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Initial bounds for certain classes of bi-univalent functions involving Rabotnov function subordinated to Horadam polynomial

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Abstract

Recently, interesting results have been obtained by studying Rabotnov functions. In this article, using the normalised Rabotnov functions and the concept of subordination related with Horadam polynomial, a new subclass of analytic bi-univalent function is introduced. Further, determining the Taylor-Maclaurin coefficients for the functions belonging to this subclass, the Fekete-Szegö inequality and Hankel determinant has been investigated.

Keywords: Analytic, bi-univalent functions, Rabotnov function, Horadam polynomials, coefficient estimates, Fekete-Szegö inequality, Hankel determinant

2020 MSC: 30C45, 30C50

1. Introduction, definitions and preliminaries

Let $\mathfrak{A}$ denote the class of functions f defined in the open unit disk $\Delta = \{\zeta : \zeta \in \mathbb{C} \text{ and } |\zeta| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ with the Taylor series expansion

$$f(\zeta) = \zeta + \sum_{k\geq 2} b_k \zeta^k, \quad \zeta \in \Delta. \tag{1.1}$$

Let $\mathfrak{S}$ be the subclass of functions in $\mathfrak{A}$ which are univalent in Δ.

The class $\mathfrak{P}$ consists of functions that are analytic in the open unit disk Δ and satisfy $\Re(p(\zeta)) > 0$. Each $p \in \mathfrak{P}$ has the series expansion

$$p(\zeta) = 1 + \sum_{k\geq 1} u_k \zeta^k. \tag{1.2}$$

For functions $f \in \mathfrak{A}$ given by (1.1) and $g \in \mathfrak{A}$ given by

$$g(\zeta) = \zeta + \sum_{k\geq 2} a_k \zeta^k, \quad \zeta \in \Delta,$$

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we define the Hadamard product (or convolution) of f and g by

$$(f * g)(\zeta) = \zeta + \sum_{k\geq 2} a_k b_k \zeta^k, \quad z \in \Delta.$$

Suppose that $f, g \in \mathfrak{A}$ then f is said to be subordinate to g if there exists an analytic function ω such that

$$\omega(0) = 0, \ |\omega(\zeta)| < 1 \quad \text{and} \quad f(\zeta) = g(\omega(\zeta)), \quad (\zeta \in \Delta).$$

This subordination is denoted by

$$f \prec g \quad \text{or} \quad f(\zeta) \prec g(\zeta), \quad (\zeta \in \Delta).$$

In particular if g is univalent in Δ,

$$f \prec g, \ (\zeta \in \Delta) \implies f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

The l^{th} Hankel determinant $H_l(k)$, $\forall k, l \in \mathbb{N}$ was studied by Noonan and Thomas [21] which is defined as

$$H_l(k) = \begin{vmatrix} b_k & b_{k+1} & \dots & b_{k+l-1} \\ b_{k+1} & b_{k+2} & \dots & b_{k+l} \\ \dots & \dots & \dots & \dots \\ b_{k+l-1} & b_{k+l} & \dots & b_{k+2l-2} \end{vmatrix}, \quad (b_1 = 1). \tag{1.3}$$

For $l = 2$ and $k = 1$, (1.3) reduces to

$$H_2(1) = \begin{vmatrix} b_1 & b_2 \\ b_2 & b_3 \end{vmatrix} = b_3 - b_2^2 \ [\because b_1 = 1],$$

which is the well known Fekete-Szegö inequality $|b_3 - \mu b_2^2|$ with the special case $\mu = 1$.

For $l = 2$ and $k = 2$, (1.3) reduces to

$$H_2(2) = \begin{vmatrix} b_2 & b_3 \\ b_3 & b_4 \end{vmatrix} = b_2 b_4 - b_3^2,$$

which is called as the second Hankel determinant. The coefficient functional has been studied by many researchers like Sokól and Sivasubramanian [12], Rajya Lakshmi et al. [19] and Ali et al. [3] for certain subclasses of $\mathfrak{S}$.

1.1. Rabotnov function

In 1948, Rabotnov [24] a renowned specialist in solid mechanics introduced a distinct function while studying problems in visco-elasticity. His wide-ranging research contributions span plasticity, creep theory, hereditary mechanics, fracture mechanics, nonelastic stability, composite materials and shell theory. The function he proposed is now known as the Rabotnov fractional exponential function or simply the Rabotnov function and is expressed as follows:

$$\mathfrak{Y}_{\wp,\ell}(\zeta) = \zeta^{\wp} \sum_{k\geq 0} \frac{\ell^k}{\Gamma((k+1)(1+\wp))} \zeta^{k(1+\wp)}, \qquad (\wp, \ell, \zeta \in \mathbb{C}) \tag{1.4}$$

(*cf.* [24]).

This function can be regarded as a special case of the classical Mittag-Leffler function which frequently appears in the analysis of fractional integral and differential equations. Their connection is given by

$$\mathfrak{Y}_{\wp,\ell}(\zeta) = \zeta^{\wp} \mathfrak{M}_{1+\wp,1+\wp}\left(\ell \zeta^{1+\wp}\right),$$

where $\mathfrak{M}$ represents the Mittag-Leffler function and $\wp, \ell, \zeta \in \mathbb{C}$ (*cf.* [20]). Extensive studies of the Mittag-Leffler function and its various generalizations may be found in [2, 6, 7].

It is evident that the Rabotnov function $\mathfrak{Y}_{\wp,\ell}(\zeta)$ does not belong to the standard class $\mathfrak{A}$. For this reason researchers have considered a normalized form of the Rabotnov function for $\wp \geq 0$ and $\ell > 0$ which is defined as follows:

$$\mathfrak{T}(\wp,\ell;\zeta) = \zeta^{\frac{1}{1+\wp}}\Gamma(1+\wp)\mathfrak{Y}_{\wp,\ell}\left(\zeta^{\frac{1}{1+\wp}}\right) = \zeta + \sum_{k\geq 2}\frac{\ell^{k-1}\Gamma(\wp+1)}{\Gamma(k(1+\wp))}\zeta^k, \qquad (\wp > -1, \ell \in \mathbb{C})$$

(*cf.* [5]).

Geometric properties such as convexity, close-to-convexity and starlikeness of this normalized function were recently examined by Eker and Ece in [10]. The above power series clearly converges for all values of the argument and when $\wp = 0$ the expression simplifies to the standard exponential function $\exp(\ell\zeta)$.

