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A note on Hermite-Hadamard type inequalities for functions related to quantum analog derivatives

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Abstract

In the note, based on $(p,q)^{\nu}$-derivative and $(p,q)^{\nu}$-integral, we prove some new Hermite-Hadamard type inequalities for functions related to quantum analog derivatives. The obtained inequalities generalize the corresponding inequalities in the literature.

Keywords: Hermite-Hadamard type inequality, convex function, $(p,q)^{\nu}$-integral

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1. Introduction

Quantum calculus is a form of calculus that doesn't rely on the concept of limits, and q-calculus is one type of quantum calculus, focusing on deriving q-analogous results without using limits (*cf.* [7]). Its appearance extends the study range of classical calculus from differentiable to non-differentiable. Post-quantum calculus, also known as (p,q)-calculus, is an extension of quantum calculus, and it plays a crucial role in natural sciences such as mathematics and physics (*cf.* [3, 6, 11]). In recent years, the q-integral and (p,q)-integral inequalities have always been a hot topic in the research of mathematicians. Especially, q-integral and (p,q)-integral inequalities of Hermite-Hadamard type for convex functions have also attracted extensive attention. Hermite-Hadamard inequality of convex function is described as follows (*cf.* [5, 15]):

Let U be an interval of real numbers and $\rho : U \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function, then

$$\rho\left(\frac{\mu+\nu}{2}\right) \leq \frac{1}{\mu-\nu}\int_{\nu}^{\mu} \rho(\tau)\mathrm{d}\tau \leq \frac{\rho(\mu)+\rho(\nu)}{2}, \tag{1.1}$$

where $\mu, \upsilon \in U$ with $\nu < \mu$.

In [1], Ali et al. applied the concept of q^{ν}-integral to generalize the classic Hermite-Hadamard inequality, and established some q^{ν}-integral inequalities for convex functions which are related to right-hand side of Hermite-Hadamard inequality, given in (1.1). In the note, based on $(p,q)^{\nu}$-derivative and $(p,q)^{\nu}$-integral, we unify and generalize some results in the literature [1]. The first section is called Introduction where we recall the Hermite-Hadamard inequality. The second section is called Preliminaries where we give the definitions of (p,q)-derivative, (p,q)-integers,

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$(p,q)_\mu$-derivative, $(p,q)_\mu$-integral, $(p,q)^\nu$-derivative, $(p,q)^\nu$-integral. We also give a couple of theorems about convex functions, (p,q)-Hölder inequality, and a rather new result given by Kunt et al. [8] about the Hermite-Hadamard type $(p,q)^\nu$-integral inequalities for convex functions in the setting of post-quantum calculus. The last section is called Main results. We first give a important lemma (see Lemma 3.1). Then we prove 6 theorems with new results.

2. Preliminaries

The below q-integers is well-known that in q-calculus or Quantum calculus:

$$[t]_q = \frac{1-q^t}{1-q},$$

where $0 < q < 1$. When $q \to 1$, one has $[t]_q = t$ (*cf.* [7]).

In Post-quantum calculus, the two parameters are considered. The (p,q)-integers is denoted by $[t]_{p,q}$ and is expressed as

$$[t]_{p,q} = \frac{p^t - q^t}{p-q},$$

where $0 < q < p \le 1$ (*cf.* [6, 9]).

Obviously, the relation between $[t]_q$ and $[t]_{p,q}$ is a different only by term p^{t-1}, that is

$$[t]_{p,q} = p^{t-1}[t]_{\frac{q}{p}}.$$

Because there is an additional parameter p in Post-quantum calculus, it can study the properties of functions with a variety of different parameter, and allow more flexibility in adjusting the model to suit different physical situations or computational needs (*cf.* [3]). Thus (p,q)-calculus provides a more general tool and q-calculus is only its special case, which makes (p,q)-calculus have a greater advantage in dealing with a wider range of mathematical objects.

Next, we introduce the relevant definitions and theorems in (p,q)-calculus.

Definition 2.1 (*cf.* [11])**.** Suppose that $\mu < \nu$, $0 < q < p \le 1$, and the function $\rho : [\mu,\nu] \subset \mathbb{R} \to \mathbb{R}$ is continuous, then (p,q)-derivative of ρ is given as

$$D_{p,q}\rho(t) = \frac{\rho(pt) - f(qt)}{(p-q)t}, \quad t \neq 0, \ t \in [\mu,\nu]. \tag{2.1}$$

For $t = 0$, we state $D_{p,q}\rho(0) = \lim_{t\to 0} D_{p,q}\rho(t)$ when the limit exists and is finite.

Definition 2.2 (*cf.* [13])**.** Suppose that $\mu < \nu$, $0 < q < p \le 1$, and the function $\rho : [\mu,\nu] \subset \mathbb{R} \to \mathbb{R}$ is continuous, then $(p,q)_\mu$-derivative of ρ is given as

$$_\mu D_{p,q}\rho(t) = \frac{\rho(pt + (1-p)\mu) - \rho(qt + (1-q)\mu)}{(p-q)(t-\mu)}, \quad t \neq \mu, \ t \in [\mu,\nu]. \tag{2.2}$$

For $t = \mu$, we state $_\mu D_{p,q}\rho(\mu) = \lim_{t\to\mu} {}_\mu D_{p,q}\rho(t)$ when the limit exists and is finite.

Definition 2.3 (*cf.* [4])**.** Suppose that $\mu < \nu$, $0 < q < p \le 1$, and the function $\rho : [\mu,\nu] \subset \mathbb{R} \to \mathbb{R}$ is continuous, then $(p,q)^\nu$-derivative of ρ is given as

$$^\nu D_{p,q}\rho(t) = \frac{\rho(qt + (1-q)\nu) - \rho(pt + (1-p)\nu)}{(p-q)(\nu-t)}, \quad t \neq \nu, \ t \in [\mu,\nu]. \tag{2.3}$$

For $t = \nu$, we state $^\nu D_{p,q}\rho(\nu) = \lim_{t\to\nu} {}^\nu D_{p,q}\rho(t)$ when the limit exists and is finite.

Remark 2.4. Taking $p = 1$ in (2.1)-(2.3), then it reduces to the concepts of q-derivative, q_μ-derivative and q^ν-derivative, respectively.

