \frac{\partial f_2}{\partial S} & \frac{\partial f_2}{\partial I} & \frac{\partial f_2}{\partial R} \\ \frac{\partial f_3}{\partial S} & \frac{\partial f_3}{\partial I} & \frac{\partial f_3}{\partial R} \end{bmatrix},$$

$$J = \begin{bmatrix} \beta\frac{I}{N} - \mu & -\beta\frac{S}{N} - q & 0 \\ \beta\frac{I}{N} & \beta\frac{S}{N} - (\mu+\gamma+k-q) & 0 \\ 0 & \gamma+k & -\mu \end{bmatrix}.$$

The Jacobian matrix of (2.1) at EE point $E_2(S^*, I^*, R^*)$ is

$$J = \begin{bmatrix} -\beta\frac{I^*}{N} - \mu & -\beta\frac{S^*}{N} - q & 0 \\ \beta\frac{I^*}{N} & \beta\frac{S^*}{N} - (\mu + \gamma + k - q) & 0 \\ 0 & \gamma + k & -\mu \end{bmatrix}. \tag{3.4}$$

The characteristic equation (3.4) is given by

$$|J - \lambda I| = 0,$$

where λ is a parameter and is,

$$(\mu + \lambda)\left[\lambda^2 - \lambda\left(\frac{\beta S^*}{N} - \frac{\beta I^*}{N} - \mu - \gamma - k + q\right) - \left(\frac{\beta S^*\mu}{N} - \frac{\beta I^*\mu}{N} - \frac{\beta I^*\gamma}{N} - \frac{\beta I^*k}{N} + \mu q - \mu k - \mu^2 - \mu\gamma\right)\right] = 0 \tag{3.5}$$

which implies

$$(\lambda + \mu) = 0$$

and

$$\lambda^2 - \lambda\left(\frac{\beta S^*}{N} - \frac{\beta I^*}{N} - \mu - \gamma - k + q\right) - \left(\frac{\beta S^*\mu}{N} - \frac{\beta I^*\mu}{N} - \frac{\beta I^*\gamma}{N} - \frac{\beta I^*k}{N} + \mu q - \mu k - \mu^2 - \mu\gamma\right) = 0.$$

From equation (3.5) we have

$$\lambda = -\mu$$

and

$$\lambda^2 + \lambda\left(\frac{\beta}{N}(I^* - S^*) + \mu + \gamma + k - q\right) + \left(\frac{\beta\mu}{N}(I^* - S^*) + \frac{\beta I^*(\gamma + k)}{N} + \mu(\mu + \gamma + k - q)\right) = 0. \tag{3.6}$$

One of the roots of the equation (3.5) is negative and two of the roots of the equation (3.6) are negative if sum of the roots of (3.6) (trace of the matrix) is negative and product of the roots of (3.6) (determinant of the matrix) is positive. Hence, the trace

$$-\left(\frac{\beta}{N}(I^* - S^*) + \mu + \gamma + k - q\right) \text{ is negative if } I^* > S^*$$

and the determinant,

$$\left(\frac{\beta\mu}{N}(I^* - S^*) + \frac{\beta I^*(\gamma + k)}{N} + \mu(\mu + \gamma + k - q)\right) \text{ is positive if } I^* > S^*.$$

Therefore, at EE point $E_2(S^*, I^*, R^*)$, (2.1) is locally asymptotically stable provided,

$$I^* > S^*.$$

□

3.4. Global stability at endemic equilibrium (EE) point

Theorem 3.3. *At EE point $E_2(S^*, I^*, R^*)$,* (2.1) *is globally asymptotically stable.*

Proof. Let the Lyapunov function be,

$$U(t) = (S - S^*)^2 + (I - I^*)^2 + (R - R^*)^2. \tag{3.7}$$

From equation (2.1) and derivative of (3.7) we get,

$$U'(t) = 2(S - S^*)\frac{dS}{dt} + 2(I - I^*)\frac{dI}{dt} + 2(R - R^*)\frac{dR}{dt}.$$

$$U'(t) = 2(S - S^*)\left[v - pv - \frac{\beta S I}{N} - \mu S - qI\right] + 2(I - I^*)\left[\frac{\beta S I}{N} - \gamma I - \mu I + qI - kI\right] + 2(R - R^*)[\gamma I - \mu R + vp + kI]. \tag{3.8}$$

By proper choice of

$$(1-p)v - qI - \beta\frac{SI}{N} = \mu S^*,$$

$$\beta\frac{SI}{N} = (\mu+\gamma+k-q)I^*$$

and

$$\gamma I + pv + kI = \mu R^*.$$

Obtained from equation (3.8)

$$U^{'}(t) = 2(S-S^*)\mu(S^*-S) + 2(I-I^*)(\mu+\gamma+k-q)(I^*-I) + 2(R-R^*)\mu(R^*-R),$$