In this context the normalized Rabotnov function can be written as

$$\begin{aligned}\mathfrak{T}(\wp,\ell;\zeta) &= \zeta + \sum_{k\geq 2}\frac{\ell^{k-1}\Gamma(\wp+1)}{\Gamma(k(1+\wp))}\zeta^k, \qquad (\wp > -1, \ell \in \mathbb{C})\\ &= \zeta + \sum_{k\geq 2}\mathfrak{Y}_k(\ell)\zeta^k \end{aligned} \tag{1.5}$$

where

$$\mathfrak{Y}_k(\ell) = \frac{\ell^{k-1}\Gamma(\wp+1)}{\Gamma(k(1+\wp))}. \tag{1.6}$$

Based on this representation, we now introduce a linear operator $\mathfrak{E}_\ell^\wp : \mathfrak{A} \to \mathfrak{A}$ defined by

$$\mathfrak{E}_\ell^\wp f(\zeta) = \mathfrak{T}(\wp,\ell;\zeta) * f(\zeta),\ \zeta \in \Delta,$$

where the symbol "$*$" denotes the Hadamard (or convolution) product. Thus if $f \in \mathfrak{A}$ has the form (1.1) then

$$\mathfrak{E}_\ell^\wp f(\zeta) = \zeta + \sum_{k\geq 2}\mathfrak{Y}_k(\wp,\ell)b_k\zeta^k, \qquad \zeta \in \Delta. \tag{1.7}$$

For simplicity of notation, we specify the following initial coefficients

$$\mathfrak{Y}_2(\wp,\ell) = \frac{\ell\Gamma(\wp+1)}{\Gamma(2(1+\wp))} \tag{1.8}$$

and

$$\mathfrak{Y}_3(\wp,\ell) = \frac{\ell^2\Gamma(\wp+1)}{\Gamma(3(1+\wp))}.$$

1.2. Horadam polynomials

The Horadam polynomials were introduced by Horadam [14] as a generalization of second- order linear recurrence relations encompassing several well-known polynomial sequences such as Fibonacci, Lucas, Pell and Chebyshev polynomials as special cases. Owing to their flexible algebraic structure these polynomials have found wide applications in number theory, combinatorics, matrix theory and study of differential equations.

The Horadam polynomials $\mathfrak{V}_k(x,c,d;s,t)$ or briefly $\mathfrak{V}_k(x)$ are given by the following recurrence relation

$$\mathfrak{V}_1(x) = c,\ \ \mathfrak{V}_2(x) = dx \quad \text{and} \quad \mathfrak{V}_k(x) = sx\mathfrak{V}_{k-1}(x) + t\mathfrak{V}_{k-2}(x), \quad (k \geq 3), \tag{1.9}$$

for some real constants c, d, s and t (*cf.* [13, 14]).

The generating function of the Horadam polynomials $\mathfrak{V}_k(x)$ is

$$\prod(x,\zeta) = \sum_{k\geq 1}\mathfrak{V}_k(x)\zeta^{k-1} = \frac{c+(d-cs)x\zeta}{1-sx\zeta-t\zeta^2} \tag{1.10}$$

(*cf.* [14]).

The following special cases of the Horadam polynomials can be obtained by appropriately choosing the parameters c, d, s and t (*cf.* [14]):

(i) For $c = d = s = t = 1$, we acquire Fibonacci polynomials $F_k(x)$.

(ii) For $c = 2$ and $d = s = t = 1$, we get the Lucas polynomials $L_k(x)$.

(iii) For $c = t = 1$ and $d = s = 2$, we obtain the Pell polynomials $P_k(x)$.

(iv) For $c = d = s = 2$ and $t = 1$, we acquire the Pell-Lucas polynomials $Q_k(x)$.

(v) For $c = d = 1,\ s = 2$ and $t = -1$, we get the Chebyshev polynomials $T_k(x)$ of the first kind.

(vi) For $c = 1,\ d = s = 2$ and $t = -1$, we have the Chebyshev polynomials $U_k(x)$ of the second kind.

1.3. Bi-univalent function class

Since univalent functions are injective they admit inverses although such inverses may not necessarily be defined throughout the unit disk Δ. However, the famous Koebe one–quarter theorem (see [9]) guarantees that the image of Δ under any $f(\zeta)$ contains a disk of radius $\frac{1}{4}$. Consequently for every $f \in \mathfrak{S}$ there exists an inverse f^{-1} satisfying

$$f^{-1}(f(\zeta)) = \zeta, \quad (\zeta \in \Delta) \text{ and } \quad g(f(\xi)) = \xi, \quad |\xi| < r(f);\ r(f) \geq 1/4.$$

Moreover the inverse function is given by

$$g(\xi) = f^{-1}(\xi) = \xi - a_2\xi^2 + \left(2a_2{}^2 - a_3\right)\xi^3 - \left(5a_3{}^2 - 5a_2a_3 + a_4\right)\xi^4 + \cdots . \tag{1.11}$$

A function $f \in \mathfrak{S}$ is called bi-univalent if both f and its inverse f^{-1} are univalent in Δ. We denote by Σ the family of all such bi-univalent functions in $\mathfrak{S}$ represented by (1.1). For instance, the functions

$$f_1(\zeta) = \frac{\zeta}{1-\zeta}, \quad f_2(\zeta) = \frac{1}{2}\log\left(\frac{1+\zeta}{1-\zeta}\right) \quad \text{and} \quad f_3(\zeta) = -\log(1-\zeta)$$

whose respective inverses are

$$f_1^{-1}(\xi) = \frac{\xi}{1+\xi}, \quad f_2^{-1}(\xi) = \frac{e^{2\xi}-1}{e^{2\xi}+1} \quad \text{and} \quad f_3^{-1}(\xi) = \frac{e^{\xi}-1}{e^{\xi}}.$$

These functions lie in Σ though the well-known Koebe function does not lie in Σ. The study of the class Σ was investigated by Lewin [17] and established the bound $|b_2| < 1.51$. Subsequently, Brannan and Clunie [8] motivated by Lewin's result conjectured that $|b_2| \leq \sqrt{2}$. Later Srivastava et al. [26] examined class Σ in connection with coefficient problems a line of research that remains open for $|b_k|$, $(k \in \mathbb{N}; k \geq 3)$. This work has motivated several researchers in the theory of bi-univalent functions in Σ. Recently coefficient estimates for functions in the class of univalent and bi-univalent functions associated with special polynomials such as the Chebyshev polynomials (see [11]), Faber polynomials (see [15]) and Horadam polynomials (see [27]) have been determined.

Definition 1.1 (*cf.* [25])**.** For $\alpha \geq 0$ and $0 \leq \beta < 1$, a function f belongs to the subclass $SD(\alpha,\beta)$ if it satisfies

$$\Re\left(\frac{f(\zeta)}{\zeta} - \beta\right) \geq \alpha\left|f'(\zeta) - \frac{f(\zeta)}{\zeta}\right|, \quad \zeta \in \Delta. \tag{1.12}$$

Motivated by the earlier study on class $SD(\alpha,\beta)$ (see [25]) and recent studies on bi-univalent functions (see [4, 1, 11, 15, 28]) we introduce a new subclass of convex functions and a related subclass of bi-univalent functions involving Rabotnov fractional exponential function subordinated with Horadam polynomial. These subclasses are given in Definitions 1.2 and 1.3. For functions belonging to them we derive estimates for the initial Taylor-Maclaurin series coefficients, Fekete-Szegö inequalities and Hankel determinants.