Definition 2.5. Suppose that $\mu < \nu$, $0 < q < p \leq 1$, and the function $\rho : [\mu, \nu] \subset \mathbb{R} \to \mathbb{R}$ is continuous, then the second $(p, q)^{\nu}$-derivative of ρ is given as

$$\begin{aligned} {}^{\nu}D^2_{p,q}\rho(t) &= {}^{\nu} D_{p,q}({}^{\nu}D_{p,q}\rho(t)) \\ &= \frac{p\rho(q^2t + (1 - q^2)\nu) - (p + q)\rho(pqt + (1 - pq)\nu) + q\rho(p^2t + (1 - p^2)\nu)}{pq(p - q)^2(\nu - t)^2}, \quad t \neq \nu, t \in [\mu, \nu]. \end{aligned} \tag{2.4}$$

For $t = \nu$, we state ${}^{\nu}D^2_{p,q}\rho(\nu) = \lim_{t\to\nu} {}^{\nu}D^2_{p,q}\rho(t)$ when the limit exists and is finite.

Definition 2.6 (*cf.* [11])**.** Suppose that $\rho : [0, c] \subset \mathbb{R} \to \mathbb{R}$ with $c > 0$ is a continuous function, then (p, q)-integral of ρ on $[0, c]$ is stated by

$$\int_0^c \rho(t)\mathrm{d}_{p,q}t = (p - q)c \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}\rho\Big(\frac{q^n}{p^{n+1}}c\Big), \tag{2.5}$$

where $0 < q < p \leq 1$.

Definition 2.7 (*cf.* [13])**.** Suppose that $\mu < \nu$, $0 < q < p \leq 1$, and the function $\rho : [\mu, \nu] \subset \mathbb{R} \to \mathbb{R}$ is continuous, then $(p, q)_{\mu}$-integral of ρ on $[\mu, \nu]$ is given as

$$\begin{aligned} \int_{\mu}^{t} \rho(s)_{\mu}\mathrm{d}_{p,q}s &= (p - q)(t - \mu) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}\rho\Big(\frac{q^n}{p^{n+1}}t + \Big(1 - \frac{q^n}{p^{n+1}}\Big)\mu\Big) \\ &= (t - \mu) \int_0^1 \rho(st + (1 - s)\mu)\mathrm{d}_{p,q}s, \end{aligned} \tag{2.6}$$

where $t \in [\mu, \nu]$.

Definition 2.8 (*cf.* [4])**.** Suppose that $\mu < \nu$, $0 < q < p \leq 1$, and the function $\rho : [\mu, \nu] \subset \mathbb{R} \to \mathbb{R}$ is continuous, then $(p, q)^{\nu}$-integral of ρ on $[\mu, \nu]$ is stated as

$$\begin{aligned} \int_{t}^{\nu} \rho(s)^{\nu}\mathrm{d}_{p,q}s &= (p - q)(\nu - t) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}\rho\Big(\frac{q^n}{p^{n+1}}t + \Big(1 - \frac{q^n}{p^{n+1}}\Big)\nu\Big) \\ &= (\nu - t) \int_0^1 \rho(st + (1 - s)\nu)\mathrm{d}_{p,q}s, \end{aligned} \tag{2.7}$$

where $t \in [\mu, \nu]$.

Remark 2.9. Taking $p = 1$ in (2.5)-(2.7), then it reduces to the concepts of q-integral, q_{μ}-integral and q^{ν}-integral, respectively.

Remark 2.10. When pick $\mu = 0$ and $t = \nu = 1$ in (2.6), it reduces to

$$\int_0^1 \rho(s)_0\mathrm{d}_{p,q}s = (p - q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}\rho\Big(\frac{q^n}{p^{n+1}}\Big).$$

When pick $t = \mu = 0$ and $\nu = 1$ in (2.7), it reduces to

$$\int_0^1 \rho(s)^1\mathrm{d}_{p,q}s = (p - q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}\rho\Big(1 - \frac{q^n}{p^{n+1}}\Big).$$

Example 2.11. For the function $f(t) = t^{\alpha}, \alpha \in \mathbb{R}$, then

$$\int_0^1 t^{\alpha}\mathrm{d}_{p,q}t = \frac{1}{[\alpha + 1]_{p,q}}, \tag{2.8}$$

where $0 < q < p \leq 1$.

Definition 2.12 (*cf.* [10])**.** Suppose that $\mu < \nu$ and $\rho : [\mu, \nu] \subset \mathbb{R} \to \mathbb{R}$ is a real function, if for some $\theta \in [0, 1]$, the following inequality

$$\rho(\theta\mu + (1-\theta)\nu) \leq \theta\rho(\mu) + (1-\theta)\rho(\nu) \tag{2.9}$$

holds, then ρ is said to be a convex function.

Theorem 2.13 (*cf.* [13])**.** *Let the function $\rho : \mathbb{R} \to \mathbb{R}$ be continuous, then the following formulas hold for $t \in \mathbb{R}$:*

(a) $D_{p,q} \int_0^t \rho(\omega)\, d_{p,q}\omega = \rho(t)$;

(b) $\int_0^t D_{p,q}\rho(\omega)\, d_{p,q}\omega = \rho(t)$;

(c) $\int_\mu^t D_{p,q}\rho(\omega)\, d_{p,q}\omega = \rho(t) - \rho(\mu), \quad for \quad \mu \in (0, t).$

Theorem 2.14 (*cf.* [13])**.** *Suppose that $\mu < \nu$ and the functions $\rho, \delta : [\mu, \nu] \subset \mathbb{R} \to \mathbb{R}$ are continuous, then the following formulas hold:*

(a) $\int_\mu^\nu [\rho(\omega) + \delta(\omega)]\, d_{p,q}\omega = \int_\mu^\nu \rho(\omega)\, d_{p,q}\omega + \int_\mu^\nu \delta(\omega)\, d_{p,q}\omega$;

(b) $\int_\mu^\nu \lambda\rho(\omega)\, d_{p,q}\omega = \lambda \int_\mu^\nu \rho(\omega)\, d_{p,q}\omega, \quad for \quad \lambda \in \mathbb{R}$;

(c) $\int_\mu^\nu \rho(q\omega + (1-q)\mu) D_{p,q}\delta(\omega)\, d_{p,q}\omega = \rho\delta(\omega) \mid_\mu^\nu - \int_\mu^\nu \delta(p\omega + (1-p)\mu) D_{p,q}\rho(\omega)\, d_{p,q}\omega$

or

(d) $\int_\mu^\nu \rho(p\omega + (1-p)\mu) D_{p,q}\delta(\omega)\, d_{p,q}\omega = \rho\delta(\omega) \mid_\mu^\nu - \int_\mu^\nu \delta(q\omega + (1-q)\mu) D_{p,q}\rho(\omega)\, d_{p,q}\omega.$

Theorem 2.15 (*cf.* [13] ((p, q)-Hölder inequality))**.** *Let $\mu < \nu$ and $\rho, \delta : [\mu, \nu] \subset \mathbb{R} \to \mathbb{R}$ be two real functions, $0 < q < p \leq 1$, $r_1, p_1 > 1$ with $\frac{1}{r_1} + \frac{1}{p_1} = 1$, then*

$$\int_\mu^\nu \mid \rho(\omega)\delta(\omega) \mid\, d_{p,q}\omega \leq \Big(\int_\mu^\nu \mid \rho(\omega) \mid^{p_1}\, d_{p,q}\omega\Big)^{\frac{1}{p_1}} \Big(\int_\mu^\nu \mid \delta(\omega) \mid^{r_1}\, d_{p,q}\omega\Big)^{\frac{1}{r_1}}.$$

In [8], the following Hermite-Hadamard type $(p, q)_\mu$-integral inequalities for convex function in the setting of post-quantum calculus are obtained by Kunt et al.

