

# Topological indices of hexagonal chain systems

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## Abstract

Many chemical molecules have acyclic structures when modeled by a graph. Most exceptions have hexagonal structure. Therefore, the study of hexagonal structures is rather useful and in this work, we studied one of the important and frequently used structure called as hexagonal chain systems. We calculated several degree based topological indices of these structures first for hexagonal chains, hexagonal double chains and hexagonal triple chains and finalized our calculations with the indices of hexagonal chain systems. At the end, we calculate the numbers of faces of these systems by means of a recently introduced graph invariant called omega.

**Keywords:** Topological index, hexagonal chain, hexagonal double chain, hexagonal triple chain, hexagonal chain systems, omega invariant

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## 1. Introduction

In this paper, unless the converse is stated, we let  $G$  to be a finite simple undirected graph with  $n$  vertices and  $m$  edges.  $n$  and  $m$  are respectively called as the order and the size of the graph. In a simple graph  $G$ , the degree of a vertex  $v$  of  $G$  is the number of edges meeting at  $v$ , and is denoted by  $d_G v$  or  $dv$ . If  $v$  has only one adjacent vertex, then it is called a pendant vertex. Naturally, a pendant vertex is of degree 1. Pendant vertices especially appear in trees as they are connected acyclic graphs.

Most of the recent graph theory research is about topological graph indices. After the first index was defined in 1947 by Wiener [15] to compare the boiling temperatures of the isomers of some molecular structures called alkanes, many mathematicians and chemists have introduced and studied over 3000 different topological graph indices registered in chemical databases. They can mainly be grouped into two large classes, degree based ones and distance based ones, but there are also many others related to several graph parameters. Our calculations are only concerned with the first class. Let us now recall the indices that we study with:

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The first and second Zagreb indices were defined in [11] by

$$M_1(G) = \sum_{u \in V(G)} du^2 \quad \text{and} \quad M_2(G) = \sum_{uv \in E(G)} dudv.$$

The forgotten index was defined in [9] by

$$F(G) = \sum_{u \in V(G)} du^3.$$

The inverse sum index is defined in [14] by

$$ISI(G) = \sum_{uv \in E(G)} \frac{dudv}{du + dv}.$$

Some of the degree based indices are known as the irregularity indices as they help us to get information about the regularness of a given graph. Sigma index is one of such indices defined in [1] by

$$\Sigma(G) = \sum_{uv \in E(G)} (du - dv)^2.$$

The Bell index is defined in [4] as another irregularity measure by

$$B(G) = \sum_{v \in V(G)} \left( du - \frac{2m}{n} \right)^2.$$

The total irregularity index is defined as another irregularity index by taking all the vertex degrees into account as

$$Irr_t(G) = \sum_{v \in V(G)} \left| du - \frac{2m}{n} \right|$$

(cf. [2]).

The harmonic index is defined by

$$H(G) = \sum_{uv \in E(G)} \frac{2}{du + dv}$$

(cf. [16]).

One of the famous degree based topological graph indices is the atom bond connectivity index which has molecular applications in terms of atoms and chemical bonds between them which are respectively modeled by vertices and edges of the graph modeling the molecule given by

$$ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{du + dv - 2}{dudv}}$$

in [7].

The geometric arithmetic index is defined in [13] as the ratio of these two means over all the edges of the graph by

$$GA(G) = \sum_{uv \in E(G)} \frac{2\sqrt{dudv}}{du + dv}.$$

The augmented Zagreb index is defined by (cf. [10]):

$$AZ(G) = \sum_{uv \in E(G)} \left( \frac{dudv}{du + dv - 2} \right)^3.$$

Another useful irregularity index is the Albertson index (cf. [3]), defined by the sum of absolute values of all the differences between degrees of pairs of vertices forming an edge

$$Alb(G) = \sum_{uv \in E(G)} |du - dv|.$$

One of the most well-known topological graph indices is known as the Randic index which is defined by

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{dudv}}$$

in [12].

The reciprocal Randic index is defined similarly to Randic index and has some advantages in chemical calculations

$$RR(G) = \sum_{uv \in E(G)} \sqrt{dudv}.$$

As  $RR(G)$  is much more easier to calculate compared with the classical Randic index, some applications required the use of reciprocal one, see [8].

