Montes Taurus J. Pure Appl. Math. 7 (1), 157–166, 2025

Montes Taurus Journal of Pure and Applied Mathematics

https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

A note on parametric Apostol-Bernoulli type polynomials

Seçil İçiuyan [a]

[a]*Department of Computer Programming, Anadolu Bil Vocational School, Istanbul Aydin University, Turkey*

Abstract

In this paper, we examine two parametric kinds of Apostol-Bernoulli, Apostol-Euler, and Apostol- Gennochi polynomials, which are generalizations of the classical Bernoulli, Euler, and Genocchi polynomials, and the second type, unified Apostol-Bernoulli type polynomials. Later, inspired by these studies, we give some polynomial expressions. We obtain some general forms of the polynomials expressed by many authors. Then, we give some theorems and results about these polynomials. Finally, we prove that the symmetric relation holds.

Keywords: Second type parametric unified Apostol-Bernoulli type polynomials, second type unified Apostol-Bernoulli type polynomials, second type parametric generalized Apostol-Bernoulli polynomials, second type parametric generalized Apostol-Euler polynomials, the second type parametric generalized Apostol-Genocchi polynomials, generalized Stirling numbers of the second kind

2020 MSC: 11B68, 11B73, 11S80

1. Introduction

In this section, we examine some generalizations to the classical Bernoulli, classical Euler, and classical Genocchi polynomials. We first focus on both classical Bernoulli, Euler, and Genocchi polynomials (*cf.* [6, 8, 13, 14, 20]) and type 2 Bernoulli, type 2 Euler, and type 2 Genocchi polynomials (*cf.* [1, 5, 12]) and then on two parametric kinds of Apostol-Bernoulli, Apostol-Euler, Apostol-Genocchi polynomials (*cf.* [7, 11, 19]). Finally, we examine the generating functions that are involved in the unification of the classical Bernoulli polynomials, Euler polynomials, and Genocchi polynomials defined by Ozden (*cf.* [14, 15]).

These generalizations show us that we can obtain a more general form of the polynomials we examined in this section. Inspired by this, in the next section, we express new polynomials and give some properties and relations provided by these polynomials.

Definition 1.1. The classical Bernoulli polynomials $B_n(x)$, Euler polynomials $E_n(x)$, Genocchi polynomials $G_n(x)$ are defined by the generating functions (*cf.* [2, 18]):

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt}, \quad (|t| < 2\pi)$$

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2}{e^t+1}e^{xt}, \quad (|t| < \pi)$$

---

†*Article ID:* MTJPAM-D-23-00058

Email address: secilbilgic@aydin.edu.tr (Seçil İçiuyan)

Received:14 December 2023, Accepted:12 May 2025, Published:25 October 2025

⋆*Corresponding Author:* Seçil İçiuyan

and

$$\sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = \frac{2t}{e^t+1} e^{xt}, \quad (|t| < \pi).$$

Definition 1.2. Type 2 Bernoulli polynomials, type 2 Euler polynomials, and type 2 Genocchi polynomials are defined by means of the following generating function, respectively, by (*cf.* [1, 5, 12]):

$$\sum_{m=0}^{\infty} \tilde{B}_m(x) \frac{t^m}{m!} = \frac{t}{e^t - e^{-t}} e^{xt}, \quad (|t| < \pi)$$

$$\sum_{m=0}^{\infty} \tilde{E}_m(x) \frac{t^m}{m!} = \frac{2}{e^t + e^{-t}} e^{xt}, \quad \left(|t| < \frac{\pi}{2}\right)$$

and

$$\sum_{m=0}^{\infty} \tilde{G}_m(x) \frac{t^m}{m!} = \frac{2t}{e^t + e^{-t}} e^{xt}, \quad \left(|t| < \frac{\pi}{2}\right).$$

The following relations are provided (*cf.* [9]):

$$\tilde{B}_m(x) = 2^{m-1} B_m\left(\frac{x+1}{2}\right), \quad (m \geq 0)$$

and

$$\tilde{E}_m(x) = 2^m E_m\left(\frac{x+1}{2}\right), \quad (m \geq 0).$$

A similar relation is also available in the following relation

$$\tilde{G}_m(x) = 2^{m-1} G_m\left(\frac{x+1}{2}\right), \quad (m \geq 0).$$

Definition 1.3. Two parametric kinds of Apostol-Bernoulli polynomials, Apostol-Euler and Apostol-Gennochi polynomials are defined as follows (*cf.* [11, 19]):

$$\begin{aligned} \sum_{m=0}^{\infty} B_m^{(c)}(r, w, \lambda) \frac{t^m}{m!} &= \frac{t}{\lambda e^t - 1} e^{rt} \cos wt, \\ \sum_{m=0}^{\infty} B_m^{(s)}(r, w, \lambda) \frac{t^m}{m!} &= \frac{t}{\lambda e^t - 1} e^{rt} \sin wt, \end{aligned}$$

$$\begin{aligned} \sum_{m=0}^{\infty} E_m^{(c)}(r, w, \lambda) \frac{t^m}{m!} &= \frac{2}{\lambda e^t + 1} e^{rt} \cos wt, \\ \sum_{m=0}^{\infty} E_m^{(s)}(r, w, \lambda) \frac{t^m}{m!} &= \frac{2}{\lambda e^t + 1} e^{rt} \sin wt \end{aligned}$$

and

$$\begin{aligned} \sum_{m=0}^{\infty} G_m^{(c)}(r, w, \lambda) \frac{t^m}{m!} &= \frac{2t}{\lambda e^t + 1} e^{rt} \cos wt, \\ \sum_{m=0}^{\infty} G_m^{(s)}(r, w, \lambda) \frac{t^m}{m!} &= \frac{2t}{\lambda e^t + 1} e^{rt} \sin wt. \end{aligned}$$

Definition 1.4. Stirling numbers of the second kind $S_2(m,k)$ are defined as follows (*cf.* [10]):

$$\sum_{m=k}^{\infty} S_2(m,k)\frac{t^m}{m!} = \frac{1}{k!}\left(e^t - 1\right)^k, \quad (k \geq 0).$$

