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Muirhead mean and its properties

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Abstract

In this paper, by finding the partial derivatives, the various convexity conditions such as the Schur, Schur geometric, and Schur harmonic convexities (concavities) of Muirhead mean are discussed and new inequalities obtained, as well as strengthening the existing inequality chain by finding the Taylor's series expansions.

Keywords: Muirhead mean, power mean, Schur convexity, Schur geometric convexity, Schur harmonic convexity

2020 MSC: 26D10, 26D15

1. Introduction

The Greek mathematicians introduced the concept of Mathematical means based on proportions in the fourth century A.D in the Pythagorean School. In literature, it is evident that Mathematical means have a lot of applications in geometry and music. Later on, several authors contributed and developed a good number of results that are applicable to various branches of science and technology. In recent years, Lokesha et al. [11], obtained the solution for an open problem raised by Rooin [24] involving means and an investigation on the homogeneous functions; as a result, some inequalities involving means are established (*cf.* [10, 14]). Studies on Greek means, the approach of new means, and their generalizations leads to several inequality results were found in [13, 20, 23]. The concept and detailed study on invariant and complementary means found in [25]. Yang et al. [30, 31] proposed the power exponential mean, and Nagaraja et al. [19, 21] studied it along with the invariant power exponential mean labeling of graph, defined respectively by

$$Z(a,b) = (a^a b^b)^{\frac{1}{a+b}} \tag{1.1}$$

and

$$Z^i(a,b) = (a^b b^a)^{\frac{1}{a+b}}. \tag{1.2}$$

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In [28, 29], authors discussed some interesting results on invariant power exponential mean and one-parameter power exponential mean. This work motivates us to develop this paper. The commonly recognized means in literature include the arithmetic mean, geometric mean and harmonic mean. The Pythagorean school defined these means on proportion, as follows:

For $a, b > 0$ the Arithmetic, Geometric, and Harmonic means are furnished by,

$$A(a,b) = \frac{a+b}{2}, \tag{1.3}$$

$$G(a,b) = \sqrt{ab} \tag{1.4}$$

and

$$H(a,b) = \frac{2ab}{a+b}. \tag{1.5}$$

About Muirhead means: Bullen [1] provided brief discussions in his book "Handbook of Means and Their Inequalities" for n arguments as well as for two arguments defined as follows (*cf.* [1]):

$$A_{2,\alpha,\beta}(a,b) = \left(\frac{a^{\alpha}b^{\beta} + a^{\beta}b^{\alpha}}{2}\right)^{\frac{1}{\alpha+\beta}} \quad \text{such that} \quad \alpha + \beta = 1. \tag{1.6}$$

The purpose of studying the Muirhead mean is a generalization of several well-known means, such as the arithmetic mean, geometric mean, and harmonic mean. It is useful for analyzing inequalities and symmetric sum expressions in mathematical analysis. Hence, the Muirhead mean is treated as the generalized mean of familiar famous means, in alignment with the Power mean, the Stolarsky mean, and the alignment chart mean (*cf.* [1]):

(i) For $\alpha = 1$ and $\beta = 0$, $A = \frac{a+b}{2}$ = Arithmetic Mean,

(ii) For $\alpha = \frac{1}{2}$ and $\beta = \dfrac{1}{2}$, $G = \sqrt{ab}$ = Geometric Mean,

(iii) For $\alpha = r$ and $\beta = 0$, $M_r = \left(\frac{a^r+b^r}{2}\right)^{\frac{1}{r}}$ = Power Mean,

(iv) For $\alpha = -1$ and $\beta = 0$, $H = \frac{2ab}{a+b}$ = Harmonic Mean.

The real-world applications of the Muirhead mean were studied by the authors in [2, 4, 9, 27], which are respectively Muirhead mean operators for decision-making with complex interval values of fuzzy numbers, complex single-valued Neutrosophic values, spherical normal fuzzy environment, and its applications to multi-attribute decision-making and for multicriteria decision-making. Here are a few more applications of the same as in the following. Muirhead's inequality is a powerful tool in Olympiad-style problems involving symmetric sums and helps prove inequalities involving symmetric functions, such as AM-GM-HM relations. In Statistics & Data Analysis: It is used in aggregation functions where multiple parameters contribute differently to an overall score and helps model diverse weighting schemes for statistical means. In Economics & Finance: In income distribution analysis, Muirhead-type means can help model different wealth distributions. Weighted means of financial indicators can be derived using Muirhead's approach. In Machine Learning & Optimization: It is used in multi-criteria optimization, where different metrics must be combined in a flexible yet systematic manner, and helps in feature scaling by normalizing variables in data processing. In thermodynamics, means like the power mean (which Muirhead mean generalizes) can be useful in entropy calculations. Engineering applications where weighted averages are needed in signal processing. Motivated by the work carried out by the authors in [28, 29]. In this paper, the Muirhead mean is developed for two positive arguments by replacing α by a and β by b, then (1.6) takes the following form:

$$A_{2,a,b}(a,b) = \left(\frac{a^b b^a + a^a b^b}{2}\right)^{\frac{1}{a+b}}. \tag{1.7}$$

Definition 1.1 (*cf.* [1])**.** For any two positive real numbers, a and b, the abbreviation for Muirhead mean throughout this paper is used as $M_h(a,b)$ and defined by

$$M_h(a,b) = \left(\frac{a^b b^a + a^a b^b}{2}\right)^{\frac{1}{a+b}}. \tag{1.8}$$

Definition 1.2 (*cf.* [25])**.** A mean is defined as a function

$$M : R_+^2 \to R_+$$

which has the property

$$\min(x_1, x_2) \le M(x_1, x_2) \le \max(x_1, x_2)$$

that x_1 and x_2 are positive real numbers.

Lemma 1.3 (*cf.* [3])**.** *Let $\Omega \subseteq R^n$ be symmetric with a non-empty interior geometrically convex set and let $\varphi : \Omega \to R_+$ be continuous on Ω and differentiable in Ω^0. If φ is symmetric on Ω and*

$$(a-b)\left(\frac{\partial\varphi}{\partial a} - \frac{\partial\varphi}{\partial b}\right) \ge 0 (\le 0) \tag{1.9}$$

$$(\ln a - \ln b)\left(a\frac{\partial\varphi}{\partial a} - b\frac{\partial\varphi}{\partial b}\right) \ge 0 (\le 0) \tag{1.10}$$

$$(a-b)\left(a^2\frac{\partial\varphi}{\partial a} - b^2\frac{\partial\varphi}{\partial b}\right) \ge 0 (\le 0) \tag{1.11}$$

holds for any $a, b \in \Omega^0$, then φ is a Schur convex (concave), Schur-geometrically convex (concave), and Schur-harmonically convex (concave function, respectively).