Definition 1.2. For $\alpha \geq 0, 0 \leq \beta < 1, \wp > -1$ and $\ell \in \mathbb{C}$, a function $f(\zeta)$ given by (1.1) belongs to the subclass $\mathfrak{V}\mathfrak{E}_\ell^\wp(\alpha, x, \zeta)$ if it satisfies

$$\frac{\mathfrak{E}_\ell^\wp f(\zeta)}{\zeta} - \alpha\left(\left(\mathfrak{E}_\ell^\wp f(\zeta)\right)' - \frac{\mathfrak{E}_\ell^\wp f(\zeta)}{\zeta}\right) \prec \prod(x,\zeta) + 1 - c, \tag{1.13}$$

where c is the real constant as in (1.9).

Definition 1.3. For $\alpha \geq 0, 0 \leq \beta < 1, \wp > -1$ and $\ell \in \mathbb{C}$, a function $f(\zeta)$ given by (1.1) belongs to the subclass $\mathfrak{V}\mathfrak{E}_{\ell}^{\wp}(\alpha, x, \zeta, \xi)$ if both the following conditions hold

$$\frac{\mathfrak{E}_{\ell}^{\wp} f(\zeta)}{\zeta} - \alpha\left(\left(\mathfrak{E}_{\ell}^{\wp} f(\zeta)\right)' - \frac{\mathfrak{E}_{\ell}^{\wp} f(\zeta)}{\zeta}\right) \prec \prod(x, \zeta) + 1 - c \tag{1.14}$$

and for $g(\xi) = f^{-1}(\zeta)$ given by (1.11)

$$\frac{\mathfrak{E}_{\ell}^{\wp} g(\xi)}{\xi} - \alpha\left(\left(\mathfrak{E}_{\ell}^{\wp} g(\xi)\right)' - \frac{\mathfrak{E}_{\ell}^{\wp} f(\xi)}{\xi}\right) \prec \prod(x, \xi) + 1 - c, \tag{1.15}$$

where c is the real constant as in (1.9).

2. Coefficient estimates of the class $\mathfrak{V}\mathfrak{E}_{\ell}^{\wp}(\alpha, x, \zeta)$ and $\mathfrak{V}\mathfrak{E}_{\ell}^{\wp}(\alpha, x, \zeta, \xi)$

This section establishes initial coefficient bounds for functions belonging to the classes $\mathfrak{V}\mathfrak{E}_{\ell}^{\wp}(\alpha, x, \zeta)$ and $\mathfrak{V}\mathfrak{E}_{\ell}^{\wp}(\alpha, x, \zeta, \xi)$. These estimates form the basis for the subsequent study on the Fekete-Szegö problem and Hankel Determinant.

Let $\mathfrak{B} = \{\omega \in \mathfrak{A} : |\omega(\zeta)| \leq 1, \zeta \in \Delta\}$ and $\mathfrak{B}_0$ be the subclass of $\mathfrak{B}$ of all ω such that $\omega(0) = 0$. The elements of $\mathfrak{B}_0$ are known as Schwarz functions.

We will need the following lemma to prove the main theorem of this section.

Lemma 2.1 (*cf.* [16])**.** *If $\omega \in \mathfrak{B}_0$ is of the form*

$$\omega(\zeta) = \sum_{k \geq 1} \omega_k \zeta^k, \quad \zeta \in \Delta \tag{2.1}$$

then for $\nu \in \mathbb{C}$

$$\left|\omega_2 - \nu\omega_1^2\right| \leq \max\{1, |\nu|\}. \tag{2.2}$$

Lemma 2.2. *If $p \in \mathfrak{P}$ and has the series of the form* (1.2) *, then*

$$|u_{k+n} - \mu u_k u_n| \leq 2 \tag{2.3}$$

$$|u_2 - \eta u_1^2| \leq 2\max\{1, |2\eta - 1|\}, \eta \in \mathbb{C} \tag{2.4}$$

$$|J u_1^3 - K u_1 u_2 + L u_3| \leq 2|J| + |K - 2J| + 2|J - K + L|. \tag{2.5}$$

We note that the inequalities (2.3) and (2.5) in the above lemma can be found in [22] and (2.4) is from [16].

Theorem 2.3. *Let the function $f(\zeta) \in \mathfrak{V}\mathfrak{E}_{\ell}^{\wp}(\alpha, x, \zeta)$. Then*

$$|b_2| \leq \frac{|dx|}{(1-\alpha)\mathfrak{Y}_2(\wp, \ell)}, \tag{2.6}$$

$$|b_3| \leq \frac{|dx|}{(1-2\alpha)\mathfrak{Y}_3(\wp, \ell)} \max\left\{1, \frac{dsx^2 + tc}{dx}\right\}, \tag{2.7}$$

$$|b_4| \leq \frac{8|dsx^2 + tc||1 + sx| + 2dx|1 + t|}{(1-3\alpha)\mathfrak{Y}_4(\wp, \ell)} \tag{2.8}$$

and

$$|b_3 - \nu {b_2}^2| \leq \frac{|dx|}{(1-2\alpha)\mathfrak{Y}_3(\wp, \ell)} \max\left\{1, \left| - sx - \frac{tc}{dx} + \frac{\nu dx(1-2\alpha)\mathfrak{Y}_3(\wp, \ell)}{(1-\alpha)^2\mathfrak{Y}_2(\wp, \ell)^2} . \right|\right\}. \tag{2.9}$$

for some ν in $\mathbb{C}$.