The generalized Stirling number of the second kind $S_2(m, v, a, b, \beta)$ is defined by the following generating function (*cf.* [16, 17]):

$$\sum_{m=0}^{\infty} S_2(m, v, a, b, \beta)\frac{t^m}{m!} = \frac{\left(\beta^b e^t - a^b\right)^v}{v!}, \qquad (v \in \mathbb{N}_0,\ a,\ b \in \mathbb{R}^+,\ \beta \in \mathbb{C}).$$

Definition 1.5. Second type generalized Apostol-Bernoulli polynomials $\tilde{B}_l^{(\alpha)}(x,\lambda)$, second type generalized Apostol-Euler polynomials $\tilde{E}_l^{(\alpha)}(x,\lambda)$, and second type generalized Apostol-Genocchi polynomials $\tilde{G}_l^{(\alpha)}(x,\lambda)$ are defined by the exponential generating functions, respectively, by (*cf.* [3]):

$$\sum_{l=0}^{\infty} \tilde{B}_l^{(\alpha)}(x,\lambda)\frac{t^l}{l!} = \left(\frac{t}{\lambda e^t - e^{-t}}\right)^{\alpha} e^{xt},$$

$$\sum_{l=0}^{\infty} \tilde{E}_l^{(\alpha)}(x,\lambda)\frac{t^l}{l!} = \left(\frac{2}{\lambda e^t + e^{-t}}\right)^{\alpha} e^{xt}$$

and

$$\sum_{l=0}^{\infty} \tilde{G}_l^{(\alpha)}(x,\lambda)\frac{t^l}{l!} = \left(\frac{2t}{\lambda e^t + e^{-t}}\right)^{\alpha} e^{xt}.$$

Definition 1.6. For $r, w \in \mathbb{R}$, $\lambda, \alpha \in \mathbb{C}$, we define the second type parametric generalized Apostol-Bernoulli, Apostol-Euler, and Apostol-Genocchi polynomials by the following generating function, respectively

$$\begin{aligned}
\sum_{f=0}^{\infty} \tilde{B}_f^{(c,\alpha)}(r,w,\lambda)\frac{t^f}{f!} &= \left(\frac{t}{\lambda e^t - e^{-t}}\right)^{\alpha} e^{rt}\cos wt,\\
\sum_{f=0}^{\infty} \tilde{B}_f^{(s,\alpha)}(r,w,\lambda)\frac{t^f}{f!} &= \left(\frac{t}{\lambda e^t - e^{-t}}\right)^{\alpha} e^{rt}\sin wt,\\
\sum_{f=0}^{\infty} \tilde{E}_f^{(c,\alpha)}(r,w,\lambda)\frac{t^f}{f!} &= \left(\frac{2}{\lambda e^t + e^{-t}}\right)^{\alpha} e^{rt}\cos wt,\\
\sum_{f=0}^{\infty} \tilde{E}_f^{(s,\alpha)}(r,w,\lambda)\frac{t^f}{f!} &= \left(\frac{2}{\lambda e^t + e^{-t}}\right)^{\alpha} e^{rt}\sin wt
\end{aligned}$$

and

$$\begin{aligned}
\sum_{f=0}^{\infty} \tilde{G}_f^{(c,\alpha)}(r,w,\lambda)\frac{t^f}{f!} &= \left(\frac{2t}{\lambda e^t + e^{-t}}\right)^{\alpha} e^{rt}\cos wt,\\
\sum_{f=0}^{\infty} \tilde{G}_f^{(s,\alpha)}(r,w,\lambda)\frac{t^f}{f!} &= \left(\frac{2t}{\lambda e^t + e^{-t}}\right)^{\alpha} e^{rt}\sin wt.
\end{aligned}$$

We get

$$\begin{aligned}
\tilde{B}_f^{(c,1)}(x,0,1) &= \tilde{B}_f(x), \\
\tilde{B}_f^{(c,1)}(0,0,1) &= \tilde{B}_f, \\
\tilde{E}_f^{(c,1)}(x,0,1) &= \tilde{E}_f(x), \\
\tilde{E}_f^{(c,1)}(0,0,1) &= \tilde{E}_f, \\
\tilde{G}_f^{(c,1)}(x,0,1) &= \tilde{G}_f(x), \\
\tilde{G}_f^{(c,1)}(0,0,1) &= \tilde{G}_f.
\end{aligned}$$

For $x \in \mathbb{R}, \beta, \alpha \in \mathbb{C}$, and $w \in \mathbb{N}_0$, second type unified Apostol-Bernoulli type polynomials are defined by (*cf.* [4, 3]):

$$\sum_{i=0}^{\infty} \tilde{R}_{i,\beta}^{(\alpha)}(x;w,a,b)\frac{t^i}{i!} = \left(\frac{2^{1-w}t^w}{\beta^b e^t - a^b e^{-t}}\right)^{\alpha} e^{xt}, \qquad \left(\left|2t + b\log\left(\frac{\beta}{a}\right)\right| < 2\pi,\ a \neq 0,\ 1^{\alpha} := 1\right). \tag{1.1}$$

Definition 1.7. For $\lambda \in \mathbb{R}$ or $\lambda \in \mathbb{C}$, the generalized sum of integer powers $\mathcal{S}_m(l,\lambda)$ is defined as follows (*cf.* [21]):

$$\sum_{m=0}^{\infty} \mathcal{S}_m(l,\lambda)\frac{t^m}{m!} = \frac{\lambda e^{(l+1)t} - 1}{\lambda e^t - 1}. \tag{1.2}$$

2. Main results

Inspired by the polynomials in the first part of this paper, we worked on some generalizations. Then we give some properties, theorems, and symmetric relations of these polynomials. When we focus on the special cases of the polynomials we define, it is seen that they satisfy the definitions and relations given by some authors.