2. Main results

In literature, the good number of results on convexity, Schur convexity, and related properties of means were discussed by Janardhana et al. [3], Kumar and Nagaraja [6], Kumar et al. [7, 8], Lokesha et al. [12], Nagaraja and Rreddy [15], Nagaraja and Sahu [16, 17], Nagaraja and Vimala [18], Nagaraja et al. [19]-[23], Silvia and Gheorghe [25], Sridevi et al. [26].

The results on ratio of difference of means are discussed in [7]. It was found that the Schur convexity results were verified and studied by several authors who provided analytical proof either by direct simplification or by substitution and, in some cases, verified by Taylor series expansion. which is the motivation for this section to discuss the Schur convexity (concavity) results of the Muirhead mean by applying the methodology of Taylor's series expansion, here fixed the variable $a = 1$ and $b = t$.

Theorem 2.1. *For $a < b$ the Muirhead mean $M_h(a,b)$ satisfy the condition of mean.*

Proof. Here it is required to verify the condition that

$$a < \left(\frac{a^b b^a + a^a b^b}{2}\right)^{\frac{1}{a+b}} < b. \tag{2.1}$$

Firstly, prove that

$$a < \left(\frac{a^b b^a + a^a b^b}{2}\right)^{\frac{1}{a+b}}.$$

For all $a < b$,

$$\left(\frac{b}{a}\right)^b + \left(\frac{b}{a}\right)^a > 1 + 1 = 2.$$

Equivalently written as

$$2a^{a+b} < \left(a^b b^a + a^a b^b\right)$$

or

$$a < \left(\frac{a^b b^a + a^a b^b}{2}\right)^{\frac{1}{a+b}}. \tag{2.2}$$

Similarly, for all $a < b$,

$$\left(\frac{a}{b}\right)^a + \left(\frac{a}{b}\right)^b < 1 + 1 = 2.$$

Equivalently written as

$$\left(a^b b^a + a^a b^b\right) < 2b^{a+b}$$

or

$$\left(\frac{a^b b^a + a^a b^b}{2}\right)^{\frac{1}{a+b}} < b. \tag{2.3}$$

Combining the Eqs. (2.2) and (2.3), gives (2.1). Thus, it verifies that $M_h(a,b)$ satisfies the condition of mean. □

Theorem 2.2. *If* $t = \frac{b}{a} > 1$, *then*

$$\frac{Z(a,b) + Z^i(a,b)}{2} = M_h(a,b).$$

Proof. Generally it is well known that

$$x^m + y^m \neq (x+y)^m.$$

For $a = 1$ and $b = t$, the Taylor's series expansions of $Z(a,b)$, $Z^i(a,b)$ and $M_h(a,b)$ respectively;

$$Z(a,b) = Z(1,t) = \left(t^t\right)^{1/t+1} = 1 + \frac{(t-1)}{2} + \frac{1}{8}(t-1)^2 - \frac{1}{16}(t-1)^3 + \frac{17}{384}(t-1)^4 + \cdots$$

$$Z^i(a,b) = Z^i(1,t) = (t)^{1/t+1} = 1 + \frac{(t-1)}{2} - \frac{3}{8}(t-1)^2 + \frac{3}{16}(t-1)^3 - \frac{23}{384}(t-1)^4 + \cdots$$

$$M_h(a,b) = M_h(1,t) = \left(\frac{t^t + t}{2}\right)^{1/t+1} = 1 + \frac{(t-1)}{2} - \frac{1}{8}(t-1)^2 + \frac{1}{16}(t-1)^3 + \frac{3}{128}(t-1)^4 + \cdots.$$

It is evident that by considering series upto the third degree terms in Taylor's series expansion,

$$\frac{Z(a,b) + Z^i(a,b)}{2} = M_h(a,b).$$

However, the fourth degree term provides

$$\frac{Z(a,b) + Z^i(a,b)}{2} < M_h(a,b).$$

□

Remark 2.3. That is the average values of power exponential mean and invariant power exponential mean equal to Muirhead mean value. Mathematically;

$$\frac{(a^a b^b)^{\frac{1}{a+b}} + (a^b b^a)^{\frac{1}{a+b}}}{2} = \left(\frac{a^b b^a + a^a b^b}{2}\right)^{\frac{1}{a+b}}.$$

Theorem 2.4. *The Muirhead mean is Schur concave for* $1 < t = \frac{b}{a} < 3$ *and Schur convex for* $t = \frac{b}{a} > 3$ *and* $t = \frac{b}{a} < 1$.

Proof. Consider

$$M_h(a,b) = \left(\frac{a^b b^a + a^a b^b}{2}\right)^{\frac{1}{a+b}}. \tag{2.4}$$

On differentiating both sides of equation (2.4) partially with respect to a and b gives:

$$\frac{\partial M_h}{\partial a} = M_h\left(\frac{(b^{a+1}a^{b-1} + b^a a^b \ln b + a^a b^b(\ln a + 1))}{(b^a a^b + a^a b^b)(a+b)} - \ln\left(\frac{b^a a^b + a^a b^b}{(a+b)^2}\right) + \ln 2\right) \tag{2.5}$$

and

$$\frac{\partial M_h}{\partial b} = M_h\left(\frac{(a^{b+1}b^{a-1} + b^a a^b \ln a + a^a b^b(\ln b + 1))}{(b^a a^b + a^a b^b)(a+b)} - \ln\left(\frac{b^a a^b + a^a b^b}{(a+b)^2}\right) + \ln 2\right). \tag{2.6}$$

On subtraction and simplification, it leads to:

$$\frac{\partial M_h}{\partial a} - \frac{\partial M_h}{\partial b} = \frac{M_h}{(a+b)(b^a a^b + a^a b^b)}\left(b^a a^a\left(\frac{b^2 - a^2}{ab}\right) + (b^a a^b - a^a b^b)\ln(b/a)\right). \tag{2.7}$$

Let

$$f(a,b) = \left(b^a a^a\left(\frac{b^2 - a^2}{ab}\right) + (b^a a^b - a^a b^b)\ln(b/a)\right). \tag{2.8}$$

To decide whether the quantity $f(a,b)$ is positive or negative, the concept of Taylor's series expansion is applied as follows:

Put $a = 1$ and $b = t$, in (2.8),

$$f(1,t) = (t^2 - 1) + t\ln t - t^t \ln t. \tag{2.9}$$

Using the Taylor's series expansion about the point a given by

$$f(t) = f(a) + \frac{(t-a)}{1!}f'(t) + \frac{(t-a)^2}{2!}f''(t) + \frac{(t-a)^3}{3!}f'''(a) + \cdots.$$

For $a = 1$, consider the terms upto fifth degree:

$$f(1,t) = 2(t-1) + (t-1)^2 - (t-1)^3 - \frac{5}{12}(t-1)^5 + \cdots.$$

Case (i) Taking into account up to the second-degree term $f(1,t) > 0$ for all $t > 1$.
That is for $a < b$ or $t > 1$, the quantity

$$(a-b)\left(\frac{\partial M_h}{\partial a} - \frac{\partial M_h}{\partial b}\right) < 0.$$

Case (ii) Considering up to third degree term $f(1,t) > 0$ for all $1 < t < 3$ and $f(1,t) < 0$ for all $t > 3$ and $t < 1$.
That is for $a < b$ or $t > 3$ and $t < 1$, the quantity

$$(a-b)\left(\frac{\partial M_h}{\partial a} - \frac{\partial M_h}{\partial b}\right) > 0$$

and for $a < b$ or $1 < t < 3$, the quantity

$$(a-b)\left(\frac{\partial M_h}{\partial a} - \frac{\partial M_h}{\partial b}\right) < 0.$$

Hence the proof of the Theorem. □

Theorem 2.5. *For* $0 < a < b$ *and* $t = \frac{b}{a} > 1$ *in* $(0,1)$*, Muirhead mean is Schur geometric convex.*

Proof. On multiplying Eq. (2.5) by a and Eq. (2.6) by b gives,

$$a\frac{\partial M_h}{\partial a} = \frac{M_h}{(a+b)^2}[\Delta_1(a,b)], \tag{2.10}$$

$$\Delta_1(a,b) = \left(a\log 2 + \left(\frac{a(a+b)(b^{a+1}a^{b-1} + b^a a^b \log b + a^a b^b(\log a + 1)}{b^a a^b + a^a b^b}\right) - a\log\left(b^a a^b + a^a b^b\right)\right)$$

and

$$b\frac{\partial M_h}{\partial b} = \frac{M_h}{(a+b)^2}[\Delta_2(a,b)], \tag{2.11}$$

$$\Delta_2(a,b) = \left(b\log 2 + \left(\frac{b(a+b)(a^{b+1}b^{a-1} + b^a a^b \log b + a^a b^b(\log b + 1)}{b^a a^b + a^a b^b}\right) - b\log\left(b^a a^b + a^a b^b\right)\right).$$

After subtraction and simplification, it leads to:

$$a\frac{\partial M_h}{\partial a} - b\frac{\partial M_h}{\partial b} = \frac{M_h}{(a+b)^2(b^a a^b + a^a b^b)}[\Delta_3(a,b)], \tag{2.12}$$

$$\Delta_3(a,b) = \left(b^a a^{b+2} - a^a b^{b+2}\right)\log b + \left(a^a b^{b+2} - a^b b^{a+2}\right)\log a + \left(b^{a+1}a^{b+1} - a^{a+1}b^{b+1}\right)(\log b - \log a)$$
$$+(b-a)\left(b^a a^b + a^a b^b\right)(\log(b^a a^b + a^a b^b) - \log 2).$$

To decide whether the quantity $\Delta_3(a,b)$, is positive or negative, the concept of Taylor's series expansion is applied as follows:

Put $a = 1$ and $b = t$, in $\Delta_3(a,b)$, gives

$$\Delta_3(1,t) = (1+t)[(t-1)(t-t^t) + (t-t^{t+1})\ln t] + (1-t)[\ln 2 - \ln(t+t^t)](t+t^t).$$

Therefore, the series expansion of $\Delta_3(1,t)$ upto the fifth degree term is:

$$\Delta_3(1,t) = -4(t-1)^3 - 4(t-1)^4 - \frac{9}{4}(t-1)^5 + \cdots.$$

That is for $a < b$ or $t > 1$, the quantity $\Delta_3(1,t) < 0$.

Thus, for $a < b$ or $t > 1$, the quantity.

$$(\ln a - \ln b)\left(a\frac{\partial M_h}{\partial a} - b\frac{\partial M_h}{\partial b}\right) > 0.$$

Hence the proof of the Theorem. □

Theorem 2.6. *For* $0 < a < b$ *and* $t = \frac{b}{a} > 1$ *in* $(0,1)$, *Muirhead mean is Schur-harmonic convex.*

Proof. On multiplying Eq. (2.5) by a^2 and Eq. (2.6) by b^2, then

$$a^2\frac{\partial M_h}{\partial a} = \frac{M_h}{(a+b)^2}[\nabla_1(a,b)], \tag{2.13}$$

$$\nabla_1(a,b) = a^2\log 2 + a^2(a+b)\left(\frac{b^{a+1}a^{b-1} + b^a a^b \log(b) + a^a b^b(\log(a) + 1))}{b^a a^b + a^a b^b}\right) - a^2\log\left(b^a a^b + a^a b^b\right)$$

and

$$b^2\frac{\partial M_h}{\partial b} = \frac{M_h}{(a+b)^2}[\nabla_2(a,b)], \tag{2.14}$$

$$\nabla_2(a,b) = b^2\log 2 + b^2(a+b)\left(\frac{a^{b+1}b^{a-1} + b^a a^b \log b + a^a b^b(\log(b) + 1))}{b^a a^b + a^a b^b}\right) - b^2\log\left(b^a a^b + a^a b^b\right).$$

On subtraction and simplification, it leads to;

$$a^2\frac{\partial M_h}{\partial a} - b^2\frac{\partial M_h}{\partial b} = \frac{M_h}{(a+b)^2(b^a a^b + a^a b^b)}[\nabla_3(a,b)], \tag{2.15}$$

$$\begin{aligned}\nabla_3(a,b) = (a^2-b^2)\log 2 + (b^2-a^2)\log(b^a a^b + a^a b^b) + b^a a^b(a^3 logb - b^3\log a) + a^a b^b(a^3\log a - b^3 logb)\\ + a^a b^b(a^3-b^3) + b^a a^b(a^2 b\log b - ab^2\log a) + a^a b^b(a^2 b\log a - ab^2\log b)\\ + (\log 2(a^2-b^2) - (b^2-a^2)log(b^a a^b + a^a b^b)(b^a a^b + a^a b^b)).\end{aligned}$$

To decide whether the quantity $\nabla_3(a,b)$, is positive or negative, the concept of Taylor's series expansion is applied as follows:

Put $a = 1$ and $b = t$, in $\nabla_3(a,b)$, gives

$$\nabla_3(1,t) = (1+t)[(t-1)(t-t^t) + (t-t^{t+1})\ln t] + (1-t)[\ln 2 - \ln(t+t^t)](t+t^t).$$

Using Taylor's series expansion up-to the fourth degree terms:

$$\nabla_3(1,t) = -4(t-1) - 8(t-1)^2 - 11(t-1)^3 - 10(t-1)^4 - \cdots.$$

That is for $a < b$ or $t > 1$, the quantity $\nabla_3(1,t) < 0$.