## 2. Topological indices of hexagonal chains

In Figure 1, we see the hexagonal chain consisting of  $k$  hexagons.

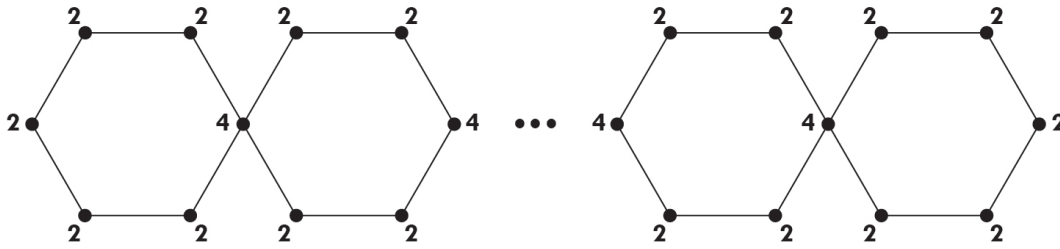


Figure 1. The hexagonal chain  $C_{6,k}^1$

In Table 1, the vertex partition of hexagonal chain  $C_{6,k}^1$  is presented. Here, the table says that there are two (the number of rows) types of vertex degrees, 2 and 4, and the total number of corresponding vertices are  $4k + 2$  and  $k - 1$ , respectively.

$d_i$	$\#v_i$
2	$4k + 2$
4	$k - 1$

Table 1. The vertex partition of  $C_{6,k}^1$

$(d_i, d_j)$	$\#(v_i, v_j)$
(2, 2)	$2k + 4$
(2, 4)	$4(k - 1)$

Table 2. The edge partition of  $C_{6,k}^1$

The size of  $C_{6,k}^1$  is  $m = 6k$  and the order is  $n = 5k + 1$  which can alternatively be obtained by adding the numbers of vertices of order 2 and order 4. Now we can calculate the chosen topological indices:

**Theorem 2.1.** *The 14 degree based topological graph indices of hexagonal chain are*

$$\begin{aligned}
 M_1(C_{6,k}^1) &= 32k - 8 \\
 M_2(C_{6,k}^1) &= 40k - 16 \\
 F(C_{6,k}^1) &= 48(2k - 1) \\
 ISI(C_{6,k}^1) &= \frac{2}{3}(11k - 2) \\
 \sigma(C_{6,k}^1) &= 16(k - 1) \\
 B(C_{6,k}^1) &= \frac{8(2k + 1)(k - 1)}{5k + 1} \\
 Irr_t(C_{6,k}^1) &= \frac{8(2k + 1)(k - 1)}{5k + 1} \\
 H(C_{6,k}^1) &= \frac{7k + 2}{3} \\
 ABC(C_{6,k}^1) &= 3\sqrt{2}k \\
 GA(C_{6,k}^1) &= \frac{(8\sqrt{2} + 6)k + 12 - 8\sqrt{2}}{3} \\
 AZ(C_{6,k}^1) &= 48k \\
 Alb(C_{6,k}^1) &= 8(k - 1) \\
 R(C_{6,k}^1) &= (1 + \sqrt{2})k + 2 - \sqrt{2} \\
 RR(C_{6,k}^1) &= (1 + 2\sqrt{2})4k + 8(1 - \sqrt{2}).
 \end{aligned}$$

*Proof.* By their definitions, all these degree based topological graph indices are defined and therefore calculated by means of vertex and edge partition of the graph  $C_{6,k}^1$ . Therefore, using Table 1 for the vertex partition, we get

$$\begin{aligned}
 M_1(C_{6,k}^1) &= (4k + 2) \cdot 2^2 + (k - 1) \cdot 4^2 \\
 &= 32k - 8,
 \end{aligned}$$

where  $\cdot$  denotes the classical multiplication, both here and throughout the paper. Now the second Zagreb index needs the use of Table 2 for the edge partition of the graph. Hence

$$\begin{aligned}
 M_2(C_{6,k}^1) &= (2k + 4) \cdot (2 \cdot 2) + 4(k - 1) \cdot (2 \cdot 4) \\
 &= 40k - 16.
 \end{aligned}$$