Definition 2.1. For $p, q \in \mathbb{R}, \beta, \alpha \in \mathbb{C}$, and $w \in \mathbb{N}_0$, second type parametric unified Apostol-Bernoulli type polynomials are defined by means of the generating function as follows:

$$\sum_{m=0}^{\infty} \tilde{R}_{m,\beta}^{(c,\alpha)}(p,q;w,a,b)\frac{t^m}{m!} = \left(\frac{2^{1-w}t^w}{\beta^b e^t - a^b e^{-t}}\right)^{\alpha} e^{pt}\cos qt, \qquad \left(\left|2t + b\log\left(\frac{\beta}{a}\right)\right| < 2\pi,\ a \neq 0,\ 1^{\alpha} := 1\right) \tag{2.1}$$

and

$$\sum_{m=0}^{\infty} \tilde{R}_{m,\beta}^{(s,\alpha)}(p,q;w,a,b)\frac{t^m}{m!} = \left(\frac{2^{1-w}t^w}{\beta^b e^t - a^b e^{-t}}\right)^{\alpha} e^{pt}\sin qt, \qquad \left(\left|2t + b\log\left(\frac{\beta}{a}\right)\right| < 2\pi,\ a \neq 0,\ 1^{\alpha} := 1\right).$$

Remark 2.2. In the expression (2.1), for $q = 0$, we have

$$\tilde{R}_{m,\beta}^{(c,\alpha)}(p,0;w,a,b) = \tilde{R}_{m,\beta}^{(\alpha)}(p;w,a,b).$$

Remark 2.3. In the expression (2.1), for $q = 0$, $\alpha = w = a = b = \beta = 1$, we have

$$\tilde{R}_{m,1}^{(c,1)}(p,0;1,1,1) = \tilde{B}_m(p).$$

Remark 2.4. In the expression (2.1), for $q = w = 0$, $a = -1$, and $\beta = \alpha = b = 1$, we have

$$\tilde{R}_{m,1}^{(c,1)}(p,0;0,-1,1) = \tilde{E}_m(p).$$

Remark 2.5. In the expression (2.1), for $q = 0$, $\alpha = w = b = 1$, $a = -\frac{1}{2}, \beta = \frac{1}{2}$, we have

$$\tilde{R}_{m,\frac{1}{2}}^{(c,1)}\left(p,0;1,-\frac{1}{2},1\right) = \tilde{G}_m(p).$$

Theorem 2.6. *For $f \geq k$, the following equations are provided between second type parametric unified Apostol-Bernoulli type polynomials:*

$$\tilde{R}_{f,\beta}^{(c,\alpha)}(p+m,q;w,a,b) = \sum_{k=0}^{f}\binom{f}{k}m^{f-k}\tilde{R}_{k,\beta}^{(c,\alpha)}(p,q;w,a,b). \tag{2.2}$$

$$\tilde{R}_{f,\beta}^{(s,\alpha)}(p+m,q;w,a,b) = \sum_{k=0}^{f}\binom{f}{k}m^{f-k}\tilde{R}_{k,\beta}^{(s,\alpha)}(p,q;w,a,b). \tag{2.3}$$

$$\tilde{R}_{f,\beta}^{(c,\alpha)}(p+m,q;w,a,b) = \sum_{k=0}^{f}\binom{f}{k}(p+m)^{f-k}\,\tilde{R}_{k,\beta}^{(c,\alpha)}(0,q;w,a,b). \tag{2.4}$$

$$\tilde{R}_{f,\beta}^{(s,\alpha)}(p+m,q;w,a,b) = \sum_{k=0}^{f}\binom{f}{k}(p+m)^{f-k}\,\tilde{R}_{k,\beta}^{(s,\alpha)}(0,q;w,a,b). \tag{2.5}$$

$$\tilde{R}_{f,\beta}^{(c,\alpha_1+\alpha_2)}(p+m,q;w,a,b) = \sum_{k=0}^{f}\binom{f}{k}\tilde{R}_{k,\beta}^{(c,\alpha_1)}(p,q;w,a,b)\,\tilde{R}_{f-k,\beta}^{(\alpha_2)}(m;w,a,b). \tag{2.6}$$

$$\tilde{R}_{f,\beta}^{(s,\alpha_1+\alpha_2)}(p+m,q;w,a,b) = \sum_{k=0}^{f}\binom{f}{k}\tilde{R}_{k,\beta}^{(s,\alpha_1)}(p,q;w,a,b)\,\tilde{R}_{f-k,\beta}^{(\alpha_2)}(m;w,a,b). \tag{2.7}$$

Proof. By using equation (2.1) and the definition of the exponential function,

$$\begin{aligned}\sum_{f=0}^{\infty}\tilde{R}_{f,\beta}^{(c,\alpha)}(p+m,q;w,a,b)\frac{t^f}{f!} &= \sum_{f=0}^{\infty}\tilde{R}_{f,\beta}^{(c,\alpha)}(p,q;w,a,b)\frac{t^f}{f!}\sum_{f=0}^{\infty}\frac{m^f t^f}{f!}\\ &= \sum_{f=0}^{\infty}\sum_{k=0}^{f}\binom{f}{k}m^{f-k}\tilde{R}_{k,\beta}^{(c,\alpha)}(p,q;w,a,b)\frac{t^f}{f!}.\end{aligned}$$

When the coefficients are equalized, equation (2.2) is obtained. Equation (2.3) is obtained similarly.

$$\begin{aligned}\sum_{f=0}^{\infty}\tilde{R}_{f,\beta}^{(c,\alpha)}(p+m,q;w,a,b)\frac{t^f}{f!} &= \sum_{f=0}^{\infty}\tilde{R}_{f,\beta}^{(c,\alpha)}(0,q;w,a,b)\frac{t^f}{f!}\sum_{f=0}^{\infty}\frac{(p+m)^f\,t^f}{f!}\\ &= \sum_{f=0}^{\infty}\sum_{k=0}^{f}\binom{f}{k}(p+m)^{f-k}\,\tilde{R}_{k,\beta}^{(c,\alpha)}(0,q;w,a,b)\frac{t^f}{f!}.\end{aligned} \tag{2.8}$$

