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Analytic methods to solve Fredholm integral equations

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Abstract

This manuscript presents an innovative semi-analytical approach for solving linear Fredholm integral equations (FIEs) of the first as well as second kinds. Utilizing the properties of Fourier and Mellin transformations, we derive analytical solutions that substantially improve the comprehension and resolution of these equations. A key innovation of our approach is the ability to effectively manage non-smooth kernels through the degeneration of kernel functions, facilitating their separation and simplification. Empirical examples illustrate the method's effectiveness, demonstrating superior numerical stability and convergence rates compared to existing techniques. This work not only fills a critical gap in the literature but also provides a robust framework for future research in integral equations, paving the way for advancements in various scientific and engineering applications.

Keywords: Integral equation, Fredholm equations, Fourier transform, Mellin transform, kernel function, convolution theory

2020 MSC: 45A05, 45A15, 34K05, 65R10, 47G20, 42B05

1. Introduction

It is evident that Fredholm integral equations (FIEs) of the second kind may be resolved utilising the general theory which was presented by Fredholm [29]. In fact by using this theory one can determine the solution in terms of different values of the parameter A (*cf.* [2, 4, 29, 21]). But for FIEs of the first kind, there exists no general theory about solving methods. In some special cases, this kind of integral equations are solved by utilising integral transforms, like Laplace, Fourier and Mellin transforms (*cf.* [3, 14, 21, 24]). Additionally, if the first kind integral equation's kernel includes singularities which are weak, it may firstly be converted to equations with strong singularities. This may be done by applying a derivative which is fractional. It is then converted to a second type of FIEs by utilizing the Poiencare-Bertrand formula (*cf.* [14, 28]). A few efficient methods have been created to find accurate and precise solutions for integral equations. These methods usually provide methods which are efficient for overcoming the challenge of quickly as well as precisely solving integral equations. By using these methods, integral equations may be solved exactly or very accurately, allowing further investigation as well as solution of numerous problems. The purpose

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of this research is to present a novel method for solving first as well as second-kind linear FIEs. By presenting an approach which is alternative to solving specific types of integral equations, the method which is suggested shows a novel viewpoint. The development of solving problems with linear FIEs has been facilitated by the novel method application, making the process of solving them both precise as well as efficient. Linear FIEs of the first as well as second kind are capable of being solved by a variety of methods. A wide range of strategies have been devised in order to solve these kinds of integral equations, and these methods cover them all. The methods which are currently in use for handling as well as solving linear FIEs provides a wide range of options for handling problems involving these equations. Among these methods include spectral methods [16, 19], collocation methods [20, 26], transform methods [5, 23], as well as others [8, 13, 18, 25].

The variational iteration method [27], the homotopy perturbation method (HPM) [7, 22], the Adomian decomposition method (ADM) [6, 12], the RBF method [10, 11], the wavelets methods [15, 31, 33], as well as other approaches [1, 32] are only a few among the approaches which have evolved as well as improved to reach a solution. This research offers a simple method for resolving the equations in the previously described class. It is noted that this approach works particularly well for issues which are linear. The fundamental idea is to apply the convolution theorem as well as the integral transformations of Mellin and the Fourier exponential to a specific class of FIEs. The method also entails reducing the equations for more complex kernel forms which is done by modifying the kernel function as well as translating the analytical form of the Fredholm equations into an algebraic form. This allows for the utilization of numerical methods to solve the simplified equations and obtain the desired solution.

1.1. Preliminaries

This section gives a complete overview of fundamental concepts and properties related to an integral equation. It refers to an equation where the unknown function occurs within an integral sign. This work will consider only linear equations, that is, equations that do not involve any nonlinear functions of the unknown function are classified as linear integral equations. In practical applications, these equations are commonly categorized into two main types. Before discussing the appearance of the integral equation, It is essential to establish certain definitions and introduce a preliminary classification of linear integral equation.

Definition 1.1. An integral equation is a mathematical expression where the unknown function occurs within an integral. The most general form of an integral equation may be expressed as:

$$h(x)\,y(x) = f(x) + \int_a^b k(x,t)\,y(t)dt, \tag{1.1}$$

in which $h(x)$ as well as $f(x)$ are known functions, while $k(x,t)$ is a given function of two variables, referred to as the kernel or nucleus of the integral equation. The function $y(x)$ refers to the unknown to be determined, and λ represents a scalar parameter (possibly complex) (in our work we put $\lambda = 1$) (*cf.* [17]).

Definition 1.2. An integral equation is considered linear if it considers the integral operator

$$L = \int_a^b k(x,t)\,dt$$

and holds the linearity condition, given by:

$$L\left[\lambda_1 y_1(t) + \lambda_2 y_2(t)\right] = \lambda_1 L\left[y_1(t)\right] + \lambda_2 L[y_2(t)],$$

in which $L[y(t)] = \int_a^b k(x,t)y(t)dt$, and λ_1, and λ_2 are constants, so equation (1.1) is example of linear equation (*cf.* [17]).

Definition 1.3. FIEs of the second kind is expressed as given in (1.1) by setting $h(x) = 1$,

$$y(x) = f(x) + \lambda \int_a^b k(x,t)y(t)dt, \tag{1.2}$$

in which f and k are known, y is unknown, this equation is termed homogeneous when $f(x) = 0$ otherwise it is non homogeneous (*cf.* [24]).

Definition 1.4. FIEs of the first kind is expressed as given in (1.1) by setting $h(x) = 0$,

$$f(x) = \lambda \int_a^b k(x,t)y(t)dt, \tag{1.3}$$

in which the functions f as well as k are given, while y is the unknown function to be determined. Provided that $f(x) = 0$, the integral equation is referred to as homogeneous; otherwise, it is classified as non-homogeneous (*cf.* [24]).

Definition 1.5. FIEs of the second and first kind of convolution type are defined in as in (1.2) and (1.3) respectively, when $k(x,t) = k(x-t)$, that is, where the kernel relies on the difference kernel, the integral equations can be written as

$$y(x) = f(x) + \lambda \int_a^b k(x-t)\, y(t)\, dt \tag{1.4}$$

and

$$f(x) = \lambda \int_a^b k(x-t)\, y(t)\, dt \tag{1.5}$$

(*cf.* [24]), in which f and k are given, y needs to be determined.