The forgotten index is very similar to the first Zagreb index and therefore by means of Table 1, we get

$$\begin{aligned}
 F(C_{6,k}^1) &= (4k + 2) \cdot 2^3 + (k - 1) \cdot 4^3 \\
 &= 48(2k - 1).
 \end{aligned}$$

The other indices are also calculated by means of these two tables to give the following results:

$$\begin{aligned}
 ISI(C_{6,k}^1) &= (2k + 4) \frac{2 \cdot 2}{2 + 2} + 4(k - 1) \frac{2 \cdot 4}{2 + 4} \\
 &= \frac{2}{3}(11k - 2).
 \end{aligned}$$

$$\begin{aligned}\sigma(C_{6,k}^1) &= (2k+4)(2-2)^2 + 4(k-1)(2-4)^2 \\ &= 16(k-1).\end{aligned}$$

$$\begin{aligned}B(C_{6,k}^1) &= (4k+2)\left(2 - \frac{2 \cdot 6k}{5k+1}\right)^2 + (k-1)\left(4 - \frac{2 \cdot 6k}{5k+1}\right)^2 \\ &= \frac{8(2k+1)(k-1)}{5k+1}.\end{aligned}$$

$$\begin{aligned}Irr_t(C_{6,k}^1) &= (4k+2)\left|2 - \frac{12k}{5k+1}\right| + (k-1)\left|4 - \frac{12k}{5k+1}\right| \\ &= \frac{8(2k+1)(k-1)}{5k+1},\end{aligned}$$

which exactly is the same with the Bell index.

$$\begin{aligned}H(C_{6,k}^1) &= (4+2k)\frac{2}{2+2} + 4(k-1)\frac{2}{2+4} \\ &= \frac{7k+2}{3}.\end{aligned}$$

$$\begin{aligned}ABC(C_{6,k}^1) &= (4+2k)\sqrt{\frac{2+2-2}{2 \cdot 2}} + 4(k-1)\sqrt{\frac{2+4-2}{2 \cdot 4}} \\ &= 3\sqrt{2}k.\end{aligned}$$

$$\begin{aligned}GA(C_{6,k}^1) &= (4+2k)\frac{2\sqrt{2 \cdot 2}}{2+2} + 4(k-1)\frac{2\sqrt{2 \cdot 4}}{2+4} \\ &= \frac{(8\sqrt{2}+6)k+12-8\sqrt{2}}{3}.\end{aligned}$$

$$\begin{aligned}AZ(C_{6,k}^1) &= (4+2k)\left(\frac{2 \cdot 2}{2+2-2}\right)^3 + 4(k-1)\left(\frac{2 \cdot 4}{2+4-2}\right)^3 \\ &= 48k.\end{aligned}$$

$$\begin{aligned}Alb(C_{6,k}^1) &= (4+2k)|2-2| + 4(k-1)|2-4| \\ &= 8(k-1).\end{aligned}$$

$$\begin{aligned}R(C_{6,k}^1) &= (4+2k)\frac{1}{\sqrt{2 \cdot 2}} + 4(k-1)\frac{1}{\sqrt{2 \cdot 4}} \\ &= (1+\sqrt{2})k+2-\sqrt{2}.\end{aligned}$$

$$\begin{aligned}
 RR(C_{6,k}^1) &= (4 + 2k)\sqrt{2 \cdot 2} + 4(k - 1)\sqrt{2 \cdot 4} \\
 &= (1 + 2\sqrt{2})4k + 8(1 - \sqrt{2}).
 \end{aligned}$$

□

### 3. Topological indices of hexagonal double chains

Hexagonal double chains are generalizations of the classical hexagonal chains obtained by adding another hexagon inside each hexagon so that two antipodal vertices of both hexagons overlap. We now obtain similar results for the topological indices of hexagonal double chains. As the calculations and methods are similar, we shall omit the proofs. Figure 2 illustrates the hexagonal double chain of length  $k$ .