Proof. Let $f(\zeta)$ be in the class $\mathfrak{V}\mathfrak{E}_{\ell}^{\wp}(\alpha, x, \zeta)$. We have two analytic functions u and v defined in the unit disk Δ such that $u(0) = v(0) = 0$

$$|u(\zeta)| = \left|u_1\zeta + u_2\zeta^2 + u_3\zeta^3 + \dots\right| < 1$$

then

$$|u_i| \leq 1 \quad \text{for} \quad i \in \mathbb{N}. \tag{2.10}$$

As $f \in \mathfrak{V}\mathfrak{E}_{\ell}^{\wp}(\alpha, x, \zeta)$ we make use of (1.13)

$$\frac{\mathfrak{E}_{\ell}^{\wp} f(\zeta)}{\zeta} - \alpha\left((\mathfrak{E}_{\ell}^{\wp} f(\zeta))' - \frac{\mathfrak{E}_{\ell}^{\wp} f(\zeta)}{\zeta}\right) \prec \prod(x, \zeta) + 1 - c$$

and it can equivalently be written as

$$\begin{aligned}(1+\alpha)\left(1 + \sum_{k\geq 2} \mathfrak{Y}_k(\wp, \ell) b_k \zeta^{k-1}\right) - \alpha\left(1 + \sum_{k\geq 2} k\mathfrak{Y}_k(\wp, \ell) b_k \zeta^{k-1}\right) &= 1 + \mathfrak{V}_1(x) + \mathfrak{V}_2(x)u(\zeta) \\ &\quad + \mathfrak{V}_3(x)(u(\zeta))^2 + \cdots - c.\end{aligned} \tag{2.11}$$

From the equality (2.11) we get

$$\begin{aligned}(1+\alpha)\left(1 + \sum_{k\geq 2} \mathfrak{Y}_k(\wp, \ell) b_k \zeta^{k-1}\right) &- \alpha\left(1 + \sum_{k\geq 2} k\mathfrak{Y}_k(\wp, \ell) b_k \zeta^{k-1}\right) \\ &= 1 + \mathfrak{V}_2(x)\zeta + \left[\mathfrak{V}_2(x)u_2 + \mathfrak{V}_3(x)u_1^2\right]\zeta^2 + \cdots - c.\end{aligned} \tag{2.12}$$

Comparing the coefficients of (2.12) we get

$$(1-\alpha)b_2\mathfrak{Y}_2(\wp, \ell) = \mathfrak{V}_2(x)u_1, \tag{2.13}$$

$$(1-2\alpha)b_3\mathfrak{Y}_3(\wp, \ell) = \mathfrak{V}_2(x)u_2 + \mathfrak{V}_3(x){u_1}^2 \tag{2.14}$$

and

$$(1-3\alpha)b_4\mathfrak{Y}_4(\wp, \ell) = \mathfrak{V}_2(x)u_3 + 2\mathfrak{V}_3(x)u_1u_2 + \mathfrak{V}_4(x)u_1^3. \tag{2.15}$$

It follows from (2.13)

$$b_2 = \frac{\mathfrak{V}_2(x)u_1}{(1-\alpha)\mathfrak{Y}_2(\wp, \ell)} \tag{2.16}$$

and

$$|b_2| \leq \frac{|dx|}{(1-\alpha)\mathfrak{Y}_2(\wp, \ell)}. \tag{2.17}$$

Now we get

$$\begin{aligned} b_3 &= \frac{\mathfrak{V}_2(x)u_2 + \mathfrak{V}_3(x){u_1}^2}{(1-2\alpha)\mathfrak{Y}_3(\wp, \ell)} \\ |b_3| &\leq \frac{|dxu_2 + 2(dsx^2 + tc)u_1^2|}{(1-2\alpha)\mathfrak{Y}_3(\wp, \ell)} \\ &= \frac{|dx|}{(1-2\alpha)\mathfrak{Y}_3(\wp, \ell)}\left|u_2 - \mu u_1^2\right| \end{aligned} \tag{2.18}$$

where $\mu = -\frac{dsx^2+tc}{dx}$.

From (2.15) we get

$$b_4 = \frac{\mathfrak{V}_2(x)u_3 + 2\mathfrak{V}_3(x)u_1u_2 + \mathfrak{V}_4(x)u_1^3}{(1-3\alpha)\mathfrak{Y}_3(\wp,\ell)}$$
$$|b_4| \leq \frac{|dxu_3 + (dsx^2+tc)u_1u_2 + (ds^2x^3 + tscx + tdx)u_1^3|}{(1-3\alpha)\mathfrak{Y}_4(\wp,\ell)}.$$

Now for some $\nu \in \mathbb{C}$ and using the values of b_2 and b_3 we get

$$\begin{aligned} b_3 - \nu b_2^2 &= \frac{\mathfrak{V}_2(x)u_2 + \mathfrak{V}_3(x)u_1^2}{(1-2\alpha)\mathfrak{Y}_3(\wp,\ell)} - \nu\frac{\mathfrak{V}_2^2(x)u_1^2}{(1-\alpha)^2\mathfrak{Y}_2(\wp,\ell)^2} \\ &= \frac{dxu_2 + dsx^2u_1^2 + tcu_1^2}{(1-2\alpha)\mathfrak{Y}_3(\wp,\ell)} - \nu\frac{d^2x^2u_1^2}{(1-\alpha)^2\mathfrak{Y}_2(\wp,\ell)^2} \\ &= \frac{dx}{(1-2\alpha)\mathfrak{Y}_3(\wp,\ell)}\left\{u_2 - \eta u_1^2\right\}, \end{aligned}$$

where $\eta = -sx - \frac{tc}{dx} + \frac{\nu dx(1-2\alpha)\mathfrak{Y}_3(\wp,\ell)}{(1-\alpha)^2\mathfrak{Y}_2(\wp,\ell)^2}$.

From Lemma 2.1 we have

$$|b_3 - \nu {b_2}^2| \leq \frac{|dx|}{(1-2\alpha)\mathfrak{Y}_3(\wp,\ell)}\max\left\{1, |-sx - \frac{tc}{dx} + \frac{\nu dx(1-2\alpha)\mathfrak{Y}_3(\wp,\ell)}{(1-\alpha)^2\mathfrak{Y}_2(\wp,\ell)^2}|\right\}. \tag{2.19}$$

□

Theorem 2.4. *Let the function $f(\zeta) \in \mathfrak{V}\mathfrak{E}_\ell^\wp(\alpha, x, \zeta, \xi)$. Then*

$$|b_2| \leq \frac{|dx|\sqrt{|dx|}}{\sqrt{\left|(dx(1-2\alpha)\mathfrak{Y}_3(\wp,\ell) - sx(1-\alpha)^2\mathfrak{Y}_2(\wp,\ell)^2)dx - tc(1-\alpha)^2\mathfrak{Y}_2(\wp,\ell)^2\right|}} \tag{2.20}$$

and

$$|b_3| \leq \frac{|dx|}{|(1-2\alpha)\mathfrak{Y}_3(\wp,\ell)|} + \frac{|d^2x^2|}{|(1-\alpha)^2\mathfrak{Y}_2(\wp,\ell)^2|}. \tag{2.21}$$

Proof. Let $f(\zeta)$ be in the class $\mathfrak{V}\mathfrak{E}_\ell^\wp(\alpha, x, \zeta, \xi)$. We have two analytic functions u and v defined in the unit disk Δ such that $u(0) = v(0) = 0$,

$$|u(\zeta)| = \left|u_1\zeta + u_2\zeta^2 + u_3\zeta^3 + \cdots\right| < 1$$

and

$$|v(\xi)| = \left|v_1w + v_2w^2 + v_3w^3 + \cdots\right| < 1,$$

then

$$|u_i| \leq 1 \quad \text{and} \quad |v_i| \leq 1 \quad \text{for} \quad i \in \mathbb{N} \tag{2.22}$$