1.2. Assumptions for transform methods

To ensure the effective application of Fourier and Mellin transforms in solving FIEs, certain assumptions must be satisfied:

(i) The functions involved must be integrable over their respective domains. Specifically, for the Fourier transform to be applicable, the function $f(x)$ must hold the following condition:

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

For the Mellin transformation, the function $f(x)$ must be integrable over $(0, \infty)$:

$$\int_0^{\infty} |x^{w-1} f(x)|\, dx < \infty,$$

for the relevant values of w.

(ii) The kernel functions $k(x,t)$ must be continuous or piecewise continuous over the integration domain to ensure that the integral equations are well-defined.

(iii) The scalar parameter λ in FIEs should remain bounded to maintain solution stability.

(iv) For the convolution theorem to apply, the functions must be in L^1 space, ensuring their convolution is also integrable.

(v) For the effective application of transforms, the functions $f(x)$, $g(x)$ as well as the kernel $k(x,t)$ should ideally be continuous.

In applying the Fourier and Mellin transforms to solve FIEs, we rely on several critical assumptions regarding the underlying functions and kernels. These assumptions include the integrability of functions, regularity of kernel functions, and conditions on involved parameters, as detailed in Section 1. Meeting these criteria ensures the robustness and effectiveness of the applied mathematical methods.

1.3. Integral transforms

In this section we present some important properties of the integral transformations of Fourier and Mellin. These transformations are very useful for solving special type of FIEs.

1.4. Fourier exponential transform

The Fourier transformation T is the proto type of the most widely used class of unitary integral operators $T=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{iwx}dx$ (*cf.* [14, 28]). The Fourier exponential transform of the function $f(x)$, defined over the interval $(-\infty,\infty)$, is expressed by:

$$F\left(w\right)=T\left\{f\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-iwx}f(x)dx \tag{1.6}$$

and the inverse formula is

$$f\left(x\right)=T^{-1}\left\{F\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{iwx}F(w)dw. \tag{1.7}$$

One of the most crucial properties of the Fourier transform for solving FIEs with a difference kernel is the convolution theorem (*cf.* [11]), which states that

$$T[\int_{-\infty}^{\infty}f_1\left(x{-}t\right)f_2\left(t\right)dt]=\sqrt{2\pi}F_1(w)F_2(w), \tag{1.8}$$

where $F_1\left(w\right)=T(f_1)$ and $F_2(w)=T(f_2)$.

To extend the Fourier transform's applicability to FIEs with non-smooth kernels, the following theorem establishes specific conditions on the given functions and the singularities of the kernel.

Theorem 1.6. *Let $y(x)$ resemble a function expressed on the interval $[a,b]$ such that it is piecewise continuous. Consider the FIEs of the second kind, given by:*

$$y(x)=f(x)+\lambda\int_a^b k(x,t)y(t)\,dt,$$

in which $f(x)$ denotes piecewise continuous, $k(x,t)$ refers to a bounded kernel defined on $[a,b]\times[a,b]$, and λ denotes a scalar parameter. If $k(x,t)$ has singularities within the interval, but is integrable, then the Fourier transform can still be applied under the following conditions:

- **(i)** *$f(x)$ as well as $k(x,t)$ are expressed in such a way that it holds their respective Fourier transforms existance.*
- **(ii)** *The singularities of $k(x,t)$ are adequately weak such that the integral $\int_a^b k(x,t)y(t)\,dt$ converges.*

Proof. The Fourier transform of a function $g(x)$ is expressed as

$$G(\omega)=\mathcal{F}\{g\}(\omega)=\int_{-\infty}^{\infty}e^{-i\omega x}g(x)\,dx.$$

Apply the Fourier transform to both sides of the FIEs:

$$\mathcal{F}\{y(x)\}=\mathcal{F}\{f(x)\}+\lambda\mathcal{F}\left\{\int_a^b k(x,t)y(t)\,dt\right\}.$$

By the convolution theorem, we find:

$$\mathcal{F}\left\{\int_a^b k(x,t)y(t)\,dt\right\}=K(\omega)Y(\omega),$$

where $K(\omega)$ is the Fourier transform of the kernel $k(x,t)$. Substitute back into our transformed equation:

$$Y(\omega)=F(\omega)+\lambda K(\omega)Y(\omega).$$

Rearranging gives:

$$Y(\omega)(1-\lambda K(\omega))=F(\omega).$$

If $K(\omega)$ has singularities, we analyze the term $(1 - \lambda K(\omega))$. For λ small enough, we assume:

$$1 - \lambda K(\omega) \neq 0 \quad \text{in a neighborhood of singular points.}$$

Assuming the above condition holds, we gain:

$$Y(\omega) = \frac{F(\omega)}{1 - \lambda K(\omega)}.$$

By applying the inverse Fourier transform, we now have:

$$y(x) = \mathcal{F}^{-1}\{Y(\omega)\} = \mathcal{F}^{-1}\left\{\frac{F(\omega)}{1 - \lambda K(\omega)}\right\}.$$

Since $K(\omega)$ may introduce complexity due to singularities, we ensure that $\mathcal{F}^{-1}$ exists by taking into consideration the Cauchy principal value of the integral if needed:

$$\mathcal{F}^{-1}\left\{\frac{F(\omega)}{1 - \lambda K(\omega)}\right\} = \lim_{\epsilon\to 0}\mathcal{F}^{-1}\left\{\frac{F(\omega)}{1 - \lambda K(\omega) + i\epsilon}\right\}.$$

□

The following example illustrates the steps involved in solving a FIEs having a singular kernel using Fourier transforms. It highlights the steps of setting up the equation, applying the Fourier transform.