Thus, by equating the coefficients of $\frac{t^f}{f!}$ on both sides of equation (2.8), we obtain equation (2.4). Equation (2.5) is obtained in a similar way as equation (2.4).

$$\begin{aligned}\sum_{f=0}^{\infty}\tilde{R}_{f,\beta}^{(c,\alpha_1+\alpha_2)}(p+m,q;w,a,b)\frac{t^f}{f!} &= \sum_{f=0}^{\infty}\tilde{R}_{f,\beta}^{(c,\alpha_1)}(p,q;w,a,b)\frac{t^f}{f!}\sum_{f=0}^{\infty}\tilde{R}_{f,\beta}^{(\alpha_2)}(m;w,a,b)\frac{t^f}{f!}\\ &= \sum_{f=0}^{\infty}\sum_{k=0}^{f}\binom{f}{k}\tilde{R}_{k,\beta}^{(c,\alpha_1)}(0,q;w,a,b)\,\tilde{R}_{f-k,\beta}^{(\alpha_2)}(m;w,a,b)\frac{t^f}{f!}.\end{aligned} \tag{2.9}$$

Finally, by comparing the coefficients of $\frac{t^f}{f!}$ on the left side of equation (2.9) with the coefficients of $\frac{t^f}{f!}$ on the equation (2.10), equation (2.6) is obtained.

For equation (2.7), when equation (2.7) is processed similarly to (2.6), the desired equality is obtained. □

Theorem 2.7. *For $n \geq w$, the following relations are provided.*

$$2^{1-w}\frac{n!}{(n-w)!}\tilde{R}^{(c,\alpha-1)}_{n-w,\beta}(p,q;w,a,b) = \beta^b\tilde{R}^{(c,\alpha)}_{n,\beta}(p+1,q;w,a,b) - a^b\tilde{R}^{(c,\alpha)}_{n,\beta}(p-1,q;w,a,b). \tag{2.10}$$

$$2^{1-w}\frac{n!}{(n-w)!}\tilde{R}^{(s,\alpha-1)}_{n-w,\beta}(p,q;w,a,b) = \beta^b\tilde{R}^{(s,\alpha)}_{n,\beta}(p+1,q;w,a,b) - a^b\tilde{R}^{(s,\alpha)}_{n,\beta}(p-1,q;w,a,b). \tag{2.11}$$

Proof. Using the definition of second type parametric unified Apostol-Bernoulli type polynomials, we have:

$$\sum_{m=0}^{\infty}\tilde{R}^{(c,\alpha-1)}_{m,\beta}(p,q;w,a,b)\frac{t^m}{m!} = \frac{1}{2^{1-w}t^w}\left\{\beta^b\sum_{m=0}^{\infty}\tilde{R}^{(c,\alpha)}_{m,\beta}(p+1,q;w,a,b)\frac{t^m}{m!} - a^b\sum_{m=0}^{\infty}\tilde{R}^{(c,\alpha)}_{m,\beta}(p-1,q;w,a,b)\frac{t^m}{m!}\right\}.$$

Then we get,

$$\frac{1}{2^{1-w}}\sum_{m=0}^{\infty}\left\{\beta^b\tilde{R}^{(c,\alpha)}_{m,\beta}(p+1,q;w,a,b) - a^b\tilde{R}^{(c,\alpha)}_{m,\beta}(p-1,q;w,a,b)\right\}\frac{t^m}{m!} = \sum_{m=0}^{\infty}\tilde{R}^{(c,\alpha-1)}_{m,\beta}(p,q;w,a,b)\frac{t^{m+w}}{m!}. \tag{2.12}$$

By putting n instead of $m+w$ on the right side of equation (2.12),

$$\sum_{n=w}^{\infty}\frac{2^{1-w}n!}{(n-w)!}\tilde{R}^{(c,\alpha-1)}_{n-w,\beta}(p,q;w,a,b)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left\{\beta^b\tilde{R}^{(c,\alpha)}_{n,\beta}(p+1,q;w,a,b) - a^b\tilde{R}^{(c,\alpha)}_{n,\beta}(p-1,q;w,a,b)\right\}\frac{t^n}{n!}. \tag{2.13}$$

Then equating the coefficients of $\frac{t^n}{n!}$ on both sides of the equation (2.13), we have (2.10). Equation (2.11) is found in a similar way. □

Corollary 2.8. *By putting $q = 0$ in (2.10), we obtain the following relation:*

$$\beta^b\tilde{R}^{(\alpha)}_{n,\beta}(x+1;w,a,b) - a^b\tilde{R}^{(\alpha)}_{n,\beta}(x-1;w,a,b) = 2^{1-w}\frac{n!}{(n-w)!}\tilde{R}^{(\alpha-1)}_{n-w,\beta}(x;w,a,b). \tag{2.14}$$

Corollary 2.9. *By putting $w = a = b = 1$, $\beta = \lambda$ in (2.14) or $q = 0$, $w = a = b = 1$, $\beta = \lambda$ in (2.10), we can get the following relation:*

$$\lambda\tilde{B}^{(\alpha)}_{n,\lambda}(x+1,\lambda) - \tilde{B}^{(\alpha)}_{n,\lambda}(x-1,\lambda) = n\tilde{B}^{(\alpha-1)}_{n-w,\lambda}(x,\lambda).$$

Theorem 2.10. *For $v \in \mathbb{N}_0$ and $m \geq wv$, we get the following identity:*

$$\tilde{R}^{(c,\alpha-v)}_{m-wv,\beta}(p,q;w,a,b) = \frac{2^{(w-1)v}v!\,(m-wv)!}{m!}\sum_{r=0}^{m}\binom{m}{r}2^r\tilde{R}^{(c,\alpha)}_{m-r,\beta}(p-v,q;w,a,b)\,S(r,v,a,b,\beta) \tag{2.15}$$

and

$$\tilde{R}^{(s,\alpha-v)}_{m-wv,\beta}(p,q;w,a,b) = \frac{2^{(w-1)v}v!\,(m-wv)!}{m!}\sum_{r=0}^{m}\binom{m}{r}2^r\tilde{R}^{(s,\alpha)}_{m-r,\beta}(p-v,q;w,a,b)\,S(r,v,a,b,\beta). \tag{2.16}$$