As $f \in \mathfrak{V}\mathfrak{E}_\ell^\wp(\alpha, x, \zeta, \xi)$ we make use of (1.14) and (1.15)

$$\frac{\mathfrak{E}_\ell^\wp f(\zeta)}{\zeta} - \alpha\left(\left(\mathfrak{E}_\ell^\wp f(\zeta)\right)' - \frac{\mathfrak{E}_\ell^\wp f(\zeta)}{\zeta}\right) \prec \prod(x,\zeta) + 1 - c \tag{2.23}$$

and for $g(\xi) = f^{-1}(\zeta)$ given by (1.11)

$$\frac{\mathfrak{E}_\ell^\wp g(\xi)}{\xi} - \alpha\left(\left(\mathfrak{E}_\ell^\wp g(\xi)\right)' - \frac{\mathfrak{E}_\ell^\wp f(\xi)}{\xi}\right) \prec \prod(x,\xi) + 1 - c. \tag{2.24}$$

Similarly

$$(1+\alpha)\left(1+\sum_{k\geq 2}\mathfrak{Y}_k(\wp,\ell)b_k\zeta^{k-1}\right)-\alpha\left(1+\sum_{k\geq 2}k\mathfrak{Y}_k(\wp,\ell)b_k\zeta^{k-1}\right) = 1+\mathfrak{V}_1(x)+\mathfrak{V}_2(x)u(\zeta) +\mathfrak{V}_3(x)(u(\zeta))^2+\cdots-c \tag{2.25}$$

and

$$(1+\alpha)\left(1+\sum_{k\geq 2}\mathfrak{Y}_k(\wp,\ell)b_k\xi^{k-1}\right)-\alpha\left(1+\sum_{k\geq 2}k\mathfrak{Y}_k(\wp,\ell)b_k\xi^{k-1}\right)=1+\mathfrak{V}_1(x)+\mathfrak{V}_2(x)v(\xi) + \mathfrak{V}_3(x)(v(\xi))^2+\cdots-c. \tag{2.26}$$

From the equations (2.25) and (2.26), we get

$$(1+\alpha)\left(1+\sum_{k\geq 2}\mathfrak{Y}_k(\wp,\ell)b_k\zeta^{k-1}\right)-\alpha\left(1+\sum_{k\geq 2}k\mathfrak{Y}_k(\wp,\ell)b_k\zeta^{k-1}\right)=1+\mathfrak{V}_2(x)\zeta+\left[\mathfrak{V}_2(x)u_2+\mathfrak{V}_3(x)u_1^2\right]\zeta^2+\cdots-c \tag{2.27}$$

and

$$(1+\alpha)\left(1+\sum_{k\geq 2}\mathfrak{Y}_k(\wp,\ell)b_k\xi^{k-1}\right)-\alpha\left(1+\sum_{k\geq 2}k\mathfrak{Y}_k(\wp,\ell)b_k\xi^{k-1}\right)=1+\mathfrak{V}_2(x)\xi + \left[\mathfrak{V}_2(x)v_2+\mathfrak{V}_3(x)v_1^2\right]\xi^2+\cdots-c. \tag{2.28}$$

Now comparing the coefficients of (2.27) and (2.28) we get

$$(1-\alpha)\mathfrak{Y}_2(\wp,\ell)b_2=\mathfrak{V}_1(x)u_1, \tag{2.29}$$

$$(1-2\alpha)\mathfrak{Y}_3(\wp,\ell)b_3=\mathfrak{V}_2(x)u_2+\mathfrak{V}_3(x){u_1}^2, \tag{2.30}$$

$$-(1-\alpha)\mathfrak{Y}_2(\wp,\ell)b_2=\mathfrak{V}_2(x)v_1 \tag{2.31}$$

and

$$(1-2\alpha)(2{b_2}^2-b_3)\mathfrak{Y}_3(\wp,\ell)=\mathfrak{V}_2(x)v_2+\mathfrak{V}_3(x){v_1}^2. \tag{2.32}$$

It follows from (2.29) and (2.31)

$$u_1=-v_1 \tag{2.33}$$

and

$$2(1-\alpha)^2\mathfrak{Y}_2(\wp,\ell)^2{b_2}^2=\mathfrak{V}_2(x)^2\left({u_1}^2+{v_1}^2\right). \tag{2.34}$$

If we add (2.30) and (2.32) we get

$$2{b_2}^2(1-2\alpha)\mathfrak{Y}_3(\wp,\ell)=\mathfrak{V}_2(x)(u_2+v_2)+\mathfrak{V}_3(x)\left({u_1}^2+{v_1}^2\right). \tag{2.35}$$

Substituting the value of $\left({u_1}^2+{v_1}^2\right)$ from (2.34) in the right-hand side of (2.35), we deduce that

$$\left[2[\mathfrak{V}_2(x)]^2(1-2\alpha)\mathfrak{Y}_3(\wp,\ell)-2\mathfrak{V}_3(x)(1-\alpha)^2\mathfrak{Y}_2(\wp,\ell)^2\right]{b_2}^2=[\mathfrak{V}_2(x)]^3(u_2+v_2) \tag{2.36}$$

$$\implies {b_2}^2=\frac{[\mathfrak{V}_2(x)]^3(u_2+v_2)}{2[\mathfrak{V}_2(x)]^2(1-2\alpha)\mathfrak{Y}_3(\wp,\ell)-2\mathfrak{V}_3(x)(1-\alpha)^2\mathfrak{Y}_2(\wp,\ell)^2}. \tag{2.37}$$

Substituting $\mathfrak{V}_2(x) = dx$ and $\mathfrak{V}_3(x) = dsx^2 + tc$ into (2.37),

$$|b_2| \leq \frac{|dx|\sqrt{|dx|}}{\sqrt{\left|(dx(1-2\alpha)\mathfrak{Y}_3(\wp,\ell) - sx(1-\alpha)^2\mathfrak{Y}_2(\wp,\ell)^2)dx - tc(1-\alpha)^2\mathfrak{Y}_2(\wp,\ell)^2\right|}}. \tag{2.38}$$

Now we subtract (2.32) and (2.30) and get

$$2(1-2\alpha)\left(b_3 - {b_2}^2\right) = \mathfrak{V}_2(x)(u_2 - v_2) + \mathfrak{V}_3(x)\left({u_1}^2 - {v_1}^2\right). \tag{2.39}$$

Substituting $u_1 = -v_1$ and $u_1^2 = v_1^2$ into the above equation, we get

$$b_3 = \frac{\mathfrak{V}_2(x)(u_2 - v_2)}{2(1-2\alpha)\mathfrak{Y}_3(\wp,\ell)} + {b_2}^2. \tag{2.40}$$

Solving (2.34), we get

$$|b_3| \leq \frac{|dx|}{|(1-2\alpha)\mathfrak{Y}_3(\wp,\ell)|} + \frac{|d^2x^2|}{|(1-\alpha)^2\mathfrak{Y}_2(\wp,\ell)^2|}. \tag{2.41}$$

□

3. Fekete-Szegö inequality for functions in $\mathfrak{V}\mathfrak{E}_\ell^\varphi(\alpha, x, \zeta, \xi)$

In this section we consider the Fekete-Szegö inequality for the functions in the class $\mathfrak{V}\mathfrak{E}_\ell^\varphi(\alpha, x, \zeta, \xi)$.