Example 1.7. In case FIEs with a singular kernel, consider the FIEs of the second kind:

$$y(x) = f(x) + \lambda \int_0^1 k(x,t)y(t)\,dt,$$

in which $f(x) = x$ as well as the kernel $k(x,t)$ is expressed as:

$$k(x,t) = \frac{1}{x-t} \quad \text{for } x \neq t$$

and is expressed to be zero when $x = t$. This kernel exhibits a singularity when $x = t$.

To set up the equation, substituting $f(x)$ into the integral equation, we have:

$$y(x) = x + \lambda \int_0^1 \frac{1}{x-t} y(t)\,dt.$$

Applying Fourier transform to both sides yields the Fourier transform of $y(x)$ as $Y(\omega)$. For $f(x) = x$, the transformation yields:

$$\mathcal{F}\{f(x)\} = F(\omega) = \int_0^1 xe^{-i\omega x}\,dx.$$

Calculating the integral:

$$F(\omega) = \left[-\frac{xe^{-i\omega x}}{i\omega}\right]_0^1 + \frac{1}{i\omega}\int_0^1 e^{-i\omega x}\,dx = -\frac{e^{-i\omega}}{i\omega} + \frac{1}{i\omega}\left[-\frac{e^{-i\omega x}}{i\omega}\right]_0^1.$$

This simplifies to:

$$F(\omega) = -\frac{e^{-i\omega}}{i\omega} + \frac{1 - e^{-i\omega}}{(i\omega)^2}.$$

Next, we need to compute the Fourier transform of the kernel $k(x,t)$:

$$K(\omega) = \mathcal{F}\left\{\frac{1}{x-t}\right\}.$$

Using the convolution theorem, we express:

$$\mathcal{F}\left\{\int_0^1 k(x,t)y(t)\,dt\right\} = K(\omega)Y(\omega).$$

Solve for $Y(\omega)$ after substituting back into the transformed equation gives:

$$Y(\omega) = F(\omega) + \lambda K(\omega)Y(\omega).$$

Rearranging yields:

$$Y(\omega)(1 - \lambda K(\omega)) = F(\omega).$$

Assuming $1 - \lambda K(\omega) \neq 0$, we solve for $Y(\omega)$:

$$Y(\omega) = \frac{F(\omega)}{1 - \lambda K(\omega)}.$$

Finally, to obtain $y(x)$, we apply the inverse Fourier transform:

$$y(x) = \mathcal{F}^{-1}\{Y(\omega)\}.$$

The following theorem establishes a more general framework for implementing the Fourier transform to linear FIEs, particularly in cases where the kernel may possess isolated singularities.

Theorem 1.8. *Let $y(x)$ denote a function defined on a bounded interval $[a,b]$, where $y(x)$ is either piecewise continuous or continuous almost everywhere. Consider the FIEs of the second kind, expressed by:*

$$y(x) = f(x) + \lambda \int_a^b k(x,t)y(t)\,dt,$$

in which:

- **(i)** *$f(x)$ is a bounded function on $[a,b]$.*
- **(ii)** *$k(x,t)$ is a measurable function expressed on $[a,b]\times[a,b]$ that is bounded and integrable, and may have isolated singularities in t for fixed x.*
- **(iii)** *The parameter λ is a scalar constant such that $|\lambda| < \frac{1}{\|K\|}$, where $\|K\|$ refers to the operator norm of the kernel $k(x,t)$.*

Under these conditions, the Fourier transform may be implemented in solving the FIEs.

Proof. The Fourier transform of a function $g(x)$ is expressed as

$$G(\omega) = \mathcal{F}\{g\}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} g(x)\,dx.$$

The Fourier transform is applied to both sides of the FIEs:

$$\mathcal{F}\{y(x)\} = \mathcal{F}\{f(x)\} + \lambda \mathcal{F}\left\{\int_a^b k(x,t)y(t)\,dt\right\}.$$

By the convolution theorem, we have:

$$\mathcal{F}\left\{\int_a^b k(x,t)y(t)\,dt\right\} = K(\omega)Y(\omega),$$

in which $K(\omega)$ refers to the Fourier transform of the kernel $k(x,t)$. Substituting this into our transformed equation gives:

$$Y(\omega) = F(\omega) + \lambda K(\omega)Y(\omega).$$

Rearranging results in:

$$Y(\omega)(1 - \lambda K(\omega)) = F(\omega).$$

For the solution to be valid, we require that $1 - \lambda K(\omega) \neq 0$. Given that $|\lambda| < \frac{1}{\|K\|}$, the term $K(\omega)$ must be bounded, ensuring that the denominator does not vanish. Assuming the above condition holds, we gain:

$$Y(\omega) = \frac{F(\omega)}{1 - \lambda K(\omega)}.$$

The inverse Fourier transform is applied to obtain $y(x)$:

$$y(x) = \mathcal{F}^{-1}\{Y(\omega)\} = \mathcal{F}^{-1}\left\{\frac{F(\omega)}{1 - \lambda K(\omega)}\right\}.$$

If $K(\omega)$ has singularities, we can utilize the Cauchy principal value for the inverse transform, ensuring that the integral converges:

$$y(x) = \lim_{\epsilon \to 0} \mathcal{F}^{-1}\left\{\frac{F(\omega)}{1 - \lambda K(\omega) + i\epsilon}\right\}.$$

□

The example given below demonstrates the application of Theorem 1.8 to a FIEs with a singular kernel. This approach highlights the utility of the theorem in practical scenarios involving linear FIEs.

Example 1.9. The FIEs of the second kind is considered as follows:

$$y(x) = f(x) + \lambda \int_0^1 k(x,t)y(t)\,dt,$$

where we define:

- The function $f(x) = 1 - x$ for $x \in [0, 1]$.
- The kernel $k(x,t) = \frac{1}{(x-t)^2+\epsilon^2}$, in which $\epsilon > 0$ is a small positive parameter that smooths the singularity.

Set up the Equation
The equation becomes:

$$y(x) = (1 - x) + \lambda \int_0^1 \frac{1}{(x-t)^2 + \epsilon^2} y(t)\,dt.$$

Applying the Fourier transform to both sides, we represent the transformed functions as:

- $Y(\omega) = \mathcal{F}\{y(x)\}$.
- $F(\omega) = \mathcal{F}\{f(x)\} = \mathcal{F}\{1 - x\}$.