Theorem 2.16 (*cf.* [8])**.** *Suppose that $\mu < \nu$ and a differentiable mapping $\rho : [\mu, \nu] \subset \mathbb{R} \to \mathbb{R}$ is convex, $0 < q < p \leq 1$, then*

$$\rho\Big(\frac{q\mu + p\nu}{[2]_{p,q}}\Big) \leq \frac{1}{p(\nu - \mu)} \int_\mu^{p\nu + (1-p)\mu} f(\omega)_\mu\, d_{p,q}\omega \leq \frac{q\rho(\mu) + p\rho(\nu)}{[2]_{p,q}}. \tag{2.10}$$

In [14], the following Hermite-Hadamard type $(p, q)^\nu$-integral inequalities for convex function in the setting of post-quantum calculus are established by Vivas-Cortez et al.

Theorem 2.17 (*cf.* [14])**.** *Suppose that $\mu < \nu$ and a differentiable mapping $\rho : [\mu, \nu] \subset \mathbb{R} \to \mathbb{R}$ is convex, $0 < q < p \leq 1$, then*

$$\rho\Big(\frac{p\mu + q\nu}{[2]_{p,q}}\Big) \leq \frac{1}{p(\nu - \mu)} \int_{p\mu + (1-p)\nu}^{\nu} \rho(\omega)^\nu\, d_{p,q}\omega \leq \frac{p\rho(\mu) + q\rho(\nu)}{[2]_{p,q}}. \tag{2.11}$$

Corollary 2.18 (*cf.* [14])**.** *Suppose that $\mu < \nu$ and a differentiable mapping $\rho : [\mu, \nu] \subset \mathbb{R} \to \mathbb{R}$ is convex, $0 < q < p \leq 1$, then*

$$\rho\Big(\frac{\mu + \nu}{2}\Big) \leq \frac{1}{2p(\mu - \nu)}\Big[\int_\mu^{p\nu + (1-p)\mu} \rho(\omega)_\mu\, d_{p,q}\omega + \int_{p\mu + (1-p)\nu}^{\nu} \rho(\omega)^\nu\, d_{p,q}\omega\Big] \leq \frac{\rho(\mu) + \rho(\nu)}{2}. \tag{2.12}$$

3. Main results

In this section, we will generalize the results with q^{ν}-integral from literature [1] to the ones with $(p,q)^{\nu}$-integral and establish some new Hermite-Hadamard type $(p,q)^{\nu}$-integral inequalities for convex function in the setting of post-quantum calculus.

Firstly, an important lemma will be given.

Lemma 3.1. *Suppose that* $0 < q < p \leq 1$, $\mu < \nu$ *and a mapping* $\rho : [\mu, \nu] \subset \mathbb{R} \to \mathbb{R}$ *is twice* $(p,q)^{\nu}$*-differentiable. If* ${}^{\nu}D^2_{p,q}\rho$ *is a continuous function on* $[\mu,\nu]$*, then one can have*

$$\begin{aligned}\frac{p^2q\rho(\nu)+p^3\rho(p\mu+(1-p)\nu)}{p+q}&-\frac{1}{\mu-\nu}\int_{p^2\mu+(1-p^2)\nu}^{\nu}\rho(\omega)^{\nu}\,\mathrm{d}_{p,q}\omega\\&=\frac{p^2q^2(\nu-\mu)^2}{p+q}\int_0^1 ps(1-qs)^{\nu}D^2_{p,q}\rho(s\mu+(1-s)\nu)\,\mathrm{d}_{p,q}s.\end{aligned}\tag{3.1}$$

Proof. Take $t = s\mu + (1-s)\nu$ in Definition 2.5, then we have

$$\begin{aligned}\int_0^1 ps(1-qs)^{\nu}D^2_{p,q}\rho(s\mu+(1-s)\nu)\mathtt{d}_{p,q}s=\int_0^1&\frac{ps(1-qs)}{pq(p-q)^2(\nu-\mu)^2s^2}[p\rho(q^2s\mu+(1-q^2s)\nu)\\&-(p+q)\rho(pqs\mu+(1-pqs)\nu)+q\rho(p^2s\mu+(1-p^2s)\nu)]\mathtt{d}_{p,q}s.\end{aligned}$$

Let

$$\begin{aligned}I_1\triangleq\int_0^1&\frac{ps}{pq(p-q)^2(\nu-\mu)^2s^2}[p\rho(q^2s\mu+(1-q^2s)\nu)-(p+q)\rho(pqs\mu+(1-pqs)\nu)\\&+q\rho(p^2s\mu+(1-p^2s)\nu)]\mathtt{d}_{p,q}s\end{aligned}$$

and

$$\begin{aligned}I_2\triangleq\int_0^1&\frac{pqs^2}{pq(p-q)^2(\nu-\mu)^2s^2}[p\rho(q^2s\mu+(1-q^2s)\nu)-(p+q)\rho(pqs\mu+(1-pqs)\upsilon)\\&+q\rho(p^2s\mu+(1-p^2s)\upsilon)]\mathtt{d}_{p,q}s,\end{aligned}$$

then

$$\int_0^1 ps(1-qs)^{\nu}D^2_{p,q}\rho(s\mu+(1-s)\nu)\mathtt{d}_{p,q}s=I_1-I_2.\tag{3.2}$$