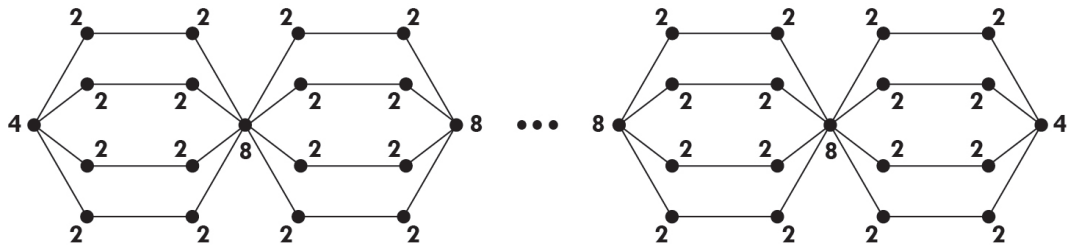


Figure 2. The hexagonal double chain  $C_{6,k}^2$

In the vertex partition of  $C_{6,k}^1$ , there were  $4k + 2$  vertices of degree 2, 2 at the top edge of each hexagon, 2 at the bottom edge of each hexagon, and two vertices at both end vertices at the left and right ends of the chain. The two vertices at both ends will have degree 4 in the hexagonal double chain and these are the only vertices of degree 4. Now, in the vertex partition of  $C_{6,k}^2$ , we have four new extra vertices of degree 2 within each hexagonal unit. Therefore, the number of vertices of degree 2 becomes  $4k + 4k = 8k$ . Finally, there are  $k - 1$  vertices of degree 8 between  $k$  hexagons giving the vertex partition of  $C_{6,k}^2$  as in Table 3. A similar argument gives the size of the hexagonal double chain as in Table 4.

$d_i$	$\#v_i$
2	$8k$
4	2
8	$k - 1$

Table 3. The vertex partition of  $C_{6,k}^2$

$(d_i, d_j)$	$\#(v_i, v_j)$
(2, 2)	$4k$
(2, 4)	8
(2, 8)	$8(k - 1)$

Table 4. The edge partition of  $C_{6,k}^2$

The size of  $C_{6,k}^2$  is  $m = 12k$  and the order is  $n = 9k + 1$ . Now we can list the values of topological graph indices of hexagonal double chains:

**Theorem 3.1.** The 14 degree based topological graph indices of a hexagonal double chain are

$$\begin{aligned}
 M_1(C_{6,k}^2) &= 32(3k - 1) \\
 M_2(C_{6,k}^2) &= 16(9k - 4) \\
 F(C_{6,k}^2) &= 192(3k - 2) \\
 ISI(C_{6,k}^2) &= \frac{84}{5}k - \frac{32}{15} \\
 \sigma(C_{6,k}^2) &= 32(9k - 8) \\
 B(C_{6,k}^2) &= \frac{1}{(9k + 1)^2} (2592k^3 - 1440k^2 - 480k - 32) \\
 Irr_1(C_{6,k}^2) &= \frac{32k}{9k + 1} (3k - 1) \\
 H(C_{6,k}^2) &= \frac{2}{15} (27k + 8) \\
 ABC(C_{6,k}^2) &= 6\sqrt{2}k \\
 GA(C_{6,k}^2) &= \frac{1}{15} [12k(5 + 8\sqrt{2}) - 16\sqrt{2}] \\
 AZ(C_{6,k}^2) &= 96k \\
 Alb(C_{6,k}^2) &= 16(3k - 2) \\
 R(C_{6,k}^2) &= 2(2k + \sqrt{2} - 1) \\
 RR(C_{6,k}^2) &= 8(5k + 2\sqrt{2} - 4).
 \end{aligned}$$

**4. Topological indices of hexagonal triple chains**

Hexagonal triple chains are similar generalizations of the classical hexagonal chains obtained by adding another two hexagons inside each hexagon so that two antipodal vertices of all three hexagons overlap, see Figure 3. The vertex and edge partitions of hexagonal triple chains are given in Tables 5 and 6.