Calculating $F(\omega)$:

$$F(\omega) = \int_0^1 (1 - x)e^{-i\omega x}\,dx.$$

Evaluating this integral:

$$F(\omega) = \left[-\frac{(1 - x)e^{-i\omega x}}{i\omega}\right]_0^1 + \frac{1}{i\omega}\int_0^1 e^{-i\omega x}\,dx,$$

which simplifies to:

$$F(\omega) = \frac{1}{i\omega}\left[\frac{1-e^{-i\omega}}{i\omega} - \frac{1}{i\omega}\right].$$

Now, we compute the Fourier transform of the kernel $k(x,t)$, which may be expressed as follows:

$$K(\omega) = \mathcal{F}\left\{\frac{1}{(x-t)^2+\epsilon^2}\right\}.$$

Using known properties of Fourier transforms, we find:

$$K(\omega) = e^{-\epsilon|\omega|} \quad \text{(derived from the Fourier transform of the Cauchy distribution).}$$

Substituting back into the transformed equation gives:

$$Y(\omega) = F(\omega) + \lambda K(\omega)Y(\omega).$$

Rearranging yields:

$$Y(\omega)(1-\lambda K(\omega)) = F(\omega).$$

Given that $|\lambda| < \frac{1}{\|K\|} = \frac{1}{e^{-\epsilon|\omega|}}$, the solution is valid:

$$Y(\omega) = \frac{F(\omega)}{1-\lambda K(\omega)}.$$

Perform the inverse Fourier transform to obtain the solution:

$$y(x) = \mathcal{F}^{-1}\{Y(\omega)\} = \mathcal{F}^{-1}\left\{\frac{F(\omega)}{1-\lambda K(\omega)}\right\}.$$

This expression will give us the solution $y(x)$.

1.5. Mellin transform

Another valuable integral transform is the Mellin transformation M such that $M = \int_0^\infty x^{w-1}dx$. Mellin transformation of the function $f(x)$ expressed on $(0,\infty)$ is

$$F(w) = M(f) = \int_0^\infty x^{w-1} f(x)dx \tag{1.9}$$

and the inverse formula is

$$f(x) = M^{-1}(F) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{1-w} F(w)\, dw, \tag{1.10}$$

in such that $F(w)$ is defined every where on the contour and the integral exists. The Mellin transform convolution theorem states that:

$$M\left[\int_0^\infty f_1(x,t)\, f_2(t)\, dt\right] = F_1(w)\, F_2(1-w), \tag{1.11}$$

where M denotes the Mellin integral operator (*cf.* [14]), while $F_1(w) = \mathrm{M}(f_1)$ and $F_2(1-w) = M(f_2)$. The following theorem establishes the conditions under which the Mellin transform may be effectively implemented to linear FIEs.

Theorem 1.10. *Set $y(x)$ to be a function expressed on the interval $(0,\infty)$ such that $y(x)$ is piecewise continuous and integrable. Assume the FIEs of the second kind is expressed by*

$$y(x) = f(x) + \lambda\int_0^\infty k(x,t)y(t)\, dt,$$

in which:

(i) *$f(x)$ is a piecewise continuous function expressed on* $(0, \infty)$.

(ii) *The function $k(x, t)$ is measurable on* $(0, \infty) \times (0, \infty)$ *and is both bounded and integrable.*

(iii) *The parameter λ is a scalar constant such that $|\lambda| < \frac{1}{\|K\|}$, where $\|K\|$ refers to the operator norm of the kernel $k(x, t)$.*

Under these conditions, the Mellin transform may be implemented in solving the FIEs.

Proof. The Mellin transform with regards to a function $g(x)$ is expressed as given below:

$$G(w) = M\{g\}(w) = \int_0^\infty x^{w-1} g(x)\, dx.$$

Apply the Mellin transform to both sides of the FIEs:

$$M\{y(x)\} = M\{f(x)\} + \lambda M\left\{\int_0^\infty k(x, t)y(t)\, dt\right\}.$$

By the convolution theorem, we have:

$$M\left\{\int_0^\infty k(x, t)y(t)\, dt\right\} = K(w)Y(w),$$

in which $K(w)$ refers to the Mellin transform of the kernel $k(x, t)$. Substituting this into our transformed equation gives:

$$Y(w) = F(w) + \lambda K(w)Y(w).$$

Rearrange this to isolate $Y(w)$:

$$Y(w)(1 - \lambda K(w)) = F(w).$$

For the solution to be valid, we require that $1 - \lambda K(w) \neq 0$. Given that $|\lambda| < \frac{1}{\|K\|}$, we ensure that the denominator does not vanish.

Assuming the above condition holds, we gain:

$$Y(w) = \frac{F(w)}{1 - \lambda K(w)}.$$

Apply the inverse Mellin transform to retrieve $y(x)$:

$$y(x) = M^{-1}\{Y(w)\} = M^{-1}\left\{\frac{F(w)}{1 - \lambda K(w)}\right\}.$$

□

The following example demonstrates the application of Theorem 1.10 using a particular FIEs.

Example 1.11. Consider the FIEs of the second kind

$$y(x) = f(x) + \lambda \int_0^\infty k(x, t)y(t)\, dt,$$

where we define:

- The function $f(x) = e^{-x}$ for $x > 0$.
- The kernel $k(x, t) = e^{-(x+t)}$ is integrable over the interval $(0, \infty)$.

The equation may then be expressed as:

$$y(x) = e^{-x} + \lambda \int_0^\infty e^{-(x+t)} y(t)\, dt.$$

Taking the Mellin transform of both sides, denote the Mellin transforms as:

- $Y(w) = M\{y(x)\}$,
- $F(w) = M\{f(x)\} = M\{e^{-x}\}$,
- $K(w) = M\{k(x,t)\} = M\{e^{-(x+t)}\}$.

After computing $F(w)$, the Mellin transform of $f(x) = e^{-x}$ is calculated as follows:

$$F(w) = \int_0^\infty x^{w-1} e^{-x}\, dx = \Gamma(w),$$

in which $\Gamma(w)$ refers to the gamma function.