By direct calculation, we obtain that

$$\begin{aligned}I_1=&p\Big[\frac{\sum_{n=0}^{\infty}f(\frac{q^{n+2}}{p^{n+1}}\mu+(1-\frac{q^{n+2}}{p^{n+1}})\nu)}{q(p-q)(\nu-\mu)^2}-\frac{\sum_{n=0}^{\infty}f(\frac{q^{n+1}}{p^{n}}\mu+(1-\frac{q^{n+1}}{p^{n}})\nu)}{q(p-q)(\nu-\mu)^2}\Big]\\&+q\Big[\frac{\sum_{n=0}^{\infty}f(\frac{q^{n}}{p^{n-1}}\mu+(1-\frac{q^{n}}{p^{n-1}})\nu)}{q(p-q)(\nu-\mu)^2}-\frac{\sum_{n=0}^{\infty}f(\frac{q^{n+1}}{p^{n}}\mu+(1-\frac{q^{n+1}}{p^{n}})\nu)}{q(p-q)(\nu-\mu)^2}\Big]\\=&\frac{p\rho(\nu)-p\rho(q\mu+(1-q)\nu)}{q(p-q)(\nu-\mu)^2}+\frac{\rho(p\mu+(1-p)\nu)-\rho(\nu)}{(p-q)(\nu-\mu)^2}\end{aligned}\tag{3.3}$$

and

$$
\begin{aligned}
I_2 =&\frac{1}{(p-q)^2(\nu-\mu)^2}\Big[\frac{p(p-q)}{q^2}\sum_{n=0}^{\infty}\frac{q^{n+2}}{p^{n+1}}\rho\Big(\frac{q^{n+2}}{p^{n+1}}\mu+\Big(1-\frac{q^{n+2}}{p^{n+1}}\Big)\nu\Big)\\
&-\frac{(p+q)(p-q)}{pq}\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n}}\rho\Big(\frac{q^{n+1}}{p^{n}}\mu+\Big(1-\frac{q^{n+1}}{p^{n}}\Big)\nu\Big)+\frac{q(p-q)}{p^2}\sum_{n=0}^{\infty}\frac{q^{n}}{p^{n-1}}\rho\Big(\frac{q^{n}}{p^{n-1}}\mu+\Big(1-\frac{q^{n}}{p^{n-1}}\Big)\nu\Big)\Big]\\
=&\frac{1}{(p-q)^2(\nu-\mu)^2}\Big[\frac{p(p-q)}{q^2}\sum_{n=0}^{\infty}\frac{q^{n}}{p^{n-1}}\rho\Big(\frac{q^{n}}{p^{n-1}}\mu+\Big(1-\frac{q^{n}}{p^{n-1}}\Big)\nu\Big)\\
&-\frac{p(p-q)}{q^2}(p\rho(p\mu+(1-p)\nu)+q\rho(q\mu+(1-q)\nu))-\frac{(p+q)(p-q)}{pq}\sum_{n=0}^{\infty}\frac{q^{n}}{p^{n-1}}\rho\Big(\frac{q^{n}}{p^{n-1}}\mu+\Big(1-\frac{q^{n}}{p^{n-1}}\Big)\nu\Big)\\
&+\frac{(p+q)(p-q)}{pq}\rho(p\mu+(1-p)\nu)+\frac{q(p-q)}{p^2}\sum_{n=0}^{\infty}\frac{q^{n}}{p^{n-1}}\rho\Big(\frac{q^{n}}{p^{n-1}}\mu+\Big(1-\frac{q^{n}}{p^{n-1}}\Big)\nu\Big)\Big]\\
=&\frac{1}{(p-q)^2(\nu-\mu)^2}\Big[\frac{(p+q)(p-q)^3}{p^2q^2}\sum_{n=0}^{\infty}\frac{q^{n}}{p^{n-1}}\rho\Big(\frac{q^{n}}{p^{n-1}}\mu+\Big(1-\frac{q^{n}}{p^{n-1}}\Big)\nu\Big)\\
&-\frac{p(p-q)}{q}\rho(q\mu+(1-q)\nu)+\frac{[q(p+q)-p^2](p-q)}{q^2}\rho(p\mu+(1-p)\nu)\Big]\\
=&\frac{p+q}{p^2q^2(\nu-\mu)^3}\int_{p^2\mu+(1-p^2)\nu}^{b}\rho(\omega)\,{}^{\nu}\mathrm{d}_{p,q}\omega-\frac{p}{q(p-q)(\nu-\mu)^2}\rho(q\mu+(1-q)\nu)\\
&+\frac{q(p+q)-p^2}{q^2(p-q)(\nu-\mu)^2}\rho(p\mu+(1-p)\nu).
\end{aligned}
\tag{3.4}
$$

Substituting (3.3) and (3.4) into (3.2), one gets

$$
\int_0^1 ps(1-qs)\,{}^{\nu}D_{p,q}^2\rho(s\mu+(1-s)\nu)\mathrm{d}_{p,q}s=\frac{q\rho(\nu)+p\rho(p\mu+(1-p)\nu)}{q^2(\nu-\mu)^2}-\frac{p+q}{p^2q^2(\nu-\mu)^3}\int_{p^2\mu+(1-p^2)\nu}^{\nu}\rho(\omega)\,{}^{\nu}\mathrm{d}_{p,q}\omega. \tag{3.5}
$$

The both sides of (3.5) are multiplied by $\frac{p^2q^2(\nu-\mu)^2}{p+q}$ at the same time and the equation (3.1) is derived. Thus the proof of Lemma 3.1 is over. □

Remark 3.2. In Lemma 3.1, one has the following:

- **(i)** We will get Lemma 1 in [1] when it picks $p = 1$;
- **(ii)** We will get Lemma 1 in [2] when it takes the limit as $p = 1, q \to 1^-$.

Next, by using Lemma 3.1 and considering the convexity of the function, we can obtain the following Theorems with new $(p,q)^{\nu}$-integral inequalities.

Theorem 3.3. *Suppose that* $0 < q < p \leq 1$, $\mu < \nu$ *and a mapping* $\rho : [\mu,\nu] \subset \mathbb{R} \to \mathbb{R}$ *is twice* $(p,q)^{\nu}$*-differentiable. If* ${}^{\nu}D_{p,q}^2\rho$ *is a continuous function and* $|{}^{\nu}D_{p,q}^2\rho|$ *is a convex function on* $[\mu,\nu]$, *then*

$$
\begin{aligned}
&\Big|\frac{p^2q\rho(\nu)+p^3\rho(p\mu+(1-p)\nu)}{p+q}-\frac{1}{\nu-\mu}\int_{p^2\mu+(1-p^2)\nu}^{\nu}\rho(\omega)\,{}^{\nu}\mathrm{d}_{p,q}\omega\Big|\\
&\qquad\leq\frac{p^2q^2(\nu-\mu)^2}{[3]_{p,q}(p+q)^2(p^2+q^2)}\Big[p^4\big|{}^{\nu}D_{p,q}^2\rho(\mu)\big|+(p^5+p^3q^2-p^4)\big|{}^{\nu}D_{p,q}^2\rho(\nu)\big|\Big].
\end{aligned}
\tag{3.6}
$$