$$U^{'}(t) = -2(S-S^*)^2\mu - 2(I-I^*)^2(\mu+\gamma+k-q) - 2(R-R^*)^2\mu < 0.$$

Hence, at its EE point (2.1) is globally asymptotically stable. □

4. Numerical simulation

Numerical simulations are performed to authenticate the analytical results. Using software tools like MATLAB, the model equations are simulated with different values of β, γ, p, q, k to observe the impact of vaccination on vertical transmitted and non-transmitted newborns on the spread of the disease. The simulations aim to observe changes in the susceptible (S), infective (I), and removable (R) populations in response to variations in specific parameters, while keeping other parameters constant. The primary focus is on examining how transmission rate (β), recovery rate (γ), vaccination rate (p), infected newborn rate (q), vaccination to infected newborn (k) influence the dynamics of the populations. Total of fifteen examples (labeled as 4.1 to 4.15) are considered to study these effects under vaccination coverage and vertical transmission. Each example consists of two types of graphical representations, Time series responses and Phase portraits. These plots show how the populations susceptible, infected, and recovered individuals changes over time and phase portraits provide a phase-space representation of the dynamics. Time series responses and phase portraits help to visualize the stability and convergence behavior of the model. For all the examples, S, I, R values are fixed and considered $S = 50$, $I = 30$, $R = 20$ to observe the change in population by varying one at a time of β, γ, p, q, k and keeping remaining parameters fixed.

Example 4.1. For $v = 10, \beta = 6, \gamma = 0.6, \mu = 0.1, p = 0.1, q = 0.4, k = 0.15$, the time series response and phase portraits obtained through simulation, which reveal the equilibrium point, are shown in Figures 2 and 3.

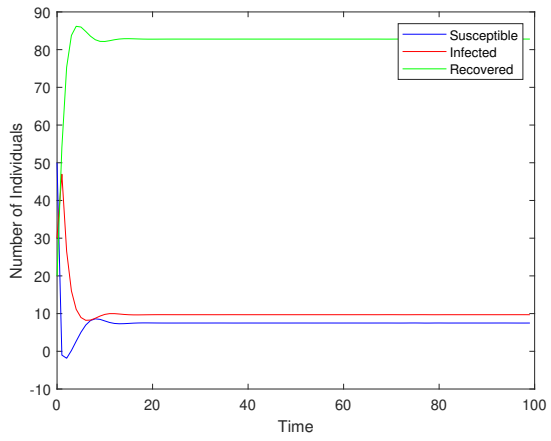

Figure 2. From the graph, it is evident that, the system oscillates asymptotically till it converges to equilibrium point (8,10,82) and exhibits stability

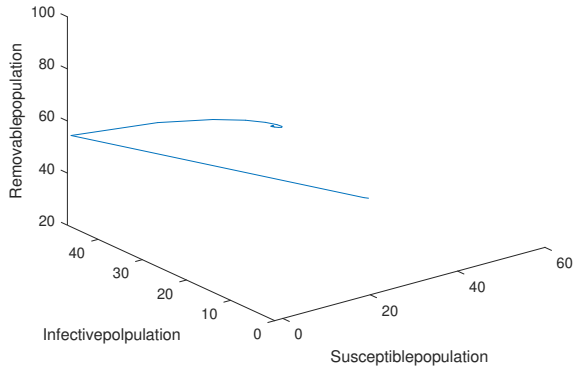

Figure 3. From the phase portrait, it is evident that, the system approaches to equilibrium point and exhibits stability

Figure 2 and Figure 3 show asymptotic behavior of the system of Example 4.1.

Example 4.2. For $v = 10, \beta = 7, \gamma = 0.6, \mu = 0.1, p = 0.1, q = 0.4, k = 0.15$, the time series response and phase portraits obtained through simulation, which reveal the equilibrium point, are shown in Figures 4 and 5.

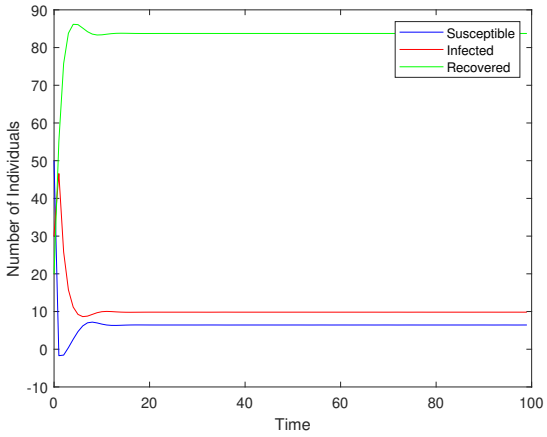

Figure 4. From the graph, it is evident that, the system oscillates asymptotically till it converges to equilibrium point (6,11,83) and exhibits stability

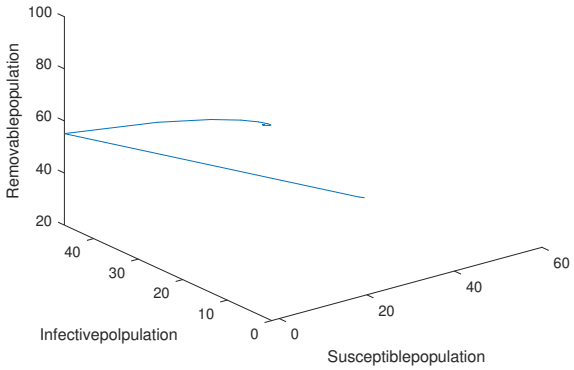

Figure 5. From the phase portrait, it is evident that, the system approaches to equilibrium point and exhibits stability

Figure 4 and Figure 5 show asymptotic behavior of the system of Example 4.2.

Example 4.3. For $v = 10, \beta = 8, \gamma = 0.6, \mu = 0.1, p = 0.1, q = 0.4, k = 0.15$, the time series response and phase portraits obtained through simulation, which reveal the equilibrium point, are shown in Figures 6 and 7.