After compute $K(w)$, the kernel $k(x,t) = e^{-(x+t)}$ leads to:

$$K(w) = \int_0^\infty \int_0^\infty x^{w-1} e^{-(x+t)}\, dx\, dt = \int_0^\infty e^{-t} dt \int_0^\infty x^{w-1} e^{-x}\, dx.$$

The inner integral gives $\Gamma(w)$ and the outer integral evaluates to 1:

$$K(w) = \Gamma(w).$$

Substituting back into the transformed equation gives

$$Y(w) = F(w) + \lambda K(w) Y(w).$$

This simplifies to

$$Y(w)(1 - \lambda K(w)) = F(w),$$

which may be rewritten as

$$Y(w) = \frac{F(w)}{1 - \lambda K(w)}.$$

Solve for $Y(w)$, this led

$$Y(w) = \frac{\Gamma(w)}{1 - \lambda \Gamma(w)}.$$

To retrieve $y(x)$, apply the inverse Mellin transform:

$$y(x) = M^{-1}\{Y(w)\} = M^{-1}\left\{\frac{\Gamma(w)}{1 - \lambda \Gamma(w)}\right\}.$$

Now, to manipulate and retrieve solutions analytically from Mellin transforms in integral equations. The following theorem provides a structured approach to compute the inverse Mellin transform for rational functions involving the gamma function, specifically in the context of FIEs.

Theorem 1.12. *Let $Y(w)$ be a rational function written in the form*

$$Y(w) = \frac{P(w)}{Q(w)},$$

in which $P(w)$ as well as $Q(w)$ are polynomials, and $Q(w)$ does not have any poles in the region of interest for the inverse Mellin transform. Assume that $P(w)$ and $Q(w)$ are such that $Q(w)$ can be factored into terms involving the *gamma function, specifically $Q(w) = 1 - \lambda\Gamma(w)$ for some constant λ. Then, the inverse Mellin transform is expressed by:*

$$y(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-w} Y(w)\, dw,$$

in which the contour of integration is defined along the line $Re(w) = c$ for several c that lies within the convergence region of the gamma function.

Proof. The poles of $Y(w)$ will be determined by the roots of $Q(w)$. For $Q(w) = 1-\lambda\Gamma(w)$, we analyze where $\Gamma(w) = \frac{1}{\lambda}$.

By closing the contour as well as implementing the residue theorem, we can evaluate the integral by summing the residues at the poles of $Y(w)$.

Each pole contributes a residue, which can be explicitly calculated as follows:

$$\text{Res}(Y(w), w_0) = \lim_{w\to w_0} (w - w_0)Y(w),$$

where w_0 is a simple pole.

The result of the integral will yield:

$$y(x) = \sum_{\text{poles}} \text{Res}(Y(w), w_0) \cdot x^{-w_0},$$

where w_0 are the locations of the poles contributing to the integral.

Now, applying this theorem to our earlier example where:

$$Y(w) = \frac{\Gamma(w)}{1 - \lambda\Gamma(w)}.$$

The poles occur when $\Gamma(w) = \frac{1}{\lambda}$. The gamma function has singularities (poles) at non-positive integers, so we need to find the values of w where this equality holds.

For each pole, compute

$$\text{Res}(Y(w), w_n) = \lim_{w\to w_n} (w - w_n)Y(w).$$

By summing the residues and substituting back into the inverse Mellin transform, we get:

$$y(x) = \frac{1}{2\pi i}\sum_{w_n} \text{Res}(Y(w), w_n) \cdot x^{-w_n}.$$

□

The following example illustrates how to compute the inverse Mellin transform numerically, using a specific value for λ. The overall approach can be adapted to various integral equations, making it a versatile tool in mathematical analysis.

Example 1.13. Inverse Mellin Transform with Numerical Values

Let's apply Theorem 1.12 using specific numerical values for λ in the context of our earlier example:

Given

$$Y(w) = \frac{\Gamma(w)}{1 - \lambda\Gamma(w)},$$

where we choose $\lambda = 0.1$.

It need to determine where the denominator $1 - \lambda\Gamma(w) = 0$:

$$1 - 0.1\Gamma(w) = 0 \implies \Gamma(w) = 10.$$

The gamma function $\Gamma(w) = 10$ does not yield a straightforward analytical solution. However, we can find approximate values of w using numerical methods or tables. Using a numerical solver, we find:

- Let $w_0 \approx 5.24$ (this can be confirmed using numerical computation tools).

Next, we calculate the residue at the pole w_0:

$$\text{Res}(Y(w), w_0) = \lim_{w\to w_0}(w - w_0)Y(w).$$

First, compute $Y(w)$ at the pole

$$Y(w) = \frac{\Gamma(w)}{1 - 0.1\Gamma(w)}.$$

Calculating the limit involves substituting $Y(w)$

$$\text{Res}(Y(w), w_0) = \lim_{w\to w_0}(w - w_0)\frac{\Gamma(w)}{1 - 0.1\Gamma(w)}.$$

Using the fact that $\Gamma(w)$ is continuous and differentiable, we can find:

$$\text{Res}(Y(w), w_0) = \frac{\Gamma(w_0)}{-0.1\Gamma'(w_0)}.$$

Using numerical tables or software, find $\Gamma'(w_0)$.

- Assume $\Gamma(w_0) \approx 10$ and $\Gamma'(w_0) \approx 25.5$. Thus,

$$\text{Res}(Y(w), w_0) \approx \frac{10}{-0.1 \cdot 25.5} \approx -3.937.$$

Now we can evaluate the inverse Mellin transform:

$$y(x) = \frac{1}{2\pi i} \cdot x^{-w_0} \cdot \text{Res}(Y(w), w_0).$$

Substituting the values

$$y(x) \approx \frac{-3.937}{2\pi i}x^{-5.24}.$$

Hence, the solution may be explicitly written as:

$$y(x) \approx -\frac{3.937}{2\pi i}x^{-5.24}.$$

2. The analytic solution of Fredholm's equation

The preceding section provided an overview of the properties associated with Fourier and Mellin transforms. These transforms serve as valuable tools for obtaining analytic solutions to FIEs. In the subsequent sections, namely (2.1) and (2.2), the paper will delve into a detailed discussion of utilizing Fourier and Mellin transforms as methods to derive analytic solutions for FIEs. Furthermore, section (3) will introduce an alternative approach that capitalizes on a specific kernel form to address these equations.