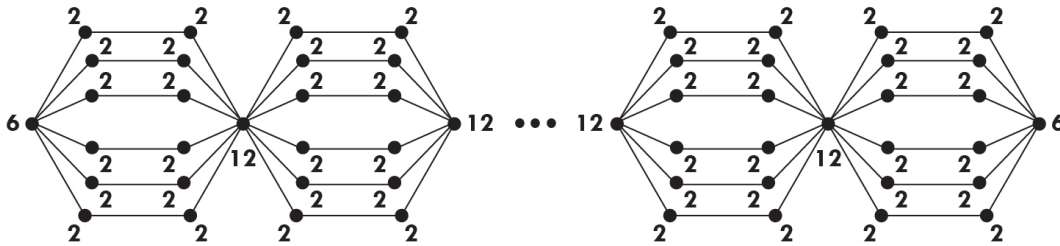


Figure 3. The hexagonal triple chain  $C_{6,k}^3$

$d_i$	$\#v_i$
2	$12k$
6	2
12	$k - 1$

Table 5. The vertex partition of  $C_{6,k}^3$

$(d_i, d_j)$	$\#(v_i, v_j)$
(2, 2)	$6k$
(2, 6)	$12$
(2, 12)	$12(k - 1)$

Table 6. The edge partition of  $C_{6,k}^3$

The size of  $C_{6,k}^3$  is  $m = 18k$  and the order is  $n = 13k + 1$ . Now we can give similar results for the topological indices of hexagonal triple chains again omitting the proofs.

**Theorem 4.1.** *The 14 degree based topological graph indices of a hexagonal triple chain are*

$$\begin{aligned}
 M_1(C_{6,k}^3) &= 24(8k - 3) \\
 M_2(C_{6,k}^3) &= 24(13k - 6) \\
 F(C_{6,k}^3) &= 48(38k - 27) \\
 ISI(C_{6,k}^3) &= \frac{186}{7}k - \frac{18}{7} \\
 \sigma(C_{6,k}^3) &= 16(75k - 63) \\
 B(C_{6,k}^3) &= \frac{1}{(13k + 1)^2} (15600k^3 - 8472k^2 - 1680k - 72) \\
 Irr_1(C_{6,k}^3) &= \frac{24}{13k + 1} (10k^2 - 2k - 1) \\
 H(C_{6,k}^3) &= \frac{3}{7}(11k + 3) \\
 ABC(C_{6,k}^3) &= 9\sqrt{2}k \\
 GA(C_{6,k}^3) &= \frac{6}{7} [(7 + 4\sqrt{6})k + \sqrt{3}(7 - 4\sqrt{2})] \\
 AZ(C_{6,k}^3) &= 144k \\
 Alb(C_{6,k}^3) &= 24(5k - 3) \\
 R(C_{6,k}^3) &= (3 + \sqrt{6})k + (2 - \sqrt{2})\sqrt{3} \\
 RR(C_{6,k}^3) &= 12(1 + 2\sqrt{6})k + 24(1 - \sqrt{2})\sqrt{3}.
 \end{aligned}$$

### 5. Topological indices of hexagonal chain systems

Finally in this section, we generalize all the above results and give the universal result for all hexagonal chain systems. In Figure 4, the most general hexagonal chain system is illustrated. As seen, it contains  $k$  hexagons in a chain each of which contains  $t - 1$  other hexagons all having two antipodal vertices overlapped.

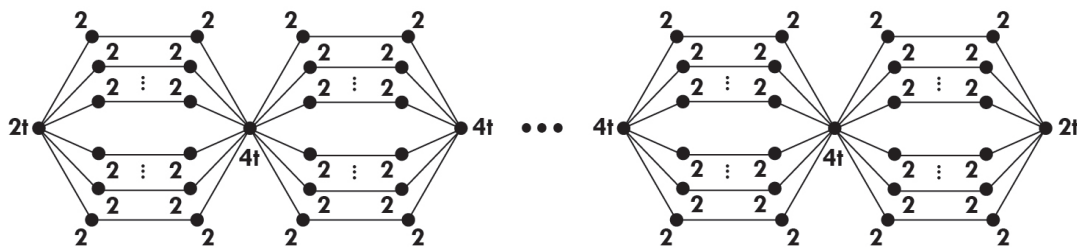


Figure 4. The hexagonal chain system  $C_{6,k}^t$

$d_i$	$\#v_i$
2	$4kt$
$2t$	2
$4t$	$k - 1$

Table 7. The vertex partition of  $C_{6,k}^t$

$(d_i, d_j)$	$\#(v_i, v_j)$
(2, 2)	$2kt$
(2, $2t$ )	$4t$
(2, $4t$ )	$4t(k - 1)$

Table 8. The edge partition of  $C_{6,k}^t$

The size of  $C_{6,k}^t$  is  $m = 6kt$  and the order is  $n = 4kt + k + 1$ . Now