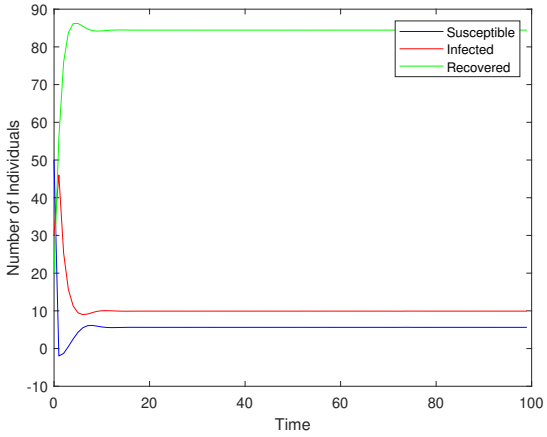

Figure 6. From the graph, it is evident that, the system oscillates asymptotically till it converges to equilibrium point (4,12,84) and exhibits stability

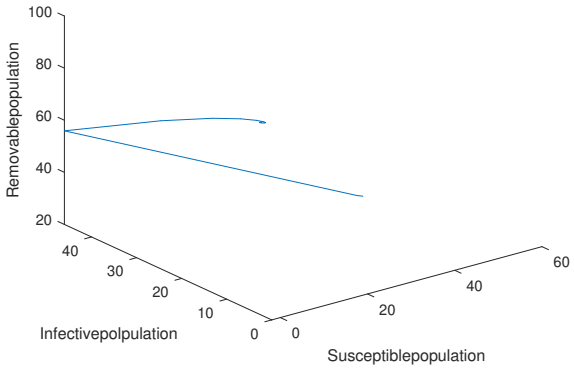

Figure 7. From the phase portrait, it is evident that, the system approaches to equilibrium point and exhibits stability

Figure 6 and Figure 7 show asymptotic behavior of the system of Example 4.3.

From Examples 4.1 to 4.3, it is observed that, there is a decrease in susceptible population and growth in infective population when there is an increase in the transmission rate (β) and remaining parameters fixed constant.

Example 4.4. For $v = 10, \beta = 6, \gamma = 0.35, \mu = 0.1, p = 0.1, q = 0.4, k = 0.15$, the time series response and phase portraits obtained through simulation, which reveal the equilibrium point, are shown in Figures 8 and 9.

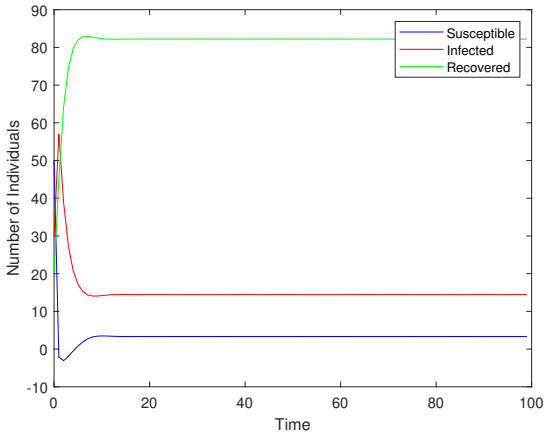

Figure 8. From the graph, it is evident that, the system oscillates asymptotically till it converges to equilibrium point (3,15,82) and exhibits stability

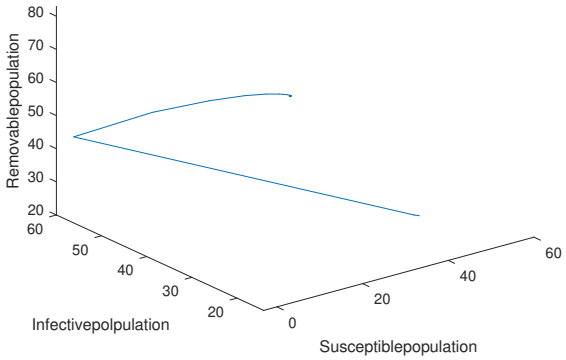

Figure 9. From the phase portrait, it is evident that, the system approaches to equilibrium point and exhibits stability

Figure 8 and Figure 9 show asymptotic behavior of the system of Example 4.4.

Example 4.5. For $v = 10, \beta = 6, \gamma = 0.45, \mu = 0.1, p = 0.1, q = 0.4, k = 0.15$, the time series response and phase portraits obtained through simulation, which reveal the equilibrium point, are shown in Figures 10 and 11.

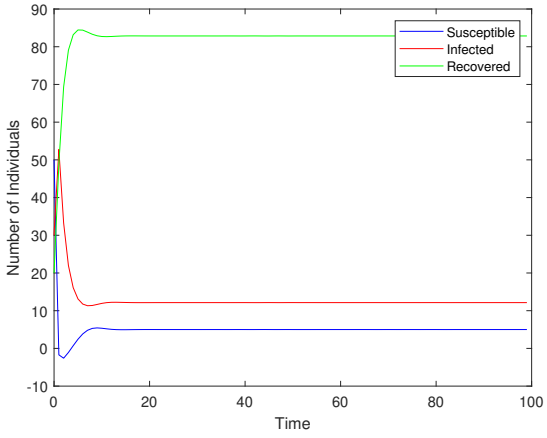

Figure 10. From the graph, it is evident that, the system oscillates asymptotically till it converges to equilibrium point (5,12,83) and exhibits stability