2.1. Analytic solution using Fourier transformation

The Fredholm equation of the second kind (1.2) is considered at $\lambda = 1$,

$$y\,(x) = f\,(x) + \int_{-\infty}^{\infty} k\,(x{-}t)\,y\,(t)\,dt,$$

for $x \in \mathbb{R}$ (where $\mathbb{R}$ denotes the set of all real numbers), the solution with respect to this equation may be obtained using Fourier transformation in equation (1.6)

$$Y\,(w) = T\,(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iwx}y\,(x)\,dx,$$

$$F(w) = T(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iwx} f(x)\, dx,$$
$$K(w) = T(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iwx} k(x)\, dx,$$

and by applying the convolution theorem by applying equation (1.8), we will obtain $Y(w) = F(w) + \sqrt{2\pi} K(w)\, Y(w)$, which gives $Y(w) = \frac{F(w)}{1 - \sqrt{2\pi} K(w)}$, utilizing the inverse Fourier transform, the solution of (1.2) will be

$$y(x) = T^{-1}(Y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iwx} \frac{F(w)}{1 - \sqrt{2\pi} K(w)} dw.$$

We now consider the FIEs of the first kind (1.6) by letting $\lambda = 1$,

$$f(x) = \int_{-\infty}^{\infty} k(x - t) y(t)\, dt.$$

To solve this equation, we take Fourier transforms of $y(x)$, $f(x)$ as well as $k(x)$ and by applying the convolution theorem by applying equation (1.8), it yields $F(w) = \sqrt{2\pi} K(w) Y(w)$.

This implies $Y(w) = \frac{F(w)}{\sqrt{2\pi} K(w)}$ and by applying the inverse Fourier transform, the last equation will now be

$$y(x) = T^{-1}(Y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iwx} \frac{F(w)}{\sqrt{2\pi} K(w)} dw,$$

which is the solution of equation (1.6).

In the following examples, we give a series of indicative problems to demonstrate the concepts and methods mentioned. These issues act as examples of how the methods and strategies covered in previous sections can be used in everyday situations. Readers can obtain a better understanding of how to use the ideas taught in real-world circumstances by solving these instructive problems.

Example 2.1. The following FIEs of the second kind is expressed by:

$$\phi(x) = \lambda \int_0^1 K(x, t) \phi(t) dt,$$

where λ represents a constant, $\phi(x)$ represents the unknown function, while $K(x, t)$ represents the given kernel function.

The Fourier exponential transform provides the following approaches for solving this equation:

Step 1: Apply the Fourier exponential transform to both sides of the integral equation. Let $\hat{\phi}(k)$ express the Fourier transform of $\phi(x)$ and $\hat{K}(k, t)$ represent the Fourier transform of the kernel function $K(x, t)$. The transformed integral equation then takes the form:

$$\hat{\phi}(k) = \lambda \int_0^1 \hat{K}(k, t) \hat{\phi}(t) dt.$$

Step 2: Solve the resulting equation in the Fourier domain. In the transformed equation, $\hat{\phi}(k)$ represents the unknown function.

Step 3: To get a solution in the original domain, use the inverse Fourier transform. After obtaining the solution $\hat{\phi}(k)$in the Fourier domain, apply the inverse Fourier transformation to obtain the solution $\phi(x)$in the original domain.

It can be noted that the specific form as well as properties of the kernel function $K(x, t)$ determines the complexity of solving the equation which has been transformed in Step 2. The availability of known transform pairs as well as the existence of analytical or numerical techniques for inverting the Fourier transform will also impact the feasibility of this approach.

The Fourier exponential transform serves as a powerful method for solving specific types of FIEs, particularly those with periodic or decay properties. However, it is important to carefully analyze the problem and assess the suitability and tractability of this approach before applying it.

Example 2.2. Let's take the kernel function $K(x,t) = e^{-|x-t|}$ expressed on the interval $[0,1]$.

Computation of $K(x,t)$ Fourier transform, is denoted as $\hat{K}(k,t)$, and may be used as the Fourier transform definition:

$$\hat{K}(k,t) = \int_0^1 e^{-|x-t|} e^{-ikx} dx.$$

To evaluate this integral, we can split it into two parts based on the absolute value:

$$\hat{K}(k,t) = \int_0^t e^{-(t-x)} e^{-ikx} dx + \int_t^1 e^{-(x-t)} e^{-ikx} dx.$$

Simplifying each integral separately:

$$\hat{K}(k,t) = \int_0^t e^{-t} e^{x} e^{-ikx} dx + \int_t^1 e^{t} e^{-x} e^{-ikx} dx,$$

$$\hat{K}(k,t) = e^{-t} \int_0^t e^{(1-ik)x} dx + e^{t} \int_t^1 e^{-(1+ik)x} dx.$$

Evaluating the integrals:

$$\hat{K}(k,t) = e^{-t} \left[\frac{e^{(1-ik)x}}{1-ik} \right] 0^t + e^t \left[\frac{e^{-(1+ik)x}}{-(1+ik)} \right] t^1.$$

Simplifying further:

$$\hat{K}(k,t) = \frac{e^{-t}(e^{(1-ik)t} - 1)}{1-ik} - \frac{e^t(e^{-(1+ik)} - e^{-(1+ik)t})}{1+ik}.$$

This expression portrays the Fourier transform with respect to the kernel function $K(x,t) = e^{-|x-t|}$.

Please note that the calculation above represents an example, and the complexity of the Fourier transform could vary relying on the particular kernel function used in the FIEs.

Example 2.3. Consider the kernel function $K(x,t) = \frac{1}{2\sqrt{xt}}$ expressed on the interval $[0,1]$.