Thus, for $a < b$ or $t > 1$, the quantity

$$(a-b)\left(a^2\frac{\partial M_h}{\partial a} - b^2\frac{\partial M_h}{\partial b}\right) > 0.$$

Hence the proof of the Theorem. □

3. Refinement of inequality chain

It is well known that the various authors have explored the following familiar as well as famous means and their applications to strengthen the inequalities in literature refer [1, 5, 20, 31].

- **(i)** $\frac{G^2}{C}(a,b) = \frac{a^2b+ab^2}{a^2+b^2}$ = Invariant Contra harmonic mean,
- **(ii)** $C(a,b) = \frac{a^2+b^2}{a+b}$ = Contra harmonic mean,
- **(iii)** $Z(a,b) = a^{a/a+b}b^{b/a+b}$= Power exponential mean,
- **(iv)** $Z^i(a,b) = a^{b/a+b}b^{a/a+b}$= Invariant power exponential mean,
- **(v)** $A(a,b) = \frac{a+b}{2}$= Arithmetic mean,
- **(vi)** $H(a,b) = \frac{2ab}{a+b}$= Harmonic mean,
- **(vii)** $G(a,b) = \sqrt{ab}$= Geometric mean,
- **(viii)** $H_e(a,b) = \frac{a+\sqrt{ab}+b}{3}$= Heron mean,
- **(ix)** $L(a,b) = \frac{a-b}{\ln a - \ln b}$= Logarithmic mean,
- **(x)** $M_r(a,b) = \left(\frac{a^r+b^r}{2}\right)^{\frac{1}{r}}$= Power mean,
- **(xi)** $I(a,b) = \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}}$ = Identric mean.

In 1993, Kuang [5] stated and proved the inequality with usual notations:

$$G \leq L \leq M_{1/3} \leq M_{1/2} \leq H_e \leq M_{2/3} \leq A. \tag{3.1}$$

In [31], authors proved that:

$$H_e \leq M_{2/3} \leq I \tag{3.2}$$

which motivates us to strengthen and establishes the inequality chain for two positive arguments of the form (*cf.* [20]):

$$\frac{G^2}{C} \leq Z^i \leq H \leq G \leq L \leq M_{1/3} \leq H_e \leq M_{2/3} \leq I \leq A \leq Z \leq C. \tag{3.3}$$

The authors provided the analytical proof for the above inequality chain; in this section, for $a = 1$ and $b = t$, the Taylor's series expansions of the above 10 means and Miurhead mean about the point $t = 1$ are as follows, and the above inequality chain is strengthened:

$$\begin{aligned}
C(1,t) &= \frac{1+t^2}{1+t} = 1 + \frac{1}{2}(t-1) + \frac{1}{4}(t-1)^2 - \frac{1}{8}(t-1)^3 + \frac{1}{16}(t-1)^4 - \frac{1}{32}(t-1)^5 + \cdots \\
Z(1,t) &= t^{t/1+t} = 1 + \frac{1}{2}(t-1) + \frac{1}{8}(t-1)^2 - \frac{1}{16}(t-1)^3 + \frac{17}{384}(t-1)^4 - \frac{9}{256}(t-1)^5 + \cdots \\
A(1,t) &= \frac{1+t}{2} = 1 + \frac{1}{2}(t-1) \\
I(1,t) &= \frac{1}{e}\left(\frac{1}{t^t}\right)^{1/1-t} = 1 + \frac{1}{2}(t-1) - \frac{1}{24}(t-1)^2 + \frac{1}{48}(t-1)^3 - \frac{73}{5760}(t-1)^4 + \frac{11}{1280}(t-1)^5 + \cdots \\
M_{2/3}(1,t) &= \left(\frac{1+t^{2/3}}{2}\right)^{3/2} = 1 + \frac{1}{2}(t-1) - \frac{1}{24}(t-1)^2 + \frac{1}{48}(t-1)^3 - \frac{133}{10368}(t-1)^4 + \frac{61}{6912}(t-1)^5 + \cdots \\
H_e(1,t) &= \frac{1+\sqrt{t}+t}{3} = 1 + \frac{1}{2}(t-1) - \frac{1}{24}(t-1)^2 + \frac{1}{48}(t-1)^3 - \frac{5}{384}(t-1)^4 + \frac{7}{768}(t-1)^5 + \cdots \\
M_{1/3}(1,t) &= \left(\frac{1+t^{1/3}}{2}\right)^3 = 1 + \frac{1}{2}(t-1) - \frac{1}{12}(t-1)^2 + \frac{1}{24}(t-1)^3 - \frac{17}{648}(t-1)^4 + \frac{1}{54}(t-1)^5 + \cdots \\
L(1,t) &= \frac{1-t}{-\ln t} = 1 + \frac{1}{2}(t-1) - \frac{1}{12}(t-1)^2 + \frac{1}{24}(t-1)^3 - \frac{19}{720}(t-1)^4 + \frac{3}{160}(t-1)^5 + \cdots \\
M_h(1,t) &= \left(\frac{t+t^t}{2}\right)^{1/1+t} = 1 + \frac{1}{2}(t-1) - \frac{1}{8}(t-1)^2 + \frac{1}{16}(t-1)^3 + \frac{3}{128}(t-1)^4 - \frac{9}{256}(t-1)^5 + \cdots \\
G(1,t) &= \sqrt{t} = 1 + \frac{1}{2}(t-1) - \frac{1}{8}(t-1)^2 + \frac{1}{16}(t-1)^3 - \frac{5}{128}(t-1)^4 + \frac{7}{256}(t-1)^5 + \cdots \\
H(1,t) &= \frac{2t}{1+t} = 1 + \frac{1}{2}(t-1) - \frac{1}{4}(t-1)^2 + \frac{1}{8}(t-1)^3 - \frac{1}{16}(t-1)^4 + \frac{1}{32}(t-1)^5 + \cdots \\
Z_1(1,t) &= t^{1/1+t} = 1 + \frac{1}{2}(t-1) - \frac{3}{8}(t-1)^2 + \frac{3}{16}(t-1)^3 - \frac{23}{384}(t-1)^4 - \frac{1}{256}(t-1)^5 + \cdots \\
\frac{G^2}{C}(1,t) &= \frac{t+t^2}{1+t^2} = 1 + \frac{1}{2}(t-1) - \frac{1}{2}(t-1)^2 + \frac{1}{4}(t-1)^3 - 0(t-1)^4 - \frac{1}{8}(t-1)^5 + \cdots.
\end{aligned}$$