Theorem 3.1. *Let the function $f \in \mathfrak{V}\mathfrak{E}_\ell^\varphi(\alpha, x, \zeta, \xi)$. Then, for some $\mu \in \mathbb{R}$,*

$$|b_3 - \mu {b_2}^2| \leq \begin{cases} \frac{|dx|}{(1-2\alpha)\mathfrak{Y}_3(\wp,\ell)}, & 0 \leq \Omega(\mu,x) \leq \frac{1}{2(1-2\alpha)\mathfrak{Y}_3(\wp,\ell)} \\ 2|dx||\Omega(\mu,x)|, & \Omega(\mu,x) \geq \frac{1}{2(1-2\alpha)\mathfrak{Y}_3(\wp,\ell)}. \end{cases} \tag{3.1}$$

Proof. For some real number μ, using equation (2.40) we have

$$b_3 - \mu {b_2}^2 = \frac{\mathfrak{V}_2(x)(u_2 - v_2)}{2(1-2\alpha)\mathfrak{Y}_3(\wp,\ell)} + (1-\mu){b_2}^2. \tag{3.2}$$

Using the equation (2.37), we have

$$b_3 - \mu {b_2}^2 = \frac{\mathfrak{V}_2(x)(u_2 - v_2)}{2(1-2\alpha)\mathfrak{Y}_3(\wp,\ell)} + (1-\mu)\frac{[\mathfrak{V}_2(x)]^3(u_2 + v_2)}{2[\mathfrak{V}_2(x)]^2(1-2\alpha)\mathfrak{Y}_3(\wp,\ell) - 2\mathfrak{V}_3(x)(1-\alpha)^2\mathfrak{Y}_2(\wp,\ell)^2},$$

$$\begin{aligned} |b_3 - \mu {b_2}^2| &= \mathfrak{V}_2(x)\Bigg[\left[\frac{(1-\mu)\mathfrak{V}_2(x)^2}{2[\mathfrak{V}_2(x)]^2(1-2\alpha)\mathfrak{Y}_3(\wp,\ell) - 2\mathfrak{V}_3(x)(1-\alpha)^2\mathfrak{Y}_2(\wp,\ell)^2} + \frac{1}{2(1-2\alpha)\mathfrak{Y}_3(\wp,\ell)}\right]u_2 \\ &\quad + \left[\frac{(1-\mu)\mathfrak{V}_2(x)^2}{2[\mathfrak{V}_2(x)]^2(1-2\alpha)\mathfrak{Y}_3(\wp,\ell) - 2\mathfrak{V}_3(x)(1-\alpha)^2\mathfrak{Y}_2(\wp,\ell)^2} - \frac{1}{2(1-2\alpha)\mathfrak{Y}_3(\wp,\ell)}\right]v_2\Bigg] \\ &= [\mathfrak{V}_2(x)]\left[\left[\Omega(\mu,x) + \frac{1}{2(1-2\alpha)}\right]u_2 + \left[\Omega(\mu,x) + \frac{1}{2(1-2\alpha)}\right]v_2\right], \end{aligned} \tag{3.3}$$

where

$$\Omega(\mu,x) = \frac{(1-\mu)\mathfrak{V}_2(x)^2}{2[\mathfrak{V}_2(x)]^2(1-2\alpha)\mathfrak{Y}_3(\wp,\ell) - 2\mathfrak{V}_3(x)(1-\alpha)^2\mathfrak{Y}_2(\wp,\ell)^2}.$$

Hence from (1.9)

$$|b_3 - \mu {b_2}^2| \leq \begin{cases} \frac{|dx|}{(1-2\alpha)\mathfrak{Y}_3(\wp,\ell)}, & 0 \leq \Omega(\mu,x) \leq \frac{1}{2(1-2\alpha)\mathfrak{Y}_3(\wp,\ell)} \\ 2|dx||\Omega(\mu,x)|, & \Omega(\mu,x) \geq \frac{1}{2(1-2\alpha)\mathfrak{Y}_3(\wp,\ell)}. \end{cases}$$

□

4. Hankel Determinant of the subclass $\mathfrak{V}\mathfrak{E}_\ell^{\wp}(\alpha, x, \zeta)$

This section focuses on the Hankel determinant associated with functions in the subclass $\mathfrak{V}\mathfrak{E}_\ell^{\wp}(\alpha, x, \zeta)$. In order to find the estimates of the Hankel Determinant we make use of the following lemma:

Lemma 4.1 (*cf.* [18]). *Let $p(\zeta) \in P$ with $u_1 \geq 0$. Then*

$$2u_2 = u_1^2 + x(4 - u_1^2), \tag{4.1}$$

$$4u_3 = u_1^3 + 2(4 - u_1^2)u_1 x - u_1(4 - u_1^2)x^2 + 2(4 - u_1^2)(1 - |x|^2)\zeta. \tag{4.2}$$

for some x, ζ with $|x| \leq 1$, $|\zeta| \leq 1$ and $u_1 \in [0, 2]$.