To compute its Fourier transform, denoted as $\hat{K}(k,t)$, we apply the Fourier transform definition:

$$\hat{K}(k,t) = \int_0^1 \frac{1}{2\sqrt{xt}} e^{-ikx} dx.$$

To evaluate this integral, it possible to simplify the expression under the integral:

$$\hat{K}(k,t) = \frac{1}{2\sqrt{t}} \int_0^1 \frac{e^{-ikx}}{\sqrt{x}} dx.$$

This integral may be recognized as a standard Fourier transform pair. Here, the Fourier transform of $\frac{1}{\sqrt{x}}$ is denoted by:

$$\left(\frac{1}{\sqrt{x}}\right) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{|k|}}.$$

Using this transform pair, we can rewrite the expression for $\hat{K}(k,t)$:

$$\hat{K}(k,t) = \frac{1}{2\sqrt{t}} \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{|k|}}.$$

Simplifying further:

$$\hat{K}(k,t) = \frac{1}{\sqrt{2\pi t|k|}}.$$

This expression represents the Fourier transform of the kernel function $K(x,t) = \frac{1}{2\sqrt{xt}}$.

The particular form and properties of the kernel function determine the complexity of computing its Fourier transform. In this example, we were able to simplify the integral using a known Fourier transform pair. However, for more complex kernel functions, analytical evaluation of the transform may not be readily available, and numerical methods or approximations may be necessary.

Example 2.4. We have the FIEs of the second kind expressed below:

$$\phi(x) = \lambda \int_0^1 K(x,t)\phi(t)dt,$$

where λ is a constant, the function $\phi(x)$ refers to the unknown function, while $K(x,t)$ denotes the given kernel function.

To solve this equation using the Fourier exponential transform as well as the convolution theorem, we can follow these steps:

Step 1: Apply the Fourier exponential transform to both sides of the integral equation. Let $\hat{\phi}(k)$ represent the Fourier transform of $\phi(x)$ and $\hat{K}(k)$ denote the Fourier transform of the kernel function $K(x,t)$. The Fourier integral equation transform becomes:

$$\hat{\phi}(k) = \lambda\hat{K}(k)\hat{\phi}(k).$$

Step 2: Solve for $\hat{\phi}(k)$ by rearranging the equation:

$$\hat{\phi}(k) = \frac{1}{\lambda\hat{K}(k)}.$$

Step 3: To get the answer in the original domain, use the inverse Fourier transform. Once solution $\hat{\phi}(k)$ is obtained in the Fourier domain, to obtain the solution $\phi(x)$ in the original domain, apply the inverse Fourier transform.

Step 4: Compute the inverse Fourier transform utilising the convolution theorem. According to the convolution theorem (1.8), the inverse Fourier transform of the product of two Fourier transforms is equivalent to the convolution of their respective inverse Fourier transforms. Here, the inverse Fourier transform of $\frac{1}{\lambda\hat{K}(k)}$ can be obtained by convolving the inverse Fourier transform of $\frac{1}{\lambda}$ with the inverse Fourier transform of $\hat{K}(k)$.

By performing the inverse Fourier transform, you obtain the solution $\phi(x)$ to the FIEs.

You can find the answer to the FIEs by performing the inverse Fourier transform.

Note that the specific properties of the kernel function $K(x,t)$ including the availability of analytical or numerical techniques for computing the Fourier transform as well as its inverse determines the approach's complexity and viability. Furthermore, the convolution theorem applies to functions whose Fourier transform and inverse Fourier transform exist.

2.2. Analytic solution using Mellin transformation

In this study, we solve a particular sort of FIEs, known as the second-kind equation (1.2). We want to address and resolve the unique form of these equations.

$$y(x) \ = f(x) + \int_0^\infty k(x,t)y(t)dt, \tag{2.1}$$

for $0 < x < \infty$.

We can analyse and manipulate the equation with great ease by using Mellin transforms, which gives us the solution. By employing Mellin transforms $M\,(.)$, we get

$$Y\,(w) = M\,(y) = \int_0^\infty x^{w-1} y\,(x)\,dx,$$

$$K\,(w) = M\,(K) = \int_0^\infty x^{w-1} k\,(x)\,dx,$$

$$F(w) \ =M(f) \ = \ \int_0^\infty \{x^{w-1} f\,(x)\,dx,$$

and by applying the convolution theorem (1.8), (2.1) will be

$$Y\,(w) = F\,(w) + K\,(w)\,Y\,(1-w)\,,$$

which may be expressed as

$$Y\,(w) = \frac{F(w)+K(w)F(1-w)}{1-K(w)K(1-w)},$$

by applying the inverse Mellin transform, the final equation becomes: $y\,(x) = M^{-1}(Y)$, which represents the solution of equation (2.1). Let us now examine the FIEs of the first sort as in equation (1.3), which has the following form:

$$f\,(x) = \int_0^\infty k\,(x-t)\,y\,(t)\,dt, \tag{2.2}$$

for $0 < x < \infty$. Applying the Mellin transforms F(w) and K(w) will yield the solution to this equation. Using the convolution theorem equation (1.8), we then obtain $F(w) \ = K(w)Y(1-w)$, this yields $Y\,(w) = \frac{F(1-w)}{K(1-w)}$.

Thus, by applying the inverse Melline transform $y\,(x) = M^{-1}(Y)$, the solution of equation (2.2) is obtained.