Inequality 3.1. Based on the above series expansions, by considering the terms up to second order and ignoring higher order leads to the inequality

$$\frac{G^2}{C} \leq Z_1 \leq H \leq G \leq L \leq M_{1/3} \leq H_e \leq M_{2/3} \leq I \leq A \leq Z \leq C. \tag{3.4}$$

However, by considering the series up to fourth degree terms, it is clear that:

$$G \leq M_h \leq L. \tag{3.5}$$

The best possible value $b = t = 2.0098$ is determined by applying the Newton-Raphson method of finding the real roots of the transcendental equations as follows:

Newton-Raphson method (Formula):

$$t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}; \quad n = 1, 2, 3, \dots . \tag{3.6}$$

Let

$$f(t) = M_h(1,t) - L(1,t) = \left(\frac{t+t^t}{2}\right)^{1/1+t} - \left(\frac{1-t}{-\ln t}\right).$$

Choose $t_0 = 2$, find $f'(t)$, then substitute in (3.6), successively gives $t_1 = 2.0100$, $t_2 = 2.0098$, $t_3 = 2.0098 = t_4 = t_5$, thus the best possible value is $t = 2.0098$ for the inequality (3.5).

Combining (3.4) and (3.5) gives the following refined or strengthened inequality chain:

$$\frac{G^2}{C} \le Z_1 \le H \le G \le M_h \le L \le M_{1/3} \le H_e \le M_{2/3} \le I \le A \le Z \le C. \tag{3.7}$$

Inequality, given by (3.7), is the refinement of inequality given by (3.3). Further, the graphical representation of this is as shown (in Figure 1) based on the values of $a = 1$ and $b = t$.

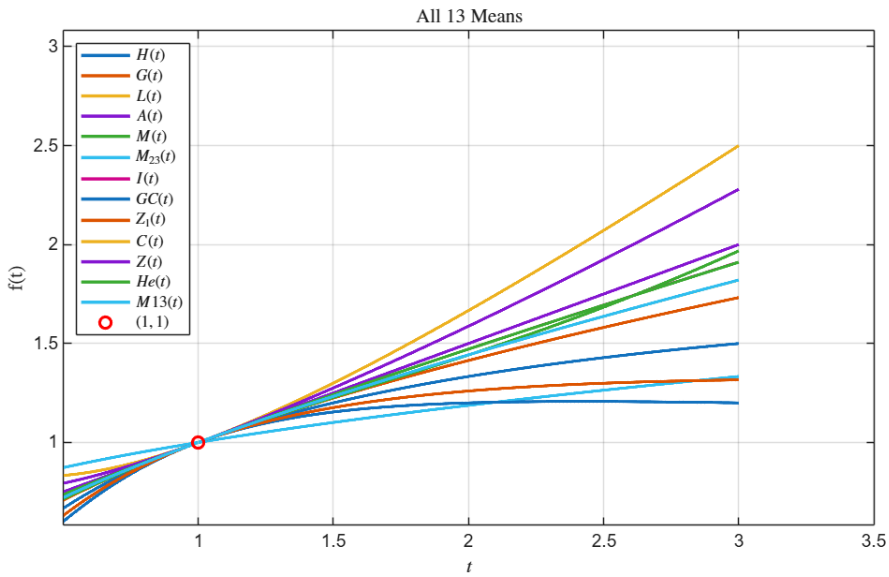

Figure 1. Graphical representation of 13 means of inequality given by (3.7)

Inequality 3.2. For $a = 1$ and $2.0098 < b = t < 2.0109$, (The inequality represented in Figure 2)

$$L \le M_h \le M_{1/3}.$$

For $t > 2.0098$ and from Inequality 3.1,

$$L \le M_h. \tag{3.8}$$

Let

$$f(t) = M_h(1,t) - M_{1/3}(1,t) = \left(\frac{t+t^t}{2}\right)^{1/1+t} - \left(\frac{1+t^{1/3}}{2}\right)^3.$$

Choose $t_0 = 2$, find $f'(t)$, then substitute in (3.6), successively gives $t_1 = 2.0111$, $t_2 = 2.0109$, $t_3 = 2.0109 = t_4 = t_5$, thus the best possible value of the inequality

$$M_h \le M_{1/3} \tag{3.9}$$

is holds for $t < 2.0098$. Combining (3.8) and (3.9) gives the Inequality 3.2.

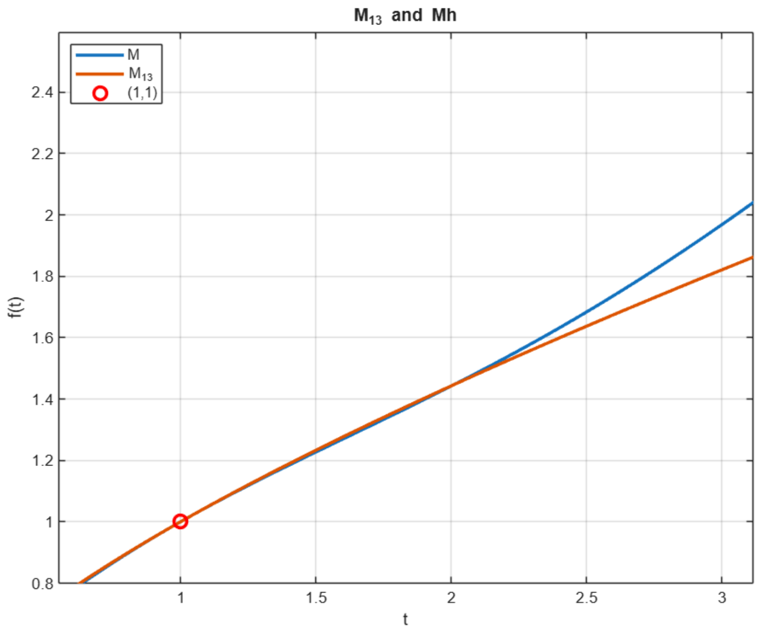

Figure 2. Graphical representation of the variation of power mean with $r = 1/3$ and Muirhead mean