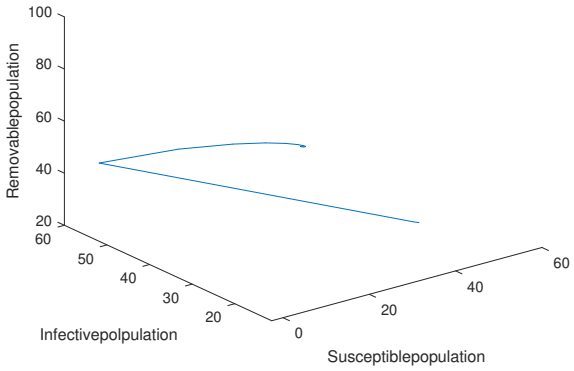

Figure 11. From the phase portrait, it is evident that, the system approaches to equilibrium point and exhibits stability

Figure 10 and Figure 11 show asymptotic behavior of the system of Example 4.5.

Example 4.6. For $v = 10, \beta = 6, \gamma = 0.55, \mu = 0.1, p = 0.1, q = 0.4, k = 0.15$, the time series response and phase portraits obtained through simulation, which reveal the equilibrium point, are shown in Figures 12 and 13.

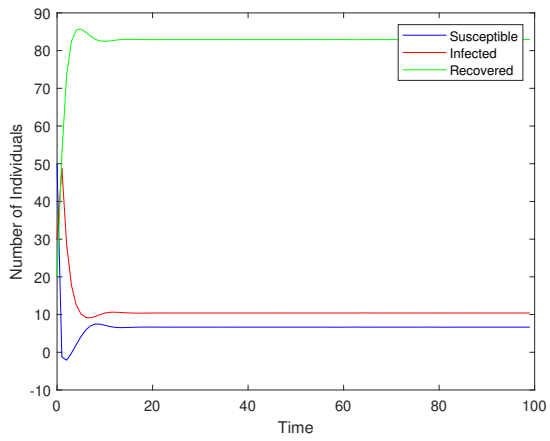

Figure 12. From the graph, it is evident that, the system oscillates asymptotically till it converges to equilibrium point (6,10,84) and exhibits stability

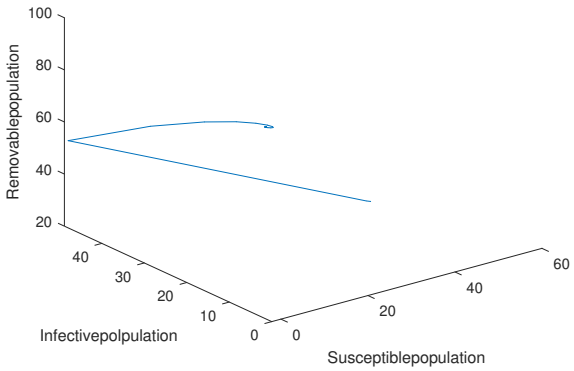

Figure 13. From the phase portrait, it is evident that, the system approaches to equilibrium point and exhibits stability

Figure 12 and Figure 13 show asymptotic behavior of the system of Example 4.6.

Examples 4.4 to 4.6, illustrates that, there is a growth in susceptible population when there is an increase in the recovery rate of infected individuals (γ) and the remaining parameters are fixed constant.

Example 4.7. For $v = 10, \beta = 6, \gamma = 0.55, \mu = 0.1, p = 0.3, q = 0.4, k = 0.15$, the time series response and phase portraits obtained through simulation, which reveal the equilibrium point, are shown in Figures 14 and 15.

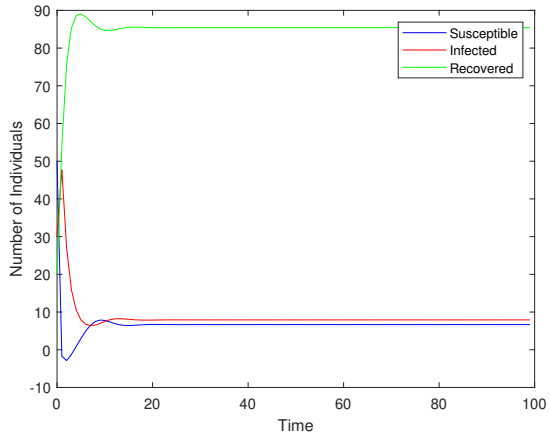

Figure 14. From the graph, it is evident that, the system oscillates asymptotically till it converges to equilibrium point (07,08,85) and exhibits stability

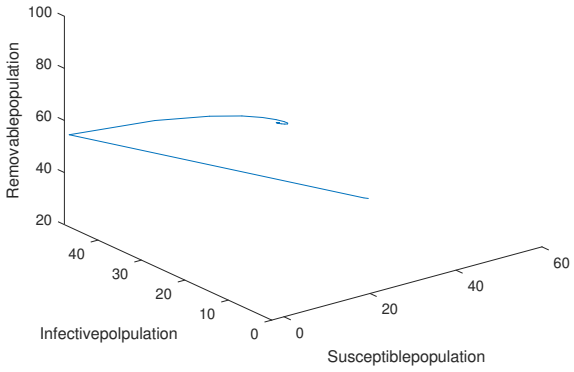

Figure 15. From the phase portrait, it is evident that, the system approaches to equilibrium point and exhibits stability

Figure 14 and Figure 15 show asymptotic behavior of the system of Example 4.7.