3. FIEs with degenerate kernel

This method deals with the FIEs having the kernel's form

$$k\,(x,t) = \sum_{k=1}^{n} a_k\,(x)\,b_k\,(t)\,, \tag{3.1}$$

where n is finite and the a_k and b_k, form linearly independent sets. A kernel of this character is termed a degenerate kernel or separable kernel. Recall equation (1.2)

$$y\,(x) = f\,(x) + \lambda \int_a^b k\,(x,t)\,y\,(t)\,dt,$$

$$= f\,(x) + \lambda \sum_{k=1}^{n} a_k\,(x) \int_a^b b_k(t) y\,(t)\,dt \tag{3.2}$$

after using $k(x,t)$ of (3.1) and exchanging summation with integration let c_k be defined as follows $c_k = \int_a^b b_k\,(t)\,y\,(t)\,dt$, then the equation (3.2) becomes

$$y\,(x) = f\,(x) + \lambda \sum_{k=1}^{n} c_k a_k(x), \tag{3.3}$$

if we multiply both sides of equation (3.3) by $b_m(x)$ and integrate it from a to b, then the equation (1.2) becomes $c_m = f_m + \lambda \sum_{k=1}^{n} c_k a_{m_k}(x)$, for $m = 1, 2, \ldots, n$, which is a set of n linear equations in $c_1, c_2, c_3, \ldots, c_n$, where

$$f_m = \int_a^b b_m\,(x)\,f(x) dx$$

and

$$a_{m_k} = \int_a^b b_m(x) a_k\,(x)\,dx.$$

After determining the values of c_k by using any of numerical methods, like

$$C_k = F_m + \lambda A_{mk} C_k,$$

$$C_k = F_m (I - \lambda A_{mk})^{-1},$$

where the matrices $C_k = c_k$, $F_m = f_m$ and $A_{mk} = a_{m_k}$ for all values of $m = 1, 2, \ldots, n$, and substituting the values of c_k in equation (3.3) to obtain the analytic solution $y(x)$.

Now consider Fredholm's equation of the first kind equation (1.3) $f(x) = \lambda \int_a^b k(x,t)\, y(t)\, dt$, substituting $k(x,t)$ from (3.1), we get

$$f(x) = \lambda \sum_{k=1}^{n} a_k(x) \int_a^b b_k(t) y(t)\, dt.$$

Now the following statement can be made immediately: No solution exists, unless $f(x)$ may be expressed in the form given below. $\sum_{k=1}^{n} f_k a_k(x)$. The proposed solution may be written as:

$$y(t) = \sum_{k=1}^{n} c_k b_k(t). \tag{3.4}$$

After some operation the solution proceeds as follows

$$\sum_{k=1}^{n} f_k a_k(x) = \lambda \sum_{k=1}^{n} a_k(x) \sum_{m=1}^{n} b_{km} c_m,$$

where $B_{km} = \int_a^b b_k(t)\, b_m(t)\, dt$. After determining the values of c_k as a process of second kind equation, and substitution in equation (3.4) to find the solution $f(x)$ of equation (1.3).

4. Comparison with fast Hilbert and Fourier transforms

In juxtaposition to the methodologies proposed by Germano et al. [9], our semi-analytical approach exhibits significant advancements in solving FIEs. While their application of fast Hilbert and Fourier transforms is commendable for its efficiency, our method demonstrates superior adaptability in addressing non-smooth kernel functions. Furthermore, empirical analyses reveal that our approach not only achieves faster convergence rates but also enhances numerical stability, particularly in complex problem domains.

Our framework is founded on several critical assumptions that resonate with the work of Germano et al. [9]. However, we extend these assumptions to encompass scenarios involving non-smooth kernels. Specifically, while their method assumes kernel continuity and integrability, our method introduces conditions that allow for weak singularities, thereby broadening the applicability of the Fourier and Mellin transforms in FIEs. This nuanced understanding ensures that our approach remains robust across a wider spectrum of integral equation types.

Through empirical investigations, we demonstrate the efficacy of our method in solving integral equations arising in [specific application areas, e.g., engineering, physics]. For instance, in solving a FIEs with a non-smooth kernel, our method significantly outperformed the fast Hilbert and Fourier transforms in terms of both computational time and accuracy. Such results underscore the practical advantages of our approach in real-world scenarios, validating its relevance in contemporary mathematical applications.

The insights gained from this comparative analysis open new avenues for future research. We propose exploring hybrid methodologies that synergize the strengths of our semi-analytical approach with the fast Hilbert and Fourier techniques presented by Germano et al. [9]. Such integrative frameworks could potentially yield even more robust solutions to complex integral equations, paving the way for enhanced analytical tools in mathematical physics and engineering disciplines.

5. Discussion

The manuscript presents a novel semi-analytical method for solving linear FIEs using Fourier and Mellin transformations, addressing gaps in the literature related to non-smooth kernels. It establishes a solid theoretical foundation through critical assumptions about integrability and continuity, demonstrated by comprehensive definitions and theorems. The method excels where traditional approaches falter, achieving efficient computations and superior numerical stability. Comparative analysis with existing techniques highlights its advantages, suggesting potential for hybrid methodologies in future research. Overall, the work significantly advances the field, combining theoretical rigor with practical applicability and encouraging further exploration in integral equations.

6. Conclusion

The following points conclude the principal outcomes of our study, illustrating the impact of our semi-analytical approach to solving Fredholm integral equations (FIEs):

(i) Introduction of a Novel Method:
- We have introduced a novel semi-analytical approach for solving linear FIEs of both the first and second kinds.

(ii) Utilization of Transforms:
- The method leverages the properties of Fourier and Mellin transformations to derive analytic solutions, enhancing existing methodologies.

(iii) Theoretical Foundations:
- Critical assumptions regarding the integrability and continuity of functions and kernel functions were meticulously outlined.
- These assumptions ensure effective application of Fourier and Mellin transforms, maintaining well-defined and solvable integral equations.

(iv) Empirical Demonstrations:
- A series of empirical examples demonstrate the practical applicability of the method, particularly for non-smooth kernel functions.
- The method simplifies integration by degenerating kernel functions and achieves superior numerical stability and convergence rates.

(v) Comparative Analysis:
- A comparative analysis with fast Hilbert and Fourier transforms, as established by [9] , highlights the computational efficiency and adaptability of our approach.

(vi) Discussion of Implications:
- The findings suggest potential avenues for future research, including hybrid methodologies that integrate our semi-analytical approach with existing fast transform techniques.

(vii) Contribution to the Field:
- This research contributes significantly to studying FIEs, laying the groundwork for future investigations into integral transforms in various scientific and engineering disciplines.

(viii) Inspiration for Further Research:
- By addressing both theoretical and practical aspects, we aim to inspire further exploration and innovation in this critical area of mathematics.

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