Proof. By using the convexity of the function $|{}^{b}D^{2}_{p,q}f|$ and Lemma 3.1, it reduces to

$$\begin{aligned}&\left|\frac{p^2q\rho(\nu)+p^3\rho(p\mu+(1-p)\nu)}{p+q}-\frac{1}{\nu-\mu}\int_{p^2\mu+(1-p^2)\nu}^{\nu}\rho(\omega)\,{}^{\nu}\mathrm{d}_{p,q}\omega\right|\\
&\leq\frac{p^2q^2(\nu-\mu)^2}{p+q}\int_0^1 ps(1-qs)\left|{}^{\nu}D^2_{p,q}\rho(s\mu+(1-s)\nu)\right|\mathrm{d}_{p,q}s\\
&\leq\frac{p^2q^2(\nu-\mu)^2}{p+q}\int_0^1 ps(1-qs)\Big[s\left|{}^{\nu}D^2_{p,q}\rho(\mu)\right|+(1-s)\left|{}^{\nu}D^2_{p,q}\rho(\nu)\right|\Big]\mathrm{d}_{p,q}s\\
&=\frac{p^2q^2(\nu-\mu)^2}{[3]_{p,q}(p+q)^2(p^2+q^2)}\Big[p^4\left|{}^{\nu}D^2_{p,q}\rho(\mu)\right|+(p^5+p^3q^2-p^4)\left|{}^{\nu}D^2_{p,q}\rho(\nu)\right|\Big].\end{aligned}$$

Thus one completes the proof of Theorem 3.3. □

Remark 3.4. In Theorem 3.3, one has the following:

- **(i)** We can obtain Theorem 4 in [1] when it picks $p=1$;
- **(ii)** We can get Proposition 2 in [12] when it takes the limit as $p=1, q\to 1^-$.

Theorem 3.5. *Suppose that* $0<q<p\leq 1$, $\mu<\nu$ *and a mapping* $\rho:[\mu,\nu]\subset\mathbb{R}\to\mathbb{R}$ *is twice* $(p,q)^{\nu}$*-differentiable. If* ${}^{\nu}D^2_{p,q}\rho$ *is a continuous function and* $|{}^{\nu}D^2_{p,q}\rho|^{p_1}$ $(p_1>1)$ *is a convex function on* $[\mu,\nu]$*, then*

$$\begin{aligned}&\left|\frac{p^2q\rho(\nu)+p^3\rho(p\mu+(1-p)\nu)}{p+q}-\frac{1}{\nu-\mu}\int_{p^2\mu+(1-p^2)\nu}^{\nu}\rho(\omega)\,{}^{\nu}d_{p,q}\omega\right|\\
\leq&\frac{p^{5-\frac{3}{p_1}}q^2(\nu-\mu)^2}{[3]_{p,q}(p+q)^2(p^2+q^2)^{1/p_1}}\Big[p^4\left|{}^{\nu}D^2_{p,q}\rho(\mu)\right|^{p_1}+(p^5+p^3q^2-p^4)\left|{}^{\nu}D^2_{p,q}\rho(\nu)\right|^{p_1}\Big]^{1/p_1}.\end{aligned}\tag{3.7}$$

Proof. By using Lemma 3.1 and (p,q)-Hölder's inequality, one can reduce to

$$\begin{aligned}&\left|\frac{p^2q\rho(\nu)+p^3\rho(p\mu+(1-p)\nu)}{p+q}-\frac{1}{\nu-\mu}\int_{p^2\mu+(1-p^2)\nu}^{\nu}\rho(\omega)\,{}^{\nu}\mathrm{d}_{p,q}\omega\right|\\
\leq&\frac{p^2q^2(\nu-\mu)^2}{p+q}\Big(\int_0^1 ps(1-qs)\mathrm{d}_{p,q}s\Big)^{1-\frac{1}{p_1}}\Big(\int_0^1 ps(1-qs)(\left|{}^{\nu}D^2_{p,q}\rho(s\mu+(1-s)\nu)\right|)^{p_1}\mathrm{d}_{p,q}s\Big)^{\frac{1}{p_1}}\\
\leq&\frac{p^2q^2(\nu-\mu)^2}{p+q}\Big(\frac{p^3}{(p+q)(p^2+pq+q^2)}\Big)^{1-\frac{1}{p_1}}\Big(\int_0^1 ps(1-qs)\Big[s\left|{}^{\nu}D^2_{p,q}\rho(\mu)\right|^{p_1}+(1-s)\left|{}^{\nu}D^2_{p,q}\rho(\nu)\right|^{p_1}\Big]\mathrm{d}_{p,q}s\Big)^{\frac{1}{p_1}}\\
=&\frac{p^{5-\frac{3}{p_1}}q^2(\nu-\mu)^2}{[3]_{p,q}(p+q)^2(p^2+q^2)^{1/p_1}}\Big[p^4\left|{}^{\nu}D^2_{p,q}\rho(\mu)\right|^{p_1}+(p^5+p^3q^2-p^4)\left|{}^{\nu}D^2_{p,q}\rho(\nu)\right|^{p_1}\Big]^{1/p_1}.\end{aligned}$$

Thus one completes the proof of Theorem 3.5. □

Remark 3.6. When pick $p=1$ in Theorem 3.5, one has

$$\left|\frac{q\rho(\nu)+\rho(\mu)}{1+q}-\frac{1}{\nu-\mu}\int_{\mu}^{\nu}\rho(\omega)\,{}^{\nu}\mathrm{d}_q\omega\right|\leq\frac{q^2(\nu-\mu)^2}{[3]_q(1+q)^2}\left[\frac{\left|{}^{\nu}D^2_q\rho(\mu)\right|^{p_1}+q^2\left|{}^{\nu}D^2_q\rho(\nu)\right|^{p_1}}{1+q^2}\right]^{1/p_1}.$$

Specially, taking the limit as $q\to 1^-$, it reduces to

$$\left|\frac{\rho(\nu)+\rho(\mu)}{2}-\frac{1}{\nu-\mu}\int_{\mu}^{\nu}\rho(\omega)\mathrm{d}\omega\right|\leq\frac{(\nu-\mu)^2}{12}\left[\frac{\left|D^2\rho(\mu)\right|^{p_1}+\left|D^2\rho(\nu)\right|^{p_1}}{2}\right]^{1/p_1}.$$