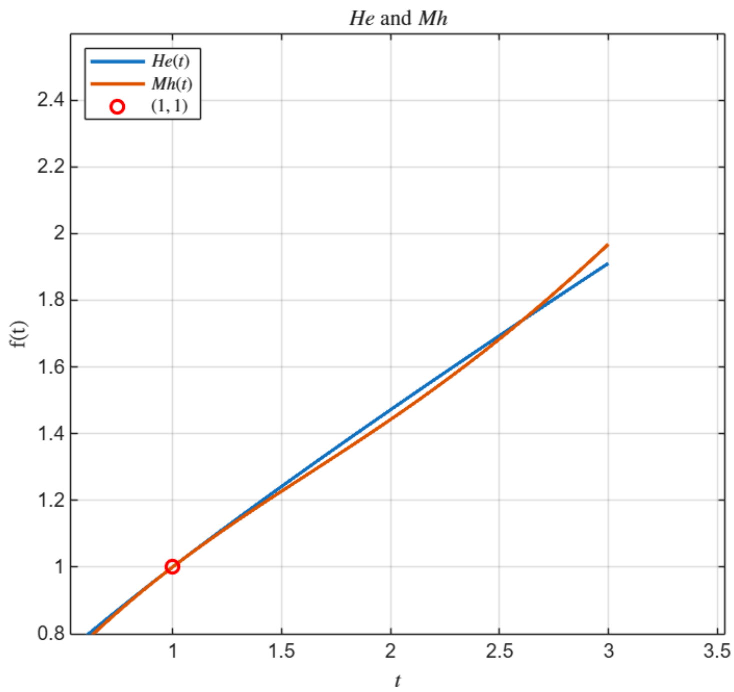

Figure 3. Graphical representation of the variation of Heron mean and Muirhead mean

Inequality 3.3. For $a = 1$ and $2.0109 < b = t < 2.6063$, (The inequality represented in Figure 3)

$$L \leq M_{1/3} \leq M_h \leq H_e.$$

For $t > 2.0109$ and from Inequality 3.2,

$$L \leq M_{1/3} \leq M_h. \tag{3.10}$$

Let

$$f(t) = M_h(1,t) - H_e(1,t) = \left(\frac{t+t^t}{2}\right)^{1/1+t} - \left(\frac{1+\sqrt{t}+t}{3}\right).$$

Choose $t_0 = 2.5$, find $f^{'}(t)$, then substitute in (3.6), successively gives $t_1 = 2.6196$, $t_2 = 2.6024$, $t_3 = 2.6063 = t_4 = t_5$, thus the best possible value of the inequality

$$M_h \leq H_e \tag{3.11}$$

is holds for $t < 2.6063$. Combining (3.10) and (3.11) gives the Inequality 3.3.

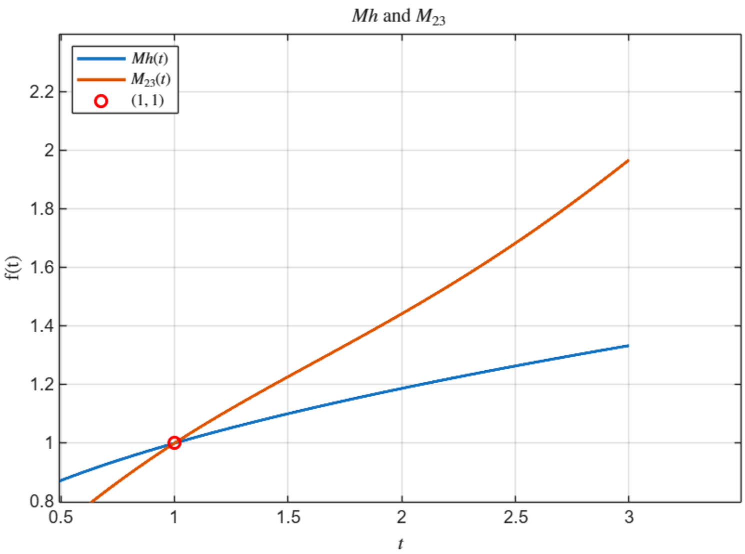

Figure 4. Graphical representation of the variation of power mean with $r = 2/3$ and Muirhead mean

Inequality 3.4. For $a = 1$ and $2.6063 < b = t < 2.6087$, (The inequality represented in Figure 4)

$$L \leq M_{1/3} \leq H_e \leq M_h \leq M_{2/3}.$$

For $t > 2.6063$ and from Inequality 3.3,

$$L \leq M_{1/3} \leq H_e \leq M_h. \tag{3.12}$$

Let

$$f(t) = M_h(1,t) - M_{2/3}(1,t) = \left(\frac{t+t^t}{2}\right)^{1/1+t} - \left(\left(\frac{1+t^{2/3}}{3/2}\right)^3\right).$$

Choose $t_0 = 2.5$, find $f^{'}(t)$, then substitute in (3.6), successively gives $t_1 = 2.6227$, $t_2 = 2.6089$, $t_3 = 2.6087 = t_4 = t_5$, thus the best possible value of the inequality

$$M_h \leq M_{2/3} \tag{3.13}$$

is holds for $t < 2.6087$. Combining (3.12) and (3.13) gives the Inequality 3.4.