Theorem 4.2. *If $f \in \mathfrak{V}\mathfrak{E}_\ell^{\wp}(\alpha, x, \zeta)$ then*

$$|b_2 b_4 - b_3^2| \leq \frac{16d^2}{(1-2\alpha)^2 \mathfrak{Y}_3(\wp, \ell)^2} + \frac{10d^2}{(1-\alpha)(1-3\alpha)\mathfrak{Y}_2(\wp, \ell)\mathfrak{Y}_4(\wp, \ell)}. \tag{4.3}$$

Proof. Using equations (2.13), (2.14) and (2.15), we get

$$b_2 b_4 - b_3^2 = \frac{\mathfrak{V}_2(x)u_1}{(1-\alpha)\mathfrak{Y}_2(\wp, \ell)}\left(\frac{\mathfrak{V}_2(x)u_3 + 2\mathfrak{V}_3(x)u_1 u_2 + \mathfrak{V}_4(x)u_1^3}{(1-3\alpha)\mathfrak{Y}_3(\wp, \ell)}\right) - \left(\frac{\mathfrak{V}_2(x)u_2 + \mathfrak{V}_3(x){u_1}^2}{(1-2\alpha)\mathfrak{Y}_3(\wp, \ell)}\right)^2. \tag{4.4}$$

Substituting $\mathfrak{V}_2(x) = dx$, $\mathfrak{V}_3(x) = dsx^2 + tc$ and $\mathfrak{V}_4(x) = ds^2x^3 + tsxc + tdx$, we get

$$\begin{aligned} b_2 b_4 - b_3^2 &= \left(\frac{dx(dsx^3 + tsxc + tdx)}{(1-\alpha)(1-3\alpha)\mathfrak{Y}_2(\wp, \ell)\mathfrak{Y}_4(\wp, \ell)} - \frac{(dsx^2 + tc)^2}{(1-2\alpha)^2\mathfrak{Y}_3(\wp, \ell)^2}\right)u_1^4 \\ &\quad + \left(\frac{2dx(dsx^2 + tc)}{(1-\alpha)(1-3\alpha)\mathfrak{Y}_2(\wp, \ell)\mathfrak{Y}_4(\wp, \ell)} - \frac{2dx(dsx^2 + tc)}{(1-2\alpha)^2\mathfrak{Y}_3(\wp, \ell)^2}\right)u_1^2 u_2 \\ &\quad + \left(\frac{d^2x^2}{(1-\alpha)(1-3\alpha)\mathfrak{Y}_2(\wp, \ell)\mathfrak{Y}_4(\wp, \ell)}\right)u_1 u_3 + \left(\frac{-d^2x^2}{(1-2\alpha)^2\mathfrak{Y}_3(\wp, \ell)^2}\right)u_2^2 \\ &= l_1 u_1^4 + l_2 u_1^2 u_2 + l_3 u_1 u_3 + l_4 u_2^2, \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} l_1 &= \frac{dx(dsx^3 + tsxc + tdx)}{(1-\alpha)(1-3\alpha)\mathfrak{Y}_2(\wp, \ell))\mathfrak{Y}_4(\wp, \ell)} - \frac{(dsx^2 + tc)^2}{(1-2\alpha)^2\mathfrak{Y}_3(\wp, \ell)^2}, \\ l_2 &= \frac{2dx(dsx^2 + tc)}{(1-\alpha)(1-3\alpha)\mathfrak{Y}_2(\wp, \ell)\mathfrak{Y}_4(\wp, \ell)} - \frac{2dx(dsx^2 + tc)}{(1-2\alpha)^2\mathfrak{Y}_3(\wp, \ell)^2}, \\ l_3 &= \frac{d^2x^2}{(1-\alpha)(1-3\alpha)\mathfrak{Y}_2(\wp, \ell)\mathfrak{Y}_4(\wp, \ell)}, \\ l_4 &= \frac{-d^2x^2}{(1-2\alpha)^2\mathfrak{Y}_3(\wp, \ell)^2}. \end{aligned}$$

Using Lemma 4.1 in (4.5)

$$\begin{aligned} b_2 b_4 - b_3^2 &= l_1 u_1^4 + l_2 u_1^2\left[\frac{1}{2}\left(u_1^2 + x\left(4 - u_1^2\right)\right)\right] + l_3 u_1\left[\frac{1}{4}\left(u_1^2 + 2\left(4 - u_1^2\right)u_1 x - u_1\left(4 - u_1^2\right)x^2 + 2\left(4 - u_1^2\right)\left(1 - |x|^2\right)\zeta\right)\right] \\ &\quad + l_4\left[\frac{1}{2}\left(u_1^2 + x\left(4 - u_1^2\right)\right)\right]^2 \\ &= \left(l_1 + \frac{l_2}{2} + \frac{l_3}{4} + \frac{l_4}{4}\right)u_1^4 + \left(\frac{l_2}{2} + \frac{l_3}{2} + \frac{l_4}{2}\right)u_1^2\left(4 - u_1^2\right)x \\ &\quad + \left(\frac{l_4}{4}(4 - u_1^2) - \frac{l_3}{4}u_1^2\right)\left(4 - u_1^2\right)x^2 + \frac{l_3 u_1}{2}\left(4 - u_1^2\right)\left(1 - |x|^2\right)\zeta \\ &= A\,u_1^4 + B\,u_1^2\left(4 - u_1^2\right)x + \left(\frac{l_4}{4}\left(4 - u_1^2\right) - \frac{l_3}{4}u_1^2\right)(4 - u_1^2)x^2 + \frac{l_3 u_1}{2}\left(4 - u_1^2\right)\left(1 - |x|^2\right)\zeta, \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} A &= l_1 + \frac{l_2}{2} + \frac{l_3}{4} + \frac{l_4}{4} \\ &= \frac{1}{4}\left(\frac{4d^2s^2x^4 + 4tsdx^2c + 4td^2x^2 + 4d^2sx^3 + 4dtcx + d^2x^2}{(1-\alpha)(1-3\alpha)\mathfrak{Y}_2(\wp,\ell)\mathfrak{Y}_4(\wp,\ell)} - \frac{4d^2s^2x^4 + 4t^2c^2 + 8ds^2xt + 4d^2sx^3 + 4dtcx + d^2x^2}{(1-2\alpha)^2\mathfrak{Y}_3(\wp,\ell)^2}\right) \end{aligned}$$

and

$$\begin{aligned} B &= \frac{l_2}{2} + \frac{l_3}{2} + \frac{l_4}{2} \\ &= \frac{1}{2}\left(\frac{2d^2sx^3 + 2tcdx + d^2x^2}{(1-\alpha)(1-3\alpha)\mathfrak{Y}_2(\wp,\ell)\mathfrak{Y}_4(\wp,\ell)} - \frac{2d^2sx^3 + 2dxtc + d^2x^2}{(1-2\alpha)^2\mathfrak{Y}_3(\wp,\ell)^2}\right). \end{aligned}$$