Example 4.8. For $v = 10, \beta = 6, \gamma = 0.55, \mu = 0.1, p = 0.6, q = 0.4, k = 0.15$, the time series response and phase portraits obtained through simulation, which reveal the equilibrium point, are shown in Figures 16 and 17.

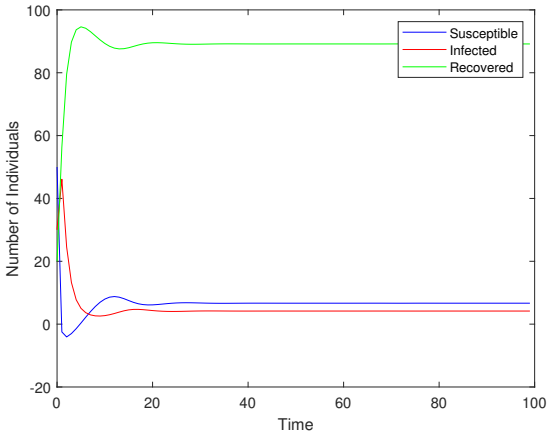

Figure 16. From the graph, it is evident that, the system oscillates asymptotically till it converges to equilibrium point (07,04,89) and exhibits stability

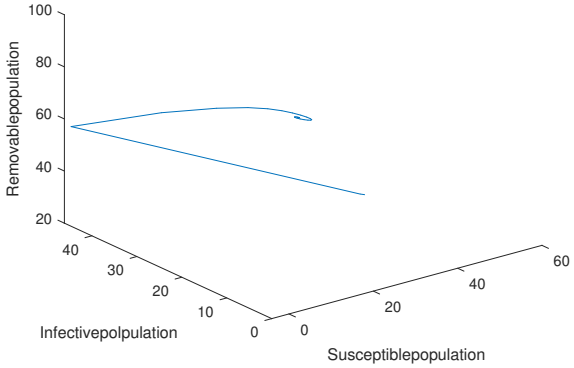

Figure 17. From the phase portrait, it is evident that, the system approaches to equilibrium point and exhibits stability

Figure 16 and Figure 17 show asymptotic behavior of the system of Example 4.8.

Example 4.9. For $v = 10, \beta = 6, \gamma = 0.55, \mu = 0.1, p = 0.9, q = 0.4, k = 0.15$, the time series response and phase portraits obtained through simulation, which reveal the equilibrium point, are shown in Figures 18 and 19.

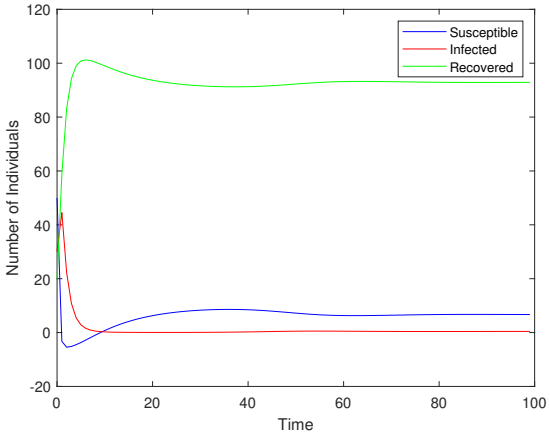

Figure 18. From the graph, it is evident that, the system oscillates asymptotically till it converges to equilibrium point (07,01,92) and exhibits stability

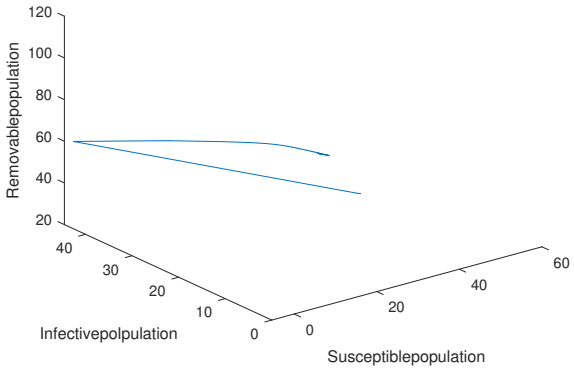

Figure 19. From the phase portrait, it is evident that, the system approaches to equilibrium point and exhibits stability

Figure 18 and Figure 19 show asymptotic behavior of the system of Example 4.9.

From Examples 4.7 to 4.9, it is observed that, the infective population almost approaches to the least with the increase in the proportion of vaccination of newborn (p) and all the other parameters fixed constant.

Example 4.10. For $v = 10, \beta = 6, \gamma = 0.55, \mu = 0.1, p = 0.2, q = 0.1, k = 0.15$, the time series response and phase portraits obtained through simulation, which reveal the equilibrium point, are shown in Figures 20 and 21.