Proof. By using the definition of second type parametric unified Apostol-Bernoulli type polynomials and the generalized Stirling number of second kind, we have:

$$\sum_{m=0}^{\infty}\tilde{R}^{(c,\alpha-v)}_{m,\beta}(p,q;w,a,b)\frac{t^m}{m!} = 2^{(w-1)v}t^{-wv}v!\sum_{m=0}^{\infty}\tilde{R}^{(c,\alpha)}_{m-wv,\beta}(p-v,q;w,a,b)\frac{t^m}{m!}\sum_{m=0}^{\infty}S(r,v,a,b,\beta)\frac{2^m t^m}{m!}$$

we get,

$$\begin{aligned}\sum_{m=0}^{\infty}\tilde{R}^{(c,\alpha-v)}_{m,\beta}(p,q;w,a,b)\frac{t^{m+wv}}{m!} &= \sum_{m=wv}^{\infty}\tilde{R}^{(c,\alpha-v)}_{m-wv,\beta}(p,q;w,a,b)\frac{t^m}{(m-wv)!} &(2.17)\\ &= 2^{(w-1)v}v!\sum_{m=0}^{\infty}\sum_{r=0}^{m}2^r\binom{m}{r}\tilde{R}^{(c,\alpha)}_{m-r,\beta}(p-v,q;w,a,b)\,S(r,v,a,b,\beta)\frac{t^m}{m!}\\ &= \sum_{m=0}^{\infty}2^{(w-1)v}v!\sum_{r=0}^{m}2^r\binom{m}{r}\tilde{R}^{(c,\alpha)}_{m-r,\beta}(p-v,q;w,a,b)\,S(r,v,a,b,\beta)\frac{t^m}{m!}. &(2.18)\end{aligned}$$

Thus, by equating the coefficients of $\frac{t^m}{m!}$ on both sides of equations (2.17) and (2.18), we get the equation (2.15). It can be seen that equation (2.16) also provides equality with a similar method. □

Corollary 2.11. *If $q = 0$ in the equation (2.15), the following equation is obtained.*

$$\tilde{R}^{(\alpha-v)}_{m-wv,\beta}(x; w, a, b) = \left\{ \frac{2^{(w-1)v} v!\,(m-wv)!}{m!} \sum_{r=0}^{m} \binom{m}{r} 2^r \tilde{R}^{(\alpha)}_{m-r,\beta}(x-v; w, a, b)\, S(r, v, a, b, \beta) \right\}. \tag{2.19}$$

Corollary 2.12. *By putting $w = a = b = 1$, $\beta = \lambda$ in equation (2.19), the following relation is obtained. Moreover, we can also find the following equation by setting $q = 0$, $w = a = b = 1$ in equation (2.15).*

$$\tilde{B}^{(\alpha-v)}_{m-v}(x, \lambda) = \frac{v!\,(m-v)!}{m!} \sum_{r=0}^{m} \binom{m}{r} 2^r \tilde{B}^{(\alpha)}_{m-r}(x-v, \lambda)\, S(r, v, 1, 1, \lambda).$$

Theorem 2.13. *The following equations are provided.*

$$\begin{aligned} \tilde{R}^{(s,\alpha_1+\alpha_2)}_{i,\beta}(p+m, 2q; w, a, b) &= 2\sum_{r=0}^{i} \binom{i}{r} \tilde{R}^{(s,\alpha_1)}_{r,\beta}(p, q; w, a, b)\, \tilde{R}^{(c,\alpha_2)}_{i-r,\beta}(m, q; w, a, b). \\ \tilde{R}^{(s,\alpha_1+\alpha_2)}_{i,\beta}(p+m, 2q; w, a, b) &= 2\sum_{r=0}^{i} \binom{i}{r} \tilde{R}^{(c,\alpha_1)}_{r,\beta}(p, q; w, a, b)\, \tilde{R}^{(s,\alpha_2)}_{i-r,\beta}(m, q; w, a, b). \end{aligned}$$

Proof. By applying trigonometric half-angle transformations in (2.1), we have

$$\begin{aligned} \sum_{i=0}^{\infty} \tilde{R}^{(s,\alpha_1+\alpha_2)}_{i,\beta}(p+m, 2q; w, a, b) \frac{t^i}{i!} &= \left(\frac{2^{1-w} t^w}{\beta^b e^t - a^b e^{-t}} \right)^{\alpha_1+\alpha_2} e^{(p+m)t} \sin 2qt \\ &= 2\left(\frac{2^{1-w} t^w}{\beta^b e^t - a^b e^{-t}} \right)^{\alpha_1+\alpha_2} e^{pt} e^{mt} \cos qt \sin qt. \end{aligned} \tag{2.20}$$

Then we get

$$2\sum_{i=0}^{\infty} \tilde{R}^{(s,\alpha_1)}_{i,\beta}(p, q; w, a, b) \frac{t^i}{i!} \sum_{i=0}^{\infty} \tilde{R}^{(c,\alpha_2)}_{i,\beta}(m, q; w, a, b) \frac{t^i}{i!} = 2\sum_{i=0}^{\infty} \sum_{r=0}^{i} \binom{i}{r} \tilde{R}^{(s,\alpha_1)}_{r,\beta}(p, q; w, a, b)\, \tilde{R}^{(c,\alpha_2)}_{i-r,\beta}(m, q; w, a, b) \frac{t^i}{i!}. \tag{2.21}$$

Finally, when the coefficients on the left side of (2.20) are compared with (2.21), the desired result is obtained. The other equation is obtained in a similar way. □