Substituting $u_1 = u$, $|x| = p$ and $|\zeta| \leq 1$ into (4.6), we assume that $0 \leq u \leq 2$. Also using the triangular inequality we get

$$\begin{aligned} \left|b_2b_4 - b_3^2\right| &\leq \frac{1}{4}\Big(\frac{4d^2s^2p^4 + 4tsdp^2c + 4td^2p^2 + 4d^2sp^3 + 4dtcp + d^2p^2}{(1-\alpha)(1-3\alpha)\mathfrak{Y}_2(\wp,\ell)\mathfrak{Y}_4(\wp,\ell)} \\ &\quad - \frac{4d^2s^2p^4 + 4t^2c^2 + 8ds^2pt + 4d^2sp^3 + 4dtcp + d^2p^2}{(1-2\alpha)^2\mathfrak{Y}_3(\wp,\ell)^2}\Big)u^4 \\ &\quad + \frac{1}{2}\Big(\frac{2d^2sp^3 + 2tcdp + d^2p^2}{(1-\alpha)(1-3\alpha)\mathfrak{Y}_2(\wp,\ell)\mathfrak{Y}_4(\wp,\ell)} - \frac{2d^2sp^3 + 2dptc + d^2p^2}{(1-2\alpha)^2\mathfrak{Y}_3(\wp,\ell)^2}\Big)u^2\left(1-u^2\right)p \\ &\quad + \Big(\frac{d^2t^2}{4(1-2\alpha)^2\mathfrak{Y}_3(\wp,\ell)^2}\left(4-u^2\right) + \frac{d^2p^2}{4(1-\alpha)(1-3\alpha)\mathfrak{Y}_2(\wp,\ell)\mathfrak{Y}_4(\wp,\ell)}\left(4-u^2\right)p^2 \\ &\quad + \frac{d^2p^2}{2(1-\alpha)(1-3\alpha)\mathfrak{Y}_2(\wp,\ell)\mathfrak{Y}_4(\wp,\ell)}\left(4-u^2\right)\left(1+p^2\right)\Big) \\ &= G(u,p). \end{aligned}$$

Differentiating $G(u,p)$ we get

$$\begin{aligned} \partial G(u,p) &\leq \frac{u^4}{4}\Big(\frac{16d^2s^2p^3 + 8tsdpc + 8td^2p + 12d^2sp^2 + 4dtc + 2d^2p}{(1-\alpha)(1-3\alpha)\mathfrak{Y}_2(\wp,\ell)\mathfrak{Y}_4(\wp,\ell)} \\ &\quad + \frac{16d^2s^2p^3 + 8ds^2t + 12d^2p^2 + 4dtc + 2d^2q}{(1-2\alpha)^2\mathfrak{Y}_3(\wp,\ell)^2}\Big) \\ &\quad + \frac{u^2(1-u^2)}{2}\Big(\frac{8d^2tp^3 + 4tcdp + 3d^2p}{(1-\alpha)(1-3\alpha)\mathfrak{Y}_2(\wp,\ell)\mathfrak{Y}_4(\wp,\ell)} + \frac{8d^2sp^3 + 4tcdp + 3d^2p}{(1-2\alpha)^2\mathfrak{Y}_3(\wp,\ell)^2}\Big) \\ &\quad + (4-u^2)\Big(\frac{4d^2p^3}{4(1-2\alpha)^2\mathfrak{Y}_3(\wp,\ell)^2}(4-u^2) + \frac{4d^2p^3u^2}{4(1-\alpha)(1-3\alpha)\mathfrak{Y}_2(\wp,\ell)\mathfrak{Y}_4(\wp,\ell)}\Big) \\ &\quad + \frac{2d^2p}{2(1-\alpha)(1-3\alpha)\mathfrak{Y}_2(\wp,\ell)\mathfrak{Y}_4(\wp,\ell)}(4-u^2) + \frac{4d^2p^3}{2(1-\alpha)(1-3\alpha)\mathfrak{Y}_2(\wp,\ell)\mathfrak{Y}_4(\wp,\ell)}\left(4-u^2\right) > 0. \end{aligned}$$

$G(u,p)$ is a function which is increasing in the interval $[0,1]$. Hence $G(u,1) \geq G(u,p)$ for $0 \leq p \leq 1$

$$\begin{aligned} G(u,p) &\leq \frac{u^4}{4}\Big(\frac{16d^2s^2 + 8tsdc + 8td^2 + 12d^2s + 4dtc + 2d^2}{(1-\alpha)(1-3\alpha)\mathfrak{Y}_2(\wp,\ell)\mathfrak{Y}_4(\wp,\ell)} + \frac{16d^2s^2 + 8ds^2t + 12d^2 + 4dtc + 2d^2}{(1-2\alpha)^2\mathfrak{Y}_3(\wp,\ell)^2}\Big) \\ &\quad + \frac{u^2(1-u^2)}{2}\Big(\frac{8d^2t + 4tcd + 3d^2}{(1-\alpha)(1-3\alpha)\mathfrak{Y}_2(\wp,\ell)\mathfrak{Y}_4(\wp,\ell)} + \frac{8d^2s + 4tcd + 3d^2}{(1-2\alpha)^2\mathfrak{Y}_3(\wp,\ell)^2}\Big) \\ &\quad + \left(4-u^2\right)\Big(\frac{4d^2}{4(1-2\alpha)^2\mathfrak{Y}_3(\wp,\ell)^2}\left(4-u^2\right) + \frac{4d^2u^2}{4(1-\alpha)(1-3\alpha)\mathfrak{Y}_2(\wp,\ell)\mathfrak{Y}_4(\wp,\ell)}\Big) \\ &\quad + \frac{d^2}{(1-\alpha)(1-3\alpha)\mathfrak{Y}_2(\wp,\ell)\mathfrak{Y}_4(\wp,\ell)}\left(4-u^2\right) + \frac{2d^2}{(1-\alpha)(1-3\alpha)\mathfrak{Y}_2(\wp,\ell)\mathfrak{Y}_4(\wp,\ell)}\left(4-u^2\right) > 0. \end{aligned}$$

Since $u \in [0, 2]$ it follows

$$|b_2 b_4 - b_3^2| \leq \frac{16d^2}{(1-2\alpha)^2 \mathfrak{Y}_3(\wp, \ell)^2} + \frac{10d^2}{(1-\alpha)(1-3\alpha)\mathfrak{Y}_2(\wp, \ell)\mathfrak{Y}_4(\wp, \ell)}. \tag{4.7}$$

□

5. Conclusion

Understanding the interplay between geometry and analysis is essential to understanding geometric function theory. This quick evolution is intimately related to the interplay between geometric behaviour and the analytic structure. In the current study we have learnt about a novel subclass of functions that are analytic with respect to the Horadam polynomials by the usage of the Rabotnov function. The first initial estimates of the coefficients are determined as well as the Fekete-Szegö and Hankel inequalities. Horadam polynomial play an important role in computer vision and image processings as well as for constructing and modifying scale space representations of images. This can effectively in applied to tasks such as edge detection, texture characterization and multi-scale image analysis. Moreover, this technique allows the study of statistical properties of textures, including gray-level distributions and spatial texture arrangements (*cf.* [23]). The same line of investigation can also be extended to other notable classes of functions, for instance meromorphic and meromorphic bi-univalent functions.

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