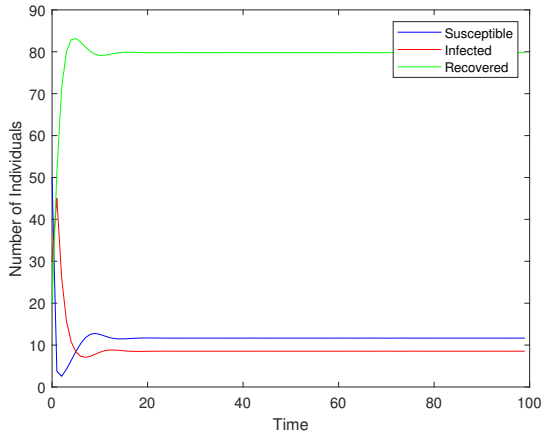

Figure 20. From the graph, it is evident that, the system oscillates asymptotically till it converges to equilibrium point (12,08,80) and exhibits stability

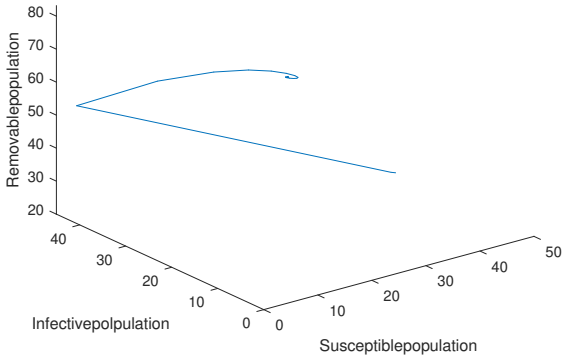

Figure 21. From the phase portrait, it is evident that, the system approaches to equilibrium point and exhibits stability

Figure 20 and Figure 21 show asymptotic behavior of the system of Example 4.10.

Example 4.11. For $v = 10, \beta = 6, \gamma = 0.55, \mu = 0.1, p = 0.2, q = 0.3, k = 0.15$, the time series response and phase portraits obtained through simulation, which reveal the equilibrium point, are shown in Figures 22 and 23.

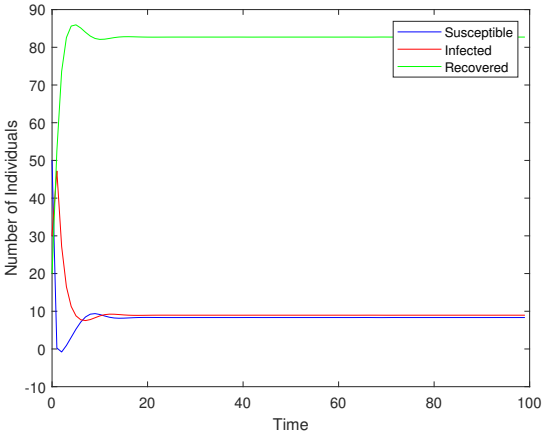

Figure 22. From the graph, it is evident that, the system oscillates asymptotically till it converges to equilibrium point (08,09,83) and exhibits stability.

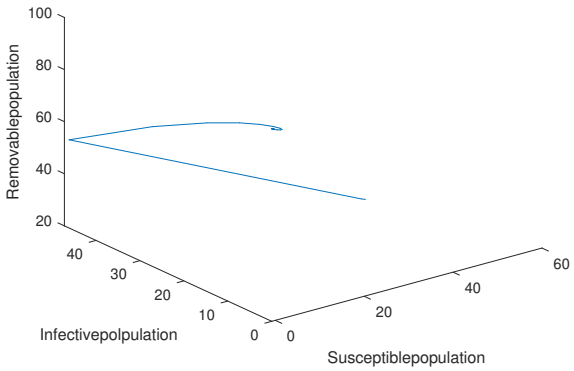

Figure 23. From the phase portrait, it is evident that, the system approaches to equilibrium point and exhibits stability.

Figure 22 and Figure 23 show asymptotic behavior of the system of Example 4.11.

Example 4.12. For $v = 10, \beta = 6, \gamma = 0.55, \mu = 0.1, p = 0.2, q = 0.6, k = 0.15$, the time series response and phase portraits obtained through simulation, which reveal the equilibrium point, are shown in Figures 24 and 25.

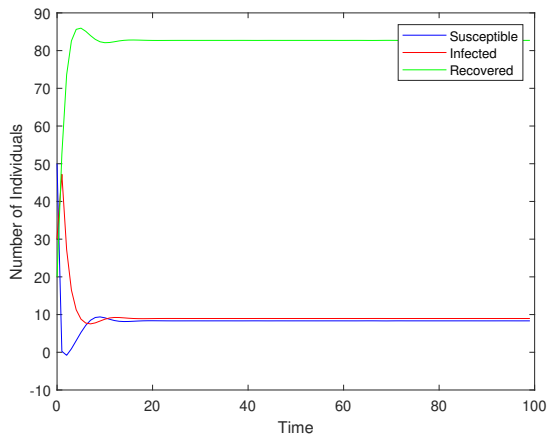

Figure 24. From the graph, it is evident that, the system oscillates asymptotically till it converges to equilibrium point (04,10,86) and exhibits stability

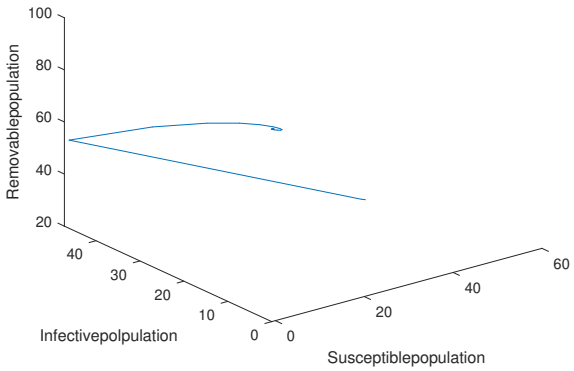

Figure 25. From the phase portrait, it is evident that, the system approaches to equilibrium point and exhibits stability

Figure 24 and Figure 25 show asymptotic behavior of the system of Example 4.12.