$$\begin{aligned} M_1(C_{6,k}^t) &= 4kt \cdot 2^2 + 2 \cdot (2t)^2 + (k - 1) \cdot (4t)^2 \\ &= 8t(-t + 2kt + 2k). \end{aligned}$$

$$\begin{aligned} M_2(C_{6,k}^t) &= 2kt \cdot (2 \cdot 2) + 4t \cdot (2 \cdot 2t) + 4t(k - 1) \cdot (2 \cdot 4t) \\ &= 8t(-2t + 4tk + k). \end{aligned}$$

$$\begin{aligned} F(C_{6,k}^t) &= 4kt \cdot 2^3 + 2 \cdot (2t)^3 + (k - 1) \cdot (4t)^3 \\ &= 16t(-3t^2 + 4t^2k + 2k). \end{aligned}$$

$$\begin{aligned} ISI(C_{6,k}^t) &= 2kt \cdot \frac{2 \cdot 2}{2 + 2} + 4t \cdot \frac{2 \cdot 2t}{2 + 2t} + 4t(k - 1) \frac{2 \cdot 4t}{2 + 4t} \\ &= \frac{2t}{(1 + 2t)(1 + t)}(10t^2k - 4t + 11tk + k). \end{aligned}$$

$$\begin{aligned} \sigma(C_{6,k}^t) &= 2kt(2 - 2)^2 + 4t(2 - 2t)^2 + 4t(k - 1)(2 - 4t)^2 \\ &= 16t(-3t^2 + 2t + 4t^2k - 4tk + k). \end{aligned}$$

$$\begin{aligned} B(C_{6,k}^t) &= 4kt \left( 2 - \frac{2 \cdot 6kt}{4kt + k + 1} \right)^2 + 2 \left( 2t - \frac{2 \cdot 6kt}{4kt + k + 1} \right)^2 + (k - 1) \left( 4t - \frac{2 \cdot 6kt}{4kt + k + 1} \right)^2 \\ &= \frac{8t}{(4kt + k + 1)^2} (2k^3 + 4k^2 + 2k + 32k^3t^3 - 24k^3t^2 + k^2t - t + 8k^2t^2 - 8kt^2 - 16k^2t^3). \end{aligned}$$

$$\begin{aligned} Irr_t(C_{6,k}^t) &= 4kt \left| 2 - \frac{2 \cdot 6kt}{4kt + k + 1} \right| + 2 \left| 2t - \frac{2 \cdot 6kt}{4kt + k + 1} \right| + (k - 1) \left| 4t - \frac{2 \cdot 6kt}{4kt + k + 1} \right| \\ &= \frac{48kt(k - 1)}{4kt + k + 1}. \end{aligned}$$

$$\begin{aligned} H(C_{6,k}^t) &= 2kt \frac{2}{2 + 2} + 4t \frac{2}{2 + 2t} + 4t(k - 1) \frac{2}{2 + 4t} \\ &= \frac{t}{(1 + t)(1 + 2t)} (5k + 7kt + 2kt^2 + 4t). \end{aligned}$$

$$\begin{aligned} ABC(C_{6,k}^t) &= 2kt \sqrt{\frac{2+2-2}{2 \cdot 2}} + 4t \sqrt{\frac{2+2t-2}{2 \cdot 2t}} + 4t(k-1) \sqrt{\frac{2+4t-2}{2 \cdot 4t}} \\ &= 3\sqrt{2}kt. \end{aligned}$$

$$\begin{aligned} GA(C_{6,k}^t) &= 2kt \cdot \frac{2\sqrt{2 \cdot 2}}{2+2} + 4t \cdot \frac{2\sqrt{2 \cdot 2t}}{2+2t} + 4t(k-1) \frac{2\sqrt{2 \cdot 4t}}{2+4t} \\ &= \frac{1}{(1+t)(1+2t)} \left[ 2kt(1+4\sqrt{2t}) + 2kt^2(3+4\sqrt{2t}) + 8t^2(1-\sqrt{2t}) + 4t^3(k+4) - 8t\sqrt{2t} \right]. \end{aligned}$$