Theorem 2.14. *The following relation is satisfied.*

$$\begin{aligned} \tilde{R}^{(s,\alpha_1+\alpha_2)}_{i,\beta}(p+m, q+l; w, a, b) &= \sum_{r=0}^{i} \binom{i}{r} \Big\{ \tilde{R}^{(s,\alpha_1)}_{r,\beta}(p, q; w, a, b)\, \tilde{R}^{(c,\alpha_2)}_{i-r,\beta}(m, l; w, a, b) \\ &\quad + \tilde{R}^{(s,\alpha_1)}_{r,\beta}(p, l; w, a, b)\, \tilde{R}^{(c,\alpha_2)}_{i-r,\beta}(m, q; w, a, b) \Big\}. \\ \tilde{R}^{(s,\alpha_1+\alpha_2)}_{i,\beta}(p+m, q+l; w, a, b) &= \sum_{r=0}^{i} \binom{i}{r} \Big\{ \tilde{R}^{(c,\alpha_1)}_{r,\beta}(p, q; w, a, b)\, \tilde{R}^{(s,\alpha_2)}_{i-r,\beta}(m, l; w, a, b) \\ &\quad + \tilde{R}^{(c,\alpha_1)}_{r,\beta}(p, l; w, a, b)\, \tilde{R}^{(s,\alpha_2)}_{i-r,\beta}(m, q; w, a, b) \Big\}. \end{aligned}$$

Proof. By using trigonometric sum and difference formulas in (2.1), we get

$$\begin{aligned}
\sum_{i=0}^{\infty} \tilde{R}_{i,\beta}^{(s,\alpha_1+\alpha_2)}(p+m,q+l;w,a,b)\frac{t^i}{i!} &= \left(\frac{2^{1-w}t^w}{\beta^b e^t - a^b e^{-t}}\right)^{\alpha_1+\alpha_2} e^{(p+m)t}\sin(q+l)t. &(2.22)\\
&= \sum_{i=0}^{\infty}\sum_{r=0}^{i}\binom{i}{r}\tilde{R}_{r,\beta}^{(s,\alpha_1)}(p,q;w,a,b)\,\tilde{R}_{i-r,\beta}^{(c,\alpha_2)}(m,l;w,a,b)\frac{t^i}{i!}\\
&\quad+\sum_{i=0}^{\infty}\sum_{r=0}^{i}\binom{i}{r}\tilde{R}_{r,\beta}^{(s,\alpha_1)}(p,l;w,a,b)\,\tilde{R}_{i-r,\beta}^{(c,\alpha_2)}(m,q;w,a,b)\frac{t^i}{i!}\\
&= \sum_{i=0}^{\infty}\sum_{r=0}^{i}\binom{i}{r}\Big\{\tilde{R}_{r,\beta}^{(s,\alpha_1)}(p,q;w,a,b)\,\tilde{R}_{i-r,\beta}^{(c,\alpha_2)}(m,l;w,a,b) &(2.23)\\
&\quad+\tilde{R}_{r,\beta}^{(s,\alpha_1)}(p,l;w,a,b)\,\tilde{R}_{i-r,\beta}^{(c,\alpha_2)}(m,q;w,a,b)\Big\}\frac{t^i}{i!}.
\end{aligned}$$

If we compare the coefficients of equations (2.22) and (2.23), the desired result is obtained. The other equality is obtained similarly. □

Theorem 2.15. *For $c \neq d$, and $c, d \neq -1$, the symmetry relation is provided.*

$$\begin{aligned}
&\sum_{r=0}^{i}\sum_{l=0}^{r}\binom{r}{l}\binom{i}{r}c^{i-r}(d+1)^r d^{r-l}\tilde{R}_{i-r,\beta}^{(c,m)}(dx;w,a,b)\,\mathcal{S}_l\left(c-1+\frac{1-c}{1+d};\left(\frac{\beta}{a}\right)^b\right)\tilde{R}_{r-l,\beta}^{(s,m-1)}(cy;w,a,b) \qquad (2.24)\\
&= \sum_{r=0}^{i}\sum_{l=0}^{r}\binom{r}{l}\binom{i}{r}d^{i-r}(c+1)^r c^{r-l}\tilde{R}_{i-r,\beta}^{(c,m)}(cx;w,a,b)\,\mathcal{S}_l\left(d-1+\frac{1-d}{1+c};\left(\frac{\beta}{a}\right)^b\right)\tilde{R}_{r-l,\beta}^{(s,m-1)}(dy;w,a,b).
\end{aligned}$$

Proof. We define $H(t)$ function that is symmetric in c and d. By using this $H(t)$ function, equation (2.24) is obtained.

$$H(t) = \frac{2^{(1-w)(2m-1)}t^{2wm-w}e^{cdxt}\left(\beta^b e^{cdt} - a^b e^{-t}\right)e^{cdyt}}{\left(\beta^b e^{ct} - a^b e^{-t}\right)^m\left(\beta^b e^{dt} - a^b e^{-t}\right)^m}\frac{1}{2}\sin 2qt.$$

First of all, we apply the following operations to prove that the $H(t)$ function gives the left side of the equation (2.24). By using equations (1.2) and (2.1), we get

$$\begin{aligned}
H(t) &= \frac{1}{c^{wm}d^{w(m-1)}}\left(\frac{2^{w-1}c^w t^w}{\beta^b e^{ct} - a^b e^{-t}}\right)^m e^{cdxt}\cos qt\left(\frac{\beta^b e^{cdt} - a^b e^{-t}}{\beta^b e^{dt} - a^b e^{-t}}\right)\left(\frac{2^{w-1}d^w t^w}{\beta^b e^{dt} - a^b e^{-t}}\right)^{m-1} e^{cdyt}\sin qt\\
&= \frac{1}{c^{wm}d^{w(m-1)}}\left(\frac{2^{w-1}c^w t^w}{\beta^b e^{ct} - a^b e^{-t}}\right)^m e^{cdxt}\cos qt\left(\frac{\beta^b e^{(cd+1)t} - a^b}{\beta^b e^{(d+1)t} - a^b}\right)\left(\frac{2^{w-1}d^w t^w}{\beta^b e^{dt} - a^b e^{-t}}\right)^{m-1} e^{cdyt}\sin qt\\
&= \frac{1}{c^{wm}d^{w(m-1)}}\left(\frac{2^{w-1}c^w t^w}{\beta^b e^{ct} - a^b e^{-t}}\right)^m e^{cdxt}\cos qt\left(\frac{\beta^b e^{\left(c+\frac{1-c}{1+d}\right)(d+1)t} - a^b}{\beta^b e^{(d+1)t} - a^b}\right)\left(\frac{2^{w-1}d^w t^w}{\beta^b e^{dt} - a^b e^{-t}}\right)^{m-1} e^{cdyt}\sin qt.
\end{aligned}$$