From Examples 4.10 to 4.12, it is observed that, the infective population increases with the increase in the rate of infected newborn (q) and all the other parameters fixed.

Example 4.13. For $v = 10, \beta = 6, \gamma = 0.55, \mu = 0.1, p = 0.2, q = 0.3, k = 0.1$, the time series response and phase portraits obtained through simulation, which reveal the equilibrium point, are shown in Figures 26 and 27.

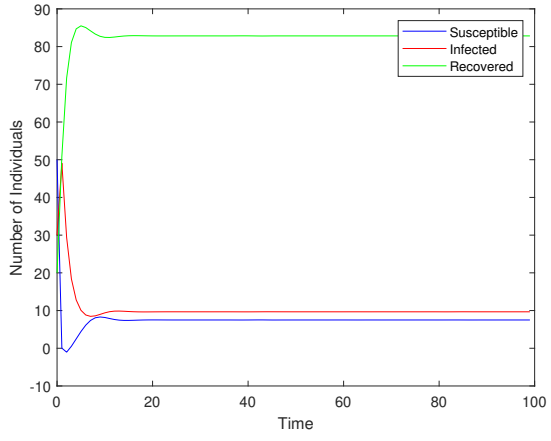

Figure 26. From the graph, it is evident that, the system oscillates asymptotically till it converges to equilibrium point (09,10,81) and exhibits stability

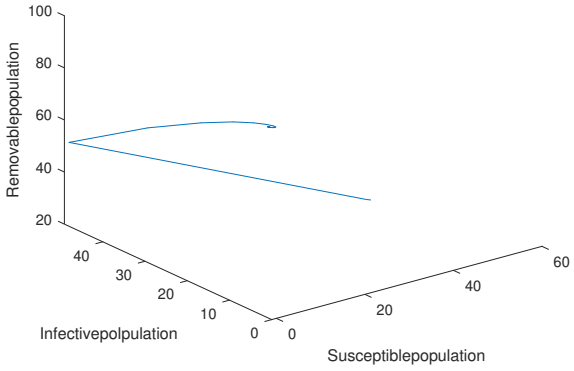

Figure 27. From the phase portrait, it is evident that, the system approaches to equilibrium point and exhibits stability

Figure 26 and Figure 27 show asymptotic behavior of the system of Example 4.13.

Example 4.14. For $v = 10$, $\beta = 6$, $\gamma = 0.55$, $\mu = 0.1$, $p = 0.2$, $q = 0.3$, $k = 0.18$, the time series response and phase portraits obtained through simulation, which reveal the equilibrium point, are shown in Figures 28 and 29.

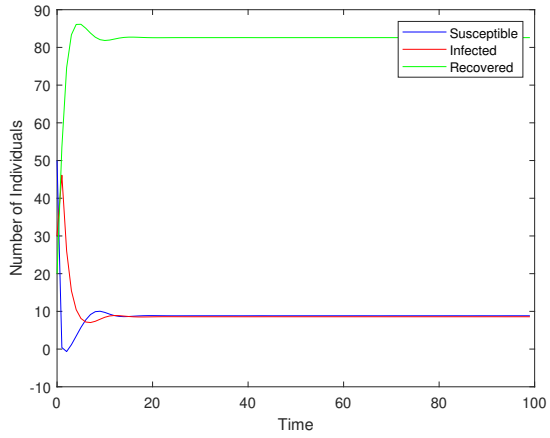

Figure 28. From the graph, it is evident that, the system oscillates asymptotically till it converges to equilibrium point (09,09,82) and exhibits stability

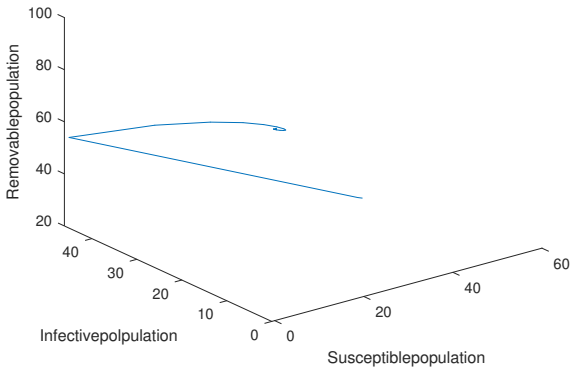

Figure 29. From the phase portrait, it is evident that, the system approaches to equilibrium point and exhibits stability.

Figure 28 and Figure 29 show asymptotic behavior of the system of Example 4.14.

Example 4.15. For $v = 10$, $\beta = 6$, $\gamma = 0.55$, $\mu = 0.1$, $p = 0.2$, $q = 0.3$, $k = 0.24$, the time series response and phase portraits obtained through simulation, which reveal the equilibrium point, are shown in Figures 30 and 31.

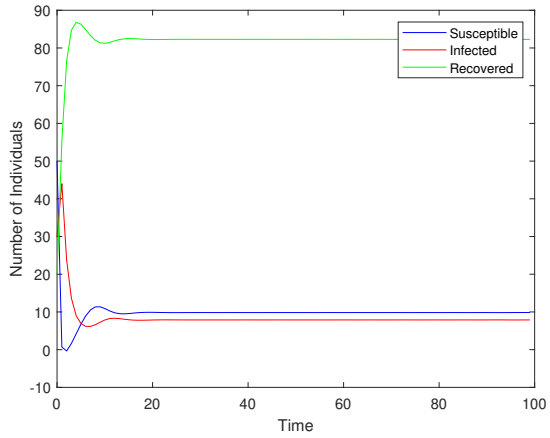

Figure 30. From the graph, it is evident that, the system oscillates asymptotically till it converges to equilibrium point (09,08,83) and exhibits stability.