Theorem 3.7. *Under the conditions of Theorem* 3.5*, we have the inequality*

$$\left|\frac{p^2q\rho(\nu)+p^3\rho(p\mu+(1-p)\nu)}{p+q}-\frac{1}{\nu-\mu}\int_{p^2\mu+(1-p^2)\nu}^{\nu}\rho(\omega)\,{}^{\nu}\mathrm{d}_{p,q}\omega\right|\\ \leq\frac{p^2q^2(\nu-\mu)^2}{p+q}(\mu_1)^{1-\frac{1}{p_1}}\left[\frac{\left|{}^{\nu}D^2_{p,q}\rho(\mu)\right|^{p_1}+(p+q-1)\left|{}^{\nu}D^2_{p,q}\rho(\nu)\right|^{p_1}}{p+q}\right]^{\frac{1}{p_1}}, \tag{3.8}$$

where

$$\mu_1=(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^n}\left(1-\frac{q^{n+1}}{p^{n+1}}\right)\right)^{\frac{p_1}{p_1-1}}.$$

Proof. By using Lemma 3.1 and (p,q)-Hölder's inequality, one can reduce to

$$\begin{aligned}&\left|\frac{p^2q\rho(\nu)+p^3\rho(p\mu+(1-p)\nu)}{p+q}-\frac{1}{\nu-\mu}\int_{p^2\mu+(1-p^2)\nu}^{\nu}\rho(\omega)\,{}^{\nu}\mathrm{d}_{p,q}\omega\right|\\ \leq&\frac{p^2q^2(\nu-\mu)^2}{p+q}\left(\int_0^1(ps(1-qs))^{\frac{p_1}{p_1-1}}\mathrm{d}_{p,q}s\right)^{1-\frac{1}{p_1}}\left(\int_0^1\left|{}^{\nu}D^2_{p,q}\rho(s\mu+(1-s)\nu)\right|^{p_1}\mathrm{d}_{p,q}s\right)^{\frac{1}{p_1}}\\ \leq&\frac{p^2q^2(\nu-\mu)^2}{p+q}\left(\int_0^1(ps(1-qs))^{\frac{p_1}{p_1-1}}\mathrm{d}_{p,q}s\right)^{1-\frac{1}{p_1}}\left(\left|{}^{\nu}D^2_{p,q}\rho(\mu)\right|^{p_1}\int_0^1 s\mathrm{d}_{p,q}s+\left|{}^{\nu}D^2_{p,q}\rho(\nu)\right|^{p_1}\int_0^1(1-s)\mathrm{d}_{p,q}s\right)^{\frac{1}{p_1}}\\ =&\frac{p^2q^2(\nu-\mu)^2}{p+q}(\mu_1)^{1-\frac{1}{p_1}}\left[\frac{\left|{}^{\nu}D^2_{p,q}\rho(\mu)\right|^{p_1}+(p+q-1)\left|{}^{\nu}D^2_{p,q}\rho(\nu)\right|^{p_1}}{p+q}\right]^{\frac{1}{p_1}}.\end{aligned}$$

Thus one completes the proof of Theorem 3.7. ☐

Remark 3.8. When pick $p=1$ in Theorem 3.7, one has

$$\left|\frac{q\rho(\nu)+\rho(\mu)}{1+q}-\frac{1}{\nu-\mu}\int_{\mu}^{\nu}\rho(\omega)\,{}^{\nu}\mathrm{d}_q\omega\right|\leq\frac{q^2(\nu-\mu)^2}{1+q}\left((1-q)\sum_{n=0}^{\infty}q^n(q^n(1-q^{n+1}))^{\frac{p_1}{p_1-1}}\right)^{1-\frac{1}{p_1}}\left[\frac{\left|{}^{\nu}D^2_q\rho(\mu)\right|^{p_1}+q\left|{}^{\nu}D^2_q\rho(\nu)\right|^{p_1}}{1+q}\right]^{\frac{1}{p_1}}.$$

Specially, taking the limit as $q\to 1^-$, it reduces to

$$\left|\frac{\rho(\nu)+\rho(\mu)}{2}-\frac{1}{\nu-\mu}\int_{\mu}^{\nu}\rho(\omega)\mathrm{d}\omega\right|\leq\frac{(\nu-\mu)^2}{2}\left[\beta\left(2,\frac{2p_1-1}{p_1-1}\right)\right]^{1/p_1}\left[\frac{\left|D^2\rho(\mu)\right|^{p_1}+\left|D^2\rho(\nu)\right|^{p_1}}{2}\right]^{1/p_1},$$

where $\beta(x,y)$ is Beta function defined as

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t.$$

Theorem 3.9. *Under the conditions of Theorem 3.5, we have the inequality*

$$\left|\frac{p^2q\rho(\nu)+p^3\rho(p\mu+(1-p)\nu)}{p+q}-\frac{1}{\nu-\mu}\int_{p^2\mu+(1-p^2)\nu}^{\nu}\rho(\omega)\,{}^{\nu}\mathrm{d}_{p,q}\omega\right|\\ \leq\frac{p^3q^2(\nu-\mu)^2}{(p+q)\left[\frac{2p_1-1}{p_1-1}\right]_{p,q}^{1-1/p_1}}\left[\mu_2\left|{}^{\nu}D^2_{p,q}\rho(\mu)\right|^{p_1}+\mu_3\left|{}^{\nu}D^2_{p,q}\rho(\nu)\right|^{p_1}\right]^{1/p_1}, \tag{3.9}$$

where

$$\mu_2=(p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^2\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^{p_1}$$

and

$$\mu_3=(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^{p_1}.$$

Proof. By using Lemma 3.1 and (p,q)-Hölder's inequality, one can reduce to

$$\begin{aligned}
&\left|\frac{p^2q\rho(\nu)+p^3\rho(p\mu+(1-p)\nu)}{p+q}-\frac{1}{\nu-\mu}\int_{p^2\mu+(1-p^2)\nu}^{\nu}\rho(\omega)^{\nu}\mathtt{d}_{p,q}\omega\right| \\
&\leq\frac{p^2q^2(\nu-\mu)^2}{p+q}\left(\int_0^1(ps)^{\frac{p_1}{p_1-1}}\mathtt{d}_{p,q}s\right)^{1-\frac{1}{p_1}}\left(\int_0^1(1-qs)^{p_1}\left|{}^{\nu}D^2_{p,q}\rho(s\mu+(1-s)\nu)\right|^{p_1}\mathtt{d}_{p,q}s\right)^{\frac{1}{p_1}} \\
&\leq\frac{p^2q^2(\nu-\mu)^2}{p+q}\left(\int_0^1(ps)^{\frac{p_1}{p_1-1}}\mathtt{d}_{p,q}s\right)^{1-\frac{1}{p_1}} \\
&\quad\times\left(\left|{}^{\nu}D^2_{p,q}\rho(\mu)\right|^{p_1}\int_0^1 s(1-qs)^{p_1}\mathtt{d}_{p,q}s+\left|{}^{\nu}D^2_{p,q}\rho(\nu)\right|^{p_1}\int_0^1(1-s)(1-qs)^{p_1}\mathtt{d}_{p,q}s\right)^{\frac{1}{p_1}} \\
&=\frac{p^3q^2(\nu-\mu)^2}{(p+q)[\frac{2p_1-1}{p_1-1}]_{p,q}^{1-1/p_1}}\left[\mu_2\left|{}^{\nu}D^2_{p,q}\rho(\mu)\right|^{p_1}+\mu_3\left|{}^{\nu}D^2_{p,q}\rho(\nu)\right|^{p_1}\right]^{1/p_1}.
\end{aligned}$$