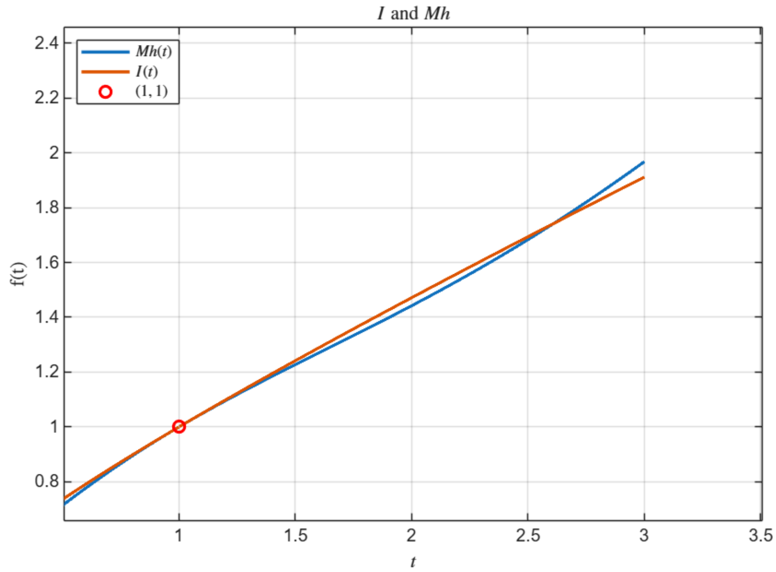

Figure 5. Graphical representation of the variation of identric mean and Muirhead mean

Inequality 3.5. For $a = 1$ and $2.6087 < b = t < 2.6122$, (The inequality represented in Figure 5)

$$L \leq M_{1/3} \leq H_e \leq M_{2/3} \leq M_h \leq I.$$

For $t > 2.6087$ and from Inequality 3.4,

$$L \leq M_{1/3} \leq H_e \leq M_{2/3} \leq M_h. \tag{3.14}$$

Let

$$f(t) = M_h(1,t) - I(1,t) = \left(\frac{t+t^t}{2}\right)^{1/1+t} - \frac{1}{e}\left(\frac{1}{t^t}\right)^{1/1-t}.$$

Choose $t_0 = 2.5$, find $f'(t)$, then substitute in (3.6), successively gives $t_1 = 2.6125$, $t_2 = 2.6122$, $t_3 = 2.6122 = t_4 = t_5$, thus the best possible value of the inequality

$$M_h \leq I \tag{3.15}$$

is holds for $t < 2.6122$. Combining (3.13) and (3.14) gives the Inequality 3.5.

Inequality 3.6. For $a = 1$ and $2.6122 < b = t < 3.2431$, (The inequality represented in Figure 6)

$$L \leq M_{1/3} \leq H_e \leq M_{2/3} \leq I \leq M_h \leq A.$$

For $t > 2.6122$ and from Inequality 3.5,

$$L \leq M_{1/3} \leq H_e \leq M_{2/3} \leq I \leq M_h. \tag{3.16}$$

Let

$$f(t) = M_h(1,t) - A(1,t) = \left(\frac{t+t^t}{2}\right)^{1/1+t} - \left(\frac{1+t}{2}\right).$$

Choose $t_0 = 3$, find $f'(t)$, then substitute in (3.6), successively gives $t_1 = 3.2844$, $t_2 = 3.2438$, $t_3 = 3.2431 = t_4 = t_5$, thus the best possible value of the inequality

$$M_h \leq A \tag{3.17}$$

is holds for $t < 3.2431$. Combining (3.15) and (3.16) gives the Inequality 3.6.

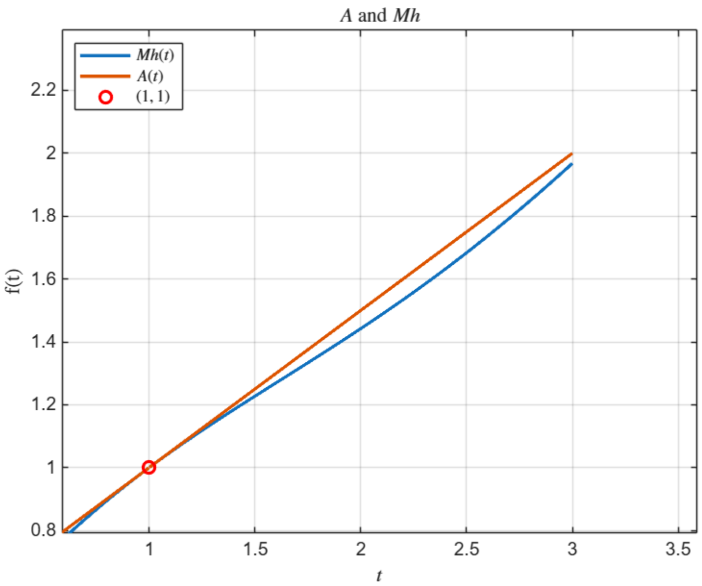

Figure 6. Graphical representation of the variation of arithmetic mean and Muirhead mean

Inequality 3.7. For $a = 1$ and $t > 3.2431$, (The inequality represented in Figure 6)

$$L \leq M_{1/3} \leq H_e \leq M_{2/3} \leq I \leq A \leq M_h \leq Z \leq C.$$

For $t > 3.2431$ and from Inequality 3.6,

$$L \leq M_{1/3} \leq H_e \leq M_{2/3} \leq I \leq M_h. \tag{3.18}$$

Combining (3.18) and (3.7) gives the Inequality 3.7.

Figure 7 below shows the graphical representation Muirhead mean and Logarithmic mean.

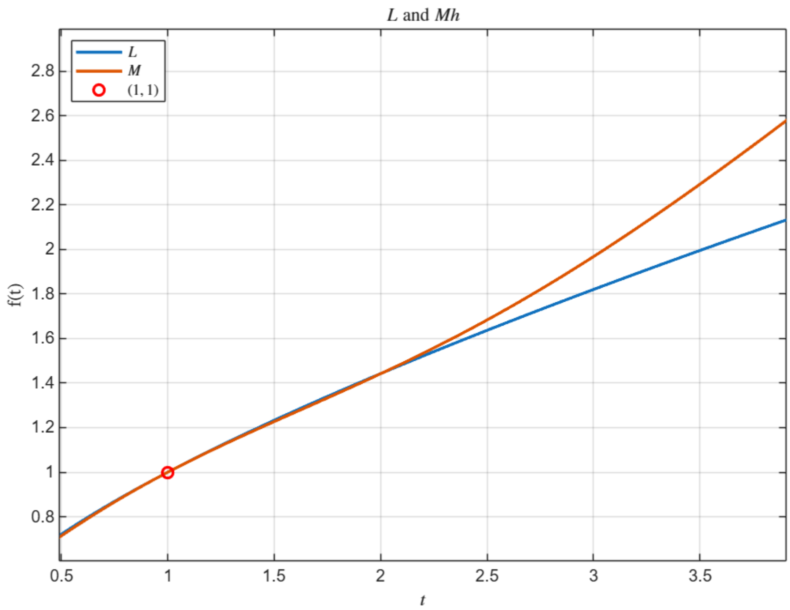

Figure 7. Graphical representation of the variation of logarithmic mean and Muirhead mean

Figure 8 below shows the graphical representation of means from Muirhead mean to Arithmetic mean.