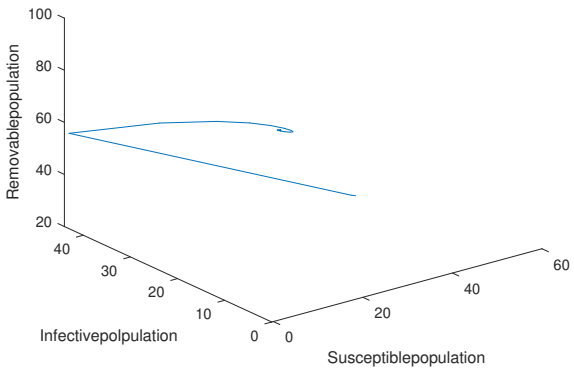

Figure 31. From the phase portrait, it is evident that, the system approaches to equilibrium point and exhibits stability.

Figure 30 and Figure 31 show asymptotic behavior of the system of Example 4.15.

From Examples 4.13 to 4.15, it is observed that, the infective population decreases and removable population increases with the increase in the proportion of vaccination to infected newborn (q) when all the other parameters fixed constant.

For the parametric values $N = 100$, $S = 50$, $I = 30$, $R = 20$, $v = 10$, $\mu = 0.1$ results obtained by varying one at a time of β, γ, p, q, k are tabulated below in the Table 3.

Example	Parameters	Equilibrium point
4.1	$\beta = 6$, $\gamma = 0.6$, $p = 0.1$, $q = 0.4$, $k = 0.15$	E(09,08,83)
4.2	$\beta = 7$, $\gamma = 0.6$, $p = 0.1$, $q = 0.4$, $k = 0.15$	E(06,11,83)
4.3	$\beta = 8$, $\gamma = 0.6$, $p = 0.1$, $q = 0.4$, $k = 0.15$	E(03,12,85)
4.4	$\beta = 6$, $\gamma = 0.35$, $p = 0.1$, $q = 0.4$, $k = 0.15$	E(02,15,83)
4.5	$\beta = 6$, $\gamma = 0.45$, $p = 0.1$, $q = 0.4$, $k = 0.15$	E(04,12,84)
4.6	$\beta = 6$, $\gamma = 0.55$, $p = 0.1$, $q = 0.4$, $k = 0.15$	E(06,10,84)
4.7	$\beta = 6$, $\gamma = 0.55$, $p = 0.3$, $q = 0.4$, $k = 0.15$	E(09,08,83)
4.8	$\beta = 6$, $\gamma = 0.55$, $p = 0.6$, $q = 0.4$, $k = 0.15$	E(14,06,80)
4.9	$\beta = 6$, $\gamma = 0.55$, $p = 0.9$, $q = 0.4$, $k = 0.15$	E(19,04,77)
4.10	$\beta = 6$, $\gamma = 0.55$, $p = 0.2$, $q = 0.1$, $k = 0.15$	E(12,08,79)
4.11	$\beta = 6$, $\gamma = 0.55$, $p = 0.2$, $q = 0.3$, $k = 0.15$	E(09,09,82)
4.12	$\beta = 6$, $\gamma = 0.55$, $p = 0.2$, $q = 0.6$, $k = 0.15$	E(04,10,86)
4.13	$\beta = 6$, $\gamma = 0.55$, $p = 0.2$, $q = 0.3$, $k = 0.1$	E(09,10,81)
4.14	$\beta = 6$, $\gamma = 0.55$, $p = 0.2$, $q = 0.3$, $k = 0.18$	E(09,09,82)
4.15	$\beta = 6$, $\gamma = 0.55$, $p = 0.2$, $q = 0.3$, $k = 0.24$	E(09,08,83)

Table 3. Table showing results obtained from graphs

5. Results and conclusions

Analyzed the stability of three compartment SIR epidemic model whose population are Susceptible (S), infected (I) and removable (R) with an assumption of the total population is constant under vaccination coverage on newborns including vertically transmitted individuals. The model is represented by the system which has a set of nonlinear ODE. It is established that, at the disease-free equilibrium point, the system becomes locally asymptotically stable, subject to the condition,

$$S^* < \frac{(\mu + \gamma + k - q)N}{\beta}$$

and at the endemic equilibrium point, the system becomes locally asymptotically stable if,

$$I^* > S^*.$$

It is proved that at the endemic equilibrium point, the system becomes globally asymptotically stable by means of suitable Lyapunov function construction. The observations that are made from the numerical simulation for different parametric values are presented below. There is a notable growth in infective and removable populations when there is an increase in the transmission rate β and remaining parameters are fixed constant. Also there is a growth in susceptible populations when there is an increase in the recovery rate of infected individuals γ. The infective population almost vanish when there is an increase in the proportion of vaccination of newborn (p) and there is a considerable growth in the infective and removable populations when there is an increase in the rate of infected newborn (q). And there is a notable growth in the removable population when there is an increase in the proportion of vaccination to the infected newborn (k).

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Conflict of Interest: Authors hereby declare that there is no conflict of interest.

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