Thus one completes the proof of Theorem 3.9. □

Remark 3.10. When pick $p=1$ in Theorem 3.9, one has

$$\begin{aligned}
&\left|\frac{q\rho(\nu)+\rho(\mu)}{1+q}-\frac{1}{\nu-\mu}\int_{\mu}^{\nu}\rho(\omega)^{\nu}\mathtt{d}_q\omega\right|\leq\frac{q^2(\nu-\mu)^2}{(1+q)[\frac{2p_1-1}{p_1-1}]_q^{1-1/p_1}} \\
&\times\left[(1-q)\sum_{n=0}^{\infty}q^{2n}(1-q^{n+1})^{p_1}\left|{}^{\nu}D^2_q\rho(\mu)\right|^{p_1}+(1-q)\sum_{n=0}^{\infty}q^{n}(1-q^{n})(1-q^{n+1})^{p_1}\left|{}^{\nu}D^2_q\rho(\nu)\right|^{p_1}\right]^{1/p_1}.
\end{aligned}$$

Specially, taking the limit as $q\to 1^-$, it reduces to

$$\left|\frac{\rho(\nu)+\rho(\mu)}{2}-\frac{1}{\nu-\mu}\int_{\mu}^{\nu}\rho(\omega)\mathtt{d}\omega\right|\leq\frac{(\nu-\mu)^2}{2(\frac{2p_1-1}{p_1-1})^{1-1/p_1}}\left[\beta(2,p_1+1)\left|D^2\rho(\mu)\right|^{p_1}+\frac{1}{p_1+2}\left|D^2\rho(\nu)\right|^{p_1}\right]^{1/p_1},$$

where $\beta(x,y)$ is Beta function.

Theorem 3.11. *Under the conditions of Theorem* 3.5, *we have the inequality*

$$\begin{aligned}
&\left|\frac{p^2q\rho(\nu)+p^3\rho(p\mu+(1-p)\nu)}{p+q}-\frac{1}{\nu-\mu}\int_{p^2\mu+(1-p^2)\nu}^{\nu}\rho(\omega)^{\nu}\mathtt{d}_{p,q}\omega\right| \\
&\leq\frac{p^{2+1/p_1}q^2(\nu-\mu)^2}{(p+q)^{1+1/p_1}}(\mu_4)^{1-\frac{1}{p_1}}\left[\frac{p\left|{}^{\nu}D^2_{p,q}\rho(\mu)\right|^{p_1}+q^2\left|{}^{\nu}D^2_{p,q}\rho(\nu)\right|^{p_1}}{[3]_{p,q}}\right]^{\frac{1}{p_1}},
\end{aligned}\tag{3.10}$$

where

$$\mu_4=\frac{p^{\frac{p_1+1}{p_1-1}}}{\left(p^{\frac{p_1}{p_1-1}}+p^{\frac{p_1}{p_1-1}-1}q+\cdots+q^{\frac{p_1}{p_1-1}}\right)\left(p^{\frac{p_1}{p_1-1}+1}+p^{\frac{p_1}{p_1-1}}q+\cdots+q^{\frac{p_1}{p_1-1}+1}\right)}.$$

Proof. By using Lemma 3.1 and (p,q)-Hölder's inequality, one can reduce to

$$\begin{aligned}
&\left|\frac{p^2q\rho(\nu)+p^3\rho(p\mu+(1-p)\nu)}{p+q}-\frac{1}{\nu-\mu}\int_{p^2\mu+(1-p^2)\nu}^{\nu}\rho(\omega)^{\nu}\mathtt{d}_{p,q}\omega\right| \\
&\leq\frac{p^2q^2(\nu-\mu)^2}{p+q}\left(\int_0^1(ps)^{\frac{p_1}{p_1-1}}(1-qs)\mathtt{d}_{p,q}s\right)^{1-\frac{1}{p_1}}\left(\int_0^1(1-qs)\left|{}^{\nu}D^2_{p,q}\rho(s\mu+(1-s)\nu)\right|^{p_1}\mathtt{d}_{p,q}s\right)^{\frac{1}{p_1}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{p^2q^2(\nu-\mu)^2}{p+q}\Big(\int_0^1 (ps)^{\frac{p_1}{p_1-1}}(1-qs)\mathrm{d}_{p,q}s\Big)^{1-\frac{1}{p_1}}\\
&\quad\times\Big(\left|{}^{\nu}D^2_{p,q}\rho(\mu)\right|^{p_1}\int_0^1 s(1-qs)\mathrm{d}_{p,q}s+\left|{}^{\nu}D^2_{p,q}\rho(\nu)\right|^{p_1}\int_0^1(1-s)(1-qs)\mathrm{d}_{p,q}s\Big)^{\frac{1}{p_1}}\\
&=\frac{p^{2+1/p_1}q^2(\nu-\mu)^2}{(p+q)^{1+1/p_1}}(\mu_4)^{1-\frac{1}{p_1}}\Big[\frac{p\left|{}^{\nu}D^2_{p,q}\rho(\mu)\right|^{p_1}+q^2\left|{}^{\nu}D^2_{p,q}\rho(\nu)\right|^{p_1}}{[3]_{p,q}}\Big]^{\frac{1}{p_1}}.
\end{aligned}
$$

Thus one completes the proof of Theorem 3.11. □

Remark 3.12. When pick $p = 1$ in Theorem 3.11, one has

$$
\left|\frac{q\rho(\nu)+\rho(\mu)}{1+q}-\frac{1}{\nu-\mu}\int_\mu^\nu \rho(\omega)^{\nu}\mathrm{d}_q\omega\right| \leq \frac{q^2(\nu-\mu)^2}{(1+q)^{1+1/p_1}}\Big[\frac{(1-q)^2}{(1-q^{\frac{2p_1-1}{p_1-1}})(1-q^{\frac{3p_1-2}{p_1-1}})}\Big]^{1-\frac{1}{p_1}}\Big[\frac{\left|{}^{\nu}D^2_{q}\rho(\mu)\right|^{p_1}+q^2\left|{}^{\nu}D^2_{q}\rho(\nu)\right|^{p_1}}{[3]_{q}}\Big]^{\frac{1}{p_1}}.
$$