We have,

$$\begin{aligned}
&= \frac{1}{c^{wm}d^{w(m-1)}}\left(\sum_{i=0}^{\infty}\tilde{R}_{i,\beta}^{(c,m)}(dx;w,a,b)\frac{(ct)^i}{i!}\right)\left(\sum_{i=0}^{\infty}\mathcal{S}_i\left(c-1+\frac{1-c}{1+d};\left(\frac{\beta}{a}\right)^b\right)\frac{(d+1)^i t^i}{i!}\right)\left(\sum_{i=0}^{\infty}\tilde{R}_{i,\beta}^{(s,m-1)}(cy;w,a,b)\frac{(dt)^i}{i!}\right)\\
&= \frac{1}{c^{wm}d^{w(m-1)}}\sum_{i=0}^{\infty}\left\{\sum_{r=0}^{i}\binom{i}{r}c^{i-r}\tilde{R}_{i-r,\beta}^{(c,m)}(dx;w,a,b)\sum_{l=0}^{r}\binom{r}{l}(d+1)^l d^{r-l}\mathcal{S}_l\left(c-1+\frac{1-c}{1+d};\left(\frac{\beta}{a}\right)^b\right)\tilde{R}_{r-l,\beta}^{(s,m-1)}(cy;w,a,b)\frac{t^i}{i!}\right\}.
\end{aligned}$$

When similar operations are also applied on the other side and then the coefficients of the left and right sides are equalized, the equation (2.24) is satisfied. □

Corollary 2.16. *For all $c, d, m \in \mathbb{N}, i \in \mathbb{N}_0$, and $c \neq d$, we have the following symmetry identity for the second type parametric generalized Apostol-Bernoulli polynomials.*

$$\begin{aligned}\sum_{r=0}^{i}\sum_{l=0}^{r}\binom{r}{l}\binom{i}{r}c^{i-r}(d+1)^{r}d^{r-l}\tilde{B}_{i-r}^{(c,m)}(dx,\lambda)\mathcal{S}_{l}\left(c-1+\frac{1-c}{1+d};\left(\frac{\beta}{a}\right)^{b}\right)\tilde{B}_{r-l}^{(s,m-1)}(cy,\lambda) \\ =\sum_{r=0}^{i}\sum_{l=0}^{r}\binom{r}{l}\binom{i}{r}d^{i-r}(c+1)^{r}c^{r-l}\tilde{B}_{i-r}^{(c,m)}(cx,\lambda)\mathcal{S}_{l}\left(d-1+\frac{1-d}{1+c};\left(\frac{\beta}{a}\right)^{b}\right)\tilde{B}_{r-l}^{(s,m-1)}(dy,\lambda).\end{aligned} \tag{2.25}$$

By setting $w = a = b = 1$ and $\beta = \lambda$ in equation (2.24)*, we obtain* (2.25).

Moreover, we can also obtain equation (2.25) using the definition of the second type parametric generalized Apostol-Bernoulli polynomials.

Corollary 2.17. *For all $c, d, m \in \mathbb{N}, i \in \mathbb{N}_0$, and $c \neq d$, we have the following symmetry identity for the second type parametric generalized Apostol-Euler polynomials.*

$$\begin{aligned}\sum_{r=0}^{i}\sum_{l=0}^{r}\binom{r}{l}\binom{i}{r}c^{i-r}(d+1)^{r}d^{r-l}\tilde{E}_{i-r}^{(c,m)}(dx,\lambda)\mathcal{S}_{l}\left(c-1+\frac{1-c}{1+d};\left(\frac{\beta}{a}\right)^{b}\right)\tilde{E}_{r-l}^{(s,m-1)}(cy,\lambda) \\ =\sum_{r=0}^{i}\sum_{l=0}^{r}\binom{r}{l}\binom{i}{r}d^{i-r}(c+1)^{r}c^{r-l}\tilde{E}_{i-r}^{(c,m)}(cx,\lambda)\mathcal{S}_{l}\left(d-1+\frac{1-d}{1+c};\left(\frac{\beta}{a}\right)^{b}\right)\tilde{E}_{r-l}^{(s,m-1)}(dy,\lambda).\end{aligned} \tag{2.26}$$

By setting $w = 0, -a = b = 1$ and $\beta = \lambda$ in equation (2.24)*, we obtain* (2.26).

Moreover, we can also obtain equation (2.26) using the definition of the second type parametric generalized Apostol-Euler polynomials.

Corollary 2.18. *For all $c, d, m \in \mathbb{N}, i \in \mathbb{N}_0$, and $c \neq d$, we have the following symmetry identity for the second type parametric generalized Apostol-Genocchi polynomials.*

$$\begin{aligned}\sum_{r=0}^{i}\sum_{l=0}^{r}\binom{r}{l}\binom{i}{r}c^{i-r}(d+1)^{r}d^{r-l}\tilde{G}_{i-r}^{(c,m)}(dx,\lambda)\mathcal{S}_{l}\left(c-1+\frac{1-c}{1+d};\left(\frac{\beta}{a}\right)^{b}\right)\tilde{G}_{r-l}^{(s,m-1)}(cy,\lambda) \\ =\sum_{r=0}^{i}\sum_{l=0}^{r}\binom{r}{l}\binom{i}{r}d^{i-r}(c+1)^{r}c^{r-l}\tilde{G}_{i-r}^{(c,m)}(cx,\lambda)\mathcal{S}_{l}\left(d-1+\frac{1-d}{1+c};\left(\frac{\beta}{a}\right)^{b}\right)\tilde{G}_{r-l}^{(s,m-1)}(dy,\lambda).\end{aligned} \tag{2.27}$$

By setting $w = 1, a = -\frac{1}{2}, b = 1, \beta = \frac{\lambda}{2}$ in (2.24)*, we obtain* (2.27).