$$\begin{aligned} AZ(C_{6,k}^t) &= 2kt \left( \frac{2 \cdot 2}{2+2-2} \right)^3 + 4t \left( \frac{2 \cdot 2t}{2+2t-2} \right)^3 + 4t(k-1) \left( \frac{2 \cdot 4t}{2+4t-2} \right)^3 \\ &= 48kt. \end{aligned}$$

$$\begin{aligned} Alb(C_{6,k}^t) &= 2kt|2-2| + 4t|2-2t| + 4t(k-1)|2-4t| \\ &= 8t(2kt - k - t). \end{aligned}$$

$$\begin{aligned} R(C_{6,k}^t) &= 2kt \frac{1}{\sqrt{2 \cdot 2}} + 4t \frac{1}{\sqrt{4t}} + 4t(k-1) \frac{1}{\sqrt{8t}} \\ &= \frac{1}{\sqrt{2t}} \left[ kt(\sqrt{2t} + 2) + 2t(\sqrt{2} - 1) \right]. \end{aligned}$$

$$\begin{aligned} RR(C_{6,k}^t) &= 2kt\sqrt{2 \cdot 2} + 4t\sqrt{2 \cdot 2t} + 4t(k-1)\sqrt{2 \cdot 4t} \\ &= 4kt(1+2\sqrt{2t}) + 8t\sqrt{t}(1-\sqrt{2}). \end{aligned}$$

### 6. Omega invariant of hexagonal chain systems

In [5], a useful graph invariant named as omega invariant was introduced by Delen and Cangul as

$$\begin{aligned} \Omega(D) &= \Omega(G) \\ &= \sum_{i=1}^{\Delta} (i-2)a_i \\ &= a_3 + 2a_4 + 3a_5 + \dots + (\Delta-2)a_{\Delta} - a_1, \end{aligned}$$

for a degree sequence  $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \dots, \Delta^{(a_{\Delta})}\}$  or any realization  $G$  of  $D$ . Same fundamental properties, its relation with Euler characteristics and the cyclomatic number are studied in [5, 6]. In particular, it is proven that  $\sigma(G) = 2(m-n)$  implying omega invariant to be an even integer which is not clear from the definition. This means that for a random set  $D$  of non-negative integers, if  $\sigma(D)$  is odd, then  $D$  is not realizable, that is, there is no graph with degree sequence  $D$ . Also for any realization  $G$  of  $D$ ,  $G$  has

$$r = \frac{\Omega(D)}{2} + c \tag{6.1}$$

faces where  $c$  is the number of components.

Now the degree sequences of the graphs  $C_{6,k}^1$ ,  $C_{6,k}^2$ ,  $C_{6,k}^3$  and  $C_{6,k}^t$  respectively are  $D(C_{6,k}^1) = \{2^{(4k+2)}, 4^{(k-1)}\}$ ,  $D(C_{6,k}^2) = \{2^{(8k)}, 4^{(2)}, 8^{(k-1)}\}$ ,  $D(C_{6,k}^3) = \{2^{(12k)}, 6^{(2)}, 12^{(k-1)}\}$  and  $D(C_{6,k}^t) = \{2^{(4kt)}, 2t^{(2)}, 4t^{(k-1)}\}$ . Therefore, using the formula in Eqn. (6.1) for the number of faces  $r$ , we reach to the following result:

**Theorem 6.1.** *The number of faces of the graph  $C_{6,k}^t$  respectively are*

$$r(C_{6,k}^t) = (2t - 1)k.$$

*Proof.* The omega invariant of  $C_{6,k}^t$  is  $4tk - 2k - 2$  and by the formula in Eq. (6.1), we obtain the result as  $c = 1$  for connected graphs.  $\square$

Hence, for hexagonal chain, hexagonal double chain and hexagonal triple chain, the number of faces is  $k$ ,  $3k$ ,  $5k$ , respectively.

## 7. Conclusion

In this work, we studied 14 degree based topological graph indices of some molecular structures called hexagonal chain systems. By means of a recently introduced graph invariant named as omega, we determined some new properties of these systems.

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