Specially, taking the limit as $q \to 1^-$, it reduces to

$$
\left|\frac{\rho(\nu)+\rho(\mu)}{2}-\frac{1}{\nu-\mu}\int_\mu^\nu \rho(\omega)\mathrm{d}\omega\right| \leq \frac{(\nu-\mu)^2}{2^{1+1/p_1}}\Big[\frac{(p_1-1)^2}{(2p_1-1)(3p_1-2)}\Big]^{1-\frac{1}{p_1}}\Big[\frac{\left|D^2\rho(\mu)\right|^{p_1}+\left|D^2\rho(\nu)\right|^{p_1}}{3}\Big]^{\frac{1}{p_1}}.
$$

Theorem 3.13. *Under the conditions of Theorem* 3.5, *we have the inequality*

$$
\begin{aligned}
&\left|\frac{p^2q\rho(\nu)+p^3\rho(p\mu+(1-p)\nu)}{p+q}-\frac{1}{\nu-\mu}\int_{p^2\mu+(1-p^2)\nu}^{\nu}\rho(\omega)^{\nu}\,d_{p,q}\omega\right|\\
&\qquad\leq\frac{p^{3-1/p_1}q^2(\nu-\mu)^2}{(p+q)^{2-1/p_1}}\Big[\mu_5\left|{}^{\nu}D^2_{p,q}\rho(\mu)\right|^{p_1}+\mu_6\left|{}^{\nu}D^2_{p,q}\rho(\nu)\right|^{p_1}\Big]^{\frac{1}{p_1}},
\end{aligned}
\tag{3.11}
$$

where

$$
\mu_5=\frac{p^{2p_1+2}}{(p^{p_1+1}+p^{p_1}q+\cdots+q^{p_1+1})(p^{p_1+2}+p^{p_1+1}q+\cdots+q^{p_1+2})}
$$

and

$$
\begin{aligned}
\mu_6=&\frac{p^{2p_1+1}}{(p^{p_1}+p^{p_1-1}q+\cdots+q^{p_1})(p^{p_1+1}+p^{p_1}q+\cdots+q^{p_1+1})}\\
&-\frac{p^{2p_1+2}}{(p^{p_1+1}+p^{p_1}q+\cdots+q^{p_1+1})(p^{p_1+2}+p^{p_1+1}q+\cdots+q^{p_1+2})}.
\end{aligned}
$$

Proof. By using Lemma 3.1 and (p,q)-Hölder's inequality, one can reduce to

$$
\begin{aligned}
&\left|\frac{p^2q\rho(\nu)+p^3\rho(p\mu+(1-p)\nu)}{p+q}-\frac{1}{\nu-\mu}\int_{p^2\mu+(1-p^2)\nu}^{\nu}\rho(\omega)^{\nu}\mathrm{d}_{p,q}\omega\right|\\
&\leq\frac{p^2q^2(\nu-\mu)^2}{p+q}\Big(\int_0^1(1-qs)\mathrm{d}_{p,q}s\Big)^{1-\frac{1}{p_1}}\Big(\int_0^1(ps)^{p_1}(1-qs)\left|{}^{\nu}D^2_{p,q}\rho(s\mu+(1-s)\nu)\right|^{p_1}\mathrm{d}_{p,q}s\Big)^{\frac{1}{p_1}}\\
&\leq\frac{p^2q^2(\nu-\mu)^2}{p+q}\Big(\int_0^1(1-qs)\mathrm{d}_{p,q}s\Big)^{1-\frac{1}{p_1}}\\
&\quad\times\Big(\left|{}^{\nu}D^2_{p,q}\rho(\mu)\right|^{p_1}\int_0^1 s(ps)^{p_1}(1-qs)\mathrm{d}_{p,q}s+\left|{}^{\nu}D^2_{p,q}\rho(\nu)\right|^{p_1}\int_0^1(1-s)(ps)^{p_1}(1-qs)\mathrm{d}_{p,q}s\Big)^{\frac{1}{p_1}}\\
&=\frac{p^{3-1/p_1}q^2(\nu-\mu)^2}{(p+q)^{2-1/p_1}}\Big[\mu_5\left|{}^{\nu}D^2_{p,q}\rho(\mu)\right|^{p_1}+\mu_6\left|{}^{\nu}D^2_{p,q}\rho(\nu)\right|^{p_1}\Big]^{\frac{1}{p_1}}.
\end{aligned}
$$

Thus one completes the proof of Theorem 3.13. □

Remark 3.14. When pick $p = 1$ in Theorem 3.13, one has

$$\left|\frac{q\rho(\nu)+\rho(\mu)}{1+q}-\frac{1}{\nu-\mu}\int_{\mu}^{\nu}\rho(\omega)\,{}^{\nu}\mathrm{d}_q\omega\right| \leq \frac{q^2(\nu-\mu)^2}{(1+q)^2}\left[\frac{(1-q)^2(1+q)}{(1-q^{p_1+1})(1-q^{p_1+2})(1-q^{p_1+3})}\right]^{\frac{1}{p_1}}$$

$$\times\left[(1-q^{p_1+1})\left|{}^{\nu}D_q^2\rho(\mu)\right|^{p_1}+q^{p_1+1}(1-q^2)\left|{}^{\nu}D_q^2\rho(\nu)\right|^{p_1}\right]^{\frac{1}{p_1}}.$$

Specially, taking the limit as $q \to 1^-$, it reduces to

$$\left|\frac{\rho(\nu)+\rho(\mu)}{2}-\frac{1}{\nu-\mu}\int_{\mu}^{\nu}\rho(\omega)\mathrm{d}\omega\right| \leq \frac{(\nu-\mu)^2}{4}\left[\frac{2(p_1+1)\left|D^2\rho(\mu)\right|^{p_1}+4\left|D^2\rho(\nu)\right|^{p_1}}{(p_1+1)(p_1+2)(p_1+3)}\right]^{1/p_1}.$$

4. Conclusion

In this note, some generalizations are given as in [1] for quantum analog derivatives and integrals. By using the new identity in Lemma, some Hermite-Hadamard type $(p,q)^{\nu}$-integral inequalities are established for convex function in the setting of post-quantum calculus. That refine and extend the results of earlier literatures. It will further enrich the inequalities theory and provide ideals and methods for solving problems.

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