Moreover, we can also obtain equation (2.27) using the definition of the second type parametric generalized Apostol-Genocchi polynomials.

3. Conclusion

Researchers in many different fields of study use generating functions, trigonometric functions, and many polynomial families in their work. We reviewed the work done by some researchers so far.

In light of these studies, we defined new polynomials. Then, we gave some equations, theorems, symmetry relations, and results provided by these polynomials. Finally, the results of this paper can be used in analytic number theory, physics, engineering, and other related fields.

Acknowledgments

This study is extracted from the Ph.D. thesis [4].

Author Contributions: This paper has only one author.

Conflict of Interest: The author declares no conflict of interest.

Funding (Financial Disclosure): There is no funding for this work.

References

[1] R. P. Agarwal, J. Y. Kang and C. S. Ryoo, *A new q-extension of Euler polynomial of the second kind and some related polynomials*, J. Comput. Anal. Appl. **27 (1)**, 136–148, 2019.

[2] T. M. Apostol, *Introduction to analytic number theory*, Springer Science and Business Media, New York, 1998.

[3] S. Bilgic, *On the second type unified Apostol-Bernoulli type polynomials*, In: Proceedings Book of the 13th Symposium on Generating Functions of Special Numbers and Polynomials and their Applications (GFSNP 2023) (Ed. by M. Alkan, I. Kucukoglu and O. Öneş), Antalya, Turkey, March 11-13, 2023, pp. 256–261; ISBN: 978-625-00-1128-7.

[4] S. Bilgiç, *Bazı dejenere özel sayılar ve polinomlar*, Akdeniz University, Graduate School of Natural and Applied Sciences, Ph.D. Thesis (Thesis no: 683293) in Mathematics, Antalya, 2021 [in Turkish].

[5] G. W. Jang and T. Kim, *A note on type 2 degenerate Euler and Bernoulli polynomials*, Adv. Stud. Contemp. Math. **29 (1)**, 147–159, 2019.

[6] N. Kilar and Y. Simsek, *Two parametric kinds of Eulerian-type polynomials associated with Euler's formula*, Symmetry **11 (9)**, 2019; Article ID: 1097.

[7] N. Kilar and Y. Simsek, *Computational formulas and identities for new classes of Hermite-based Milne–Thomson type polynomials: Analysis of generating functions with Euler's formula*, Math. Methods Appl. Sci. **44 (8)**, 6731–6762, 2021.

[8] N. Kilar, Y. Simsek and H. M. Srivastava, *Recurrence relations, associated formulas, and combinatorial sums for some parametrically generalized polynomials arising from an analysis of the Laplace transform and generating functions*, Ramanujan J. **61 (3)**, 731–756, 2023.

[9] D. S. Kim and T. Kim, *A note on a new type of degenerate Bernoulli numbers*, Russ. J. Math. Phys. **27**, 227–235, 2020.

[10] T. Kim, D. S. Kim and H. I. Kwon, *A note on degenerate Stirling numbers and their applications*, Proc. Jangjeon Math. Soc. **21 (2)**, 195–203, 2018.

[11] D. S. Kim, T. Kim and H. Lee, *A note on degenerate Euler and Bernoulli polynomials of a complex variable*, Symmetry **11 (9)**, 2019; Article ID: 1168.

[12] H. Na and J. H. Kim, *A note on type degenerate poly-Genocchi polynomials and numbers of complex variables arising from the degenerate modified poly exponential function*, Adv. Stud. Contemp. Math. **4**, 559–570, 2020.

[13] H. Ozden, *Unification of generating function of the Bernoulli, Euler and Genocchi numbers and polynomials*, AIP Conf. Proc. **1281**, 1125–1128, 2010.

[14] H. Ozden, *Generating functions of the unified representation of the Bernoulli, Euler and Genocchi polynomials of higher order*, AIP Conf. Proc. **1389**, 349–352, 2011.

[15] H. Ozden and Y. Simsek, *Modification and unification of the Apostol-type numbers and polynomials and their applications*, Appl. Math. Comput. **235**, 338–351, 2014.

[16] Y. Simsek, *Generating functions for generalized Stirling type numbers, array type polynomials, Eulerian type polynomials and their applications*, J. Fixed Point Theory Appl. **2013**, 2013; Article ID: 87.

[17] Y. Simsek, *Identities associated with generalized Stirling type numbers and Eulerian type polynomials*, Appl. Comput. Math. **18 (3)**, 251–26, 2013.

[18] H. M. Srivastava and J. Choi, *Series associated with the zeta and related functions*, Springer Science and Business Media (Volume 530), Dordrecht, 2001.

[19] H. M. Srivastava, M. Masjed-Jamei and M. R. Beyki, *A parametric type of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials.* Appl. Math. Inf. Sci. **12 (5)**, 907–916, 2018.

[20] H. M. Srivastava, M. Masjed-Jamei and M. R. Beyki, *Some new generalizations and applications of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials*, Rocky Mountain J. Math. **49 (2)**, 681–697, 2019.

[21] Z. Zhang and H.Yang, *Several identities for the generalized Apostol–Bernoulli polynomials*, Comput. Math. Appl. **56 (12)**, 2993–2999, 2008.

How to cite this article: S. İçiuyan, *A note on parametric Apostol-Bernoulli type polynomials*, Montes Taurus J. Pure Appl. Math. **7 (1)**, 157–166, 2025; Article ID: MTJPAM-D-23-00058.