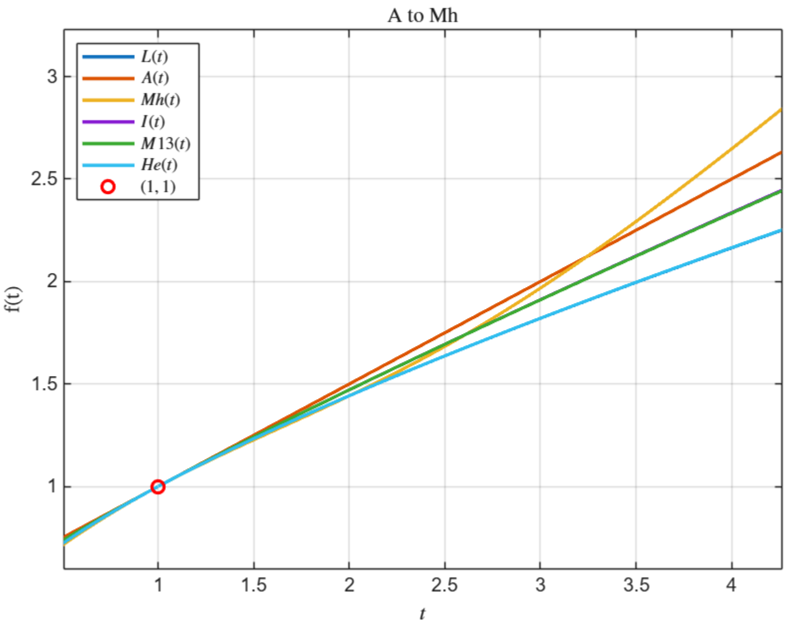

Figure 8. Graphical representation of the variation of Muirhead mean, arithmetic mean and other means of inequality given by (3.7)

Figure 9 below shows the graphical representation of means from invariant contra harmonic mean to Muirhead mean.

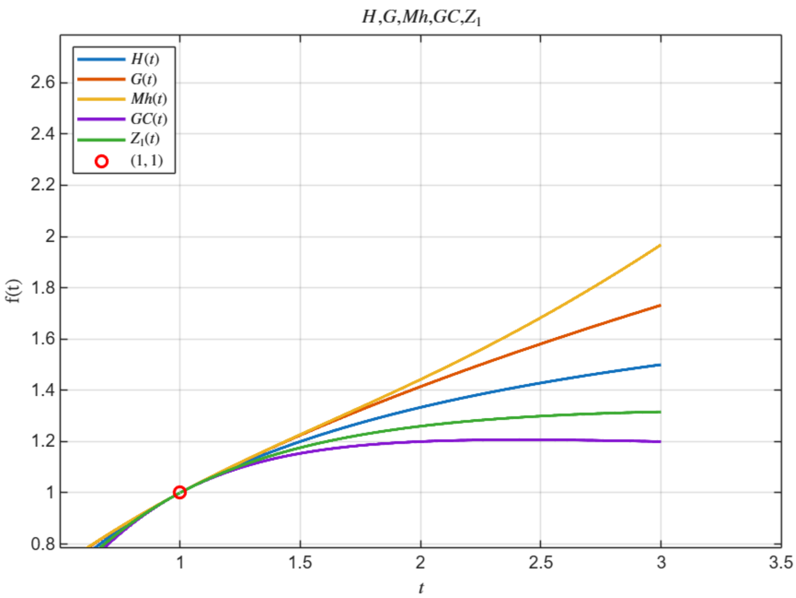

Figure 9. Graphical representation of the variation of 5 means as described in inequality given by (3.7)

Figure 10 below shows the graphical representation of means from Muirhead mean to contra harmonic mean.

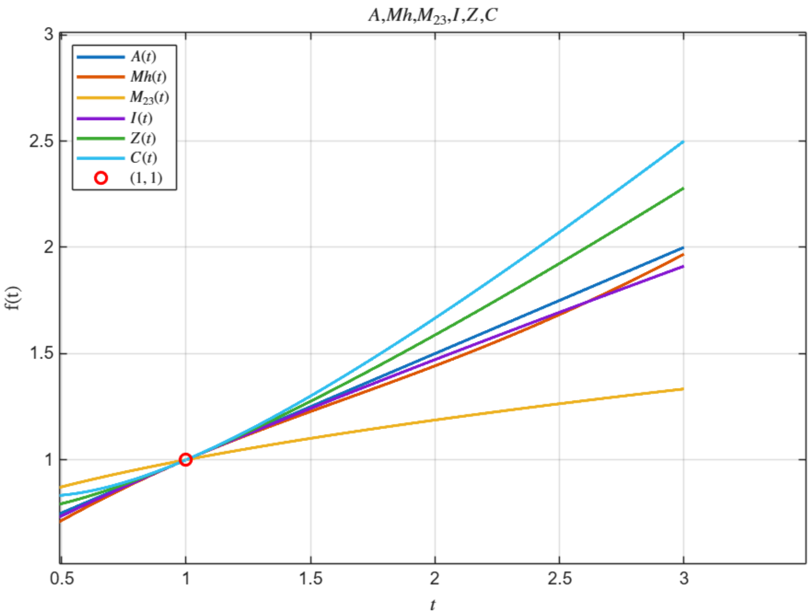

Figure 10. Graphical representation of the variation of 6 means as described in inequality given by (3.7)

4. Importance

The Muirhead mean is not as like other means; it reveals from the graphical representation that in Figure 1, the average value of Muirhead mean crosses the average values of several means; however, other means are either increasing or decreasing in nature with respect to each other.

5. Conclusion

Through this paper, the Muirhead mean is re-defined and proved to satisfy the definition of mean. Further, it was proved that the average values of the power exponential mean and invariant power exponential mean are equal to the Muirhead mean value for some $t > 1$. Also, verified and established that for some values of t the various convexity conditions, such as the Schur, Schur geometric, and Schur harmonic convexities (concavities) of the Muirhead mean. Finally, a few inequality chains were established, which strengthened the existing inequality chain by applying the concepts of the Newton-Raphson method to find the approximate solutions of algebraic equations and Taylor's series expansions.

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