Montes Taurus J. Pure Appl. Math. 8 (1), 1–28, 2026

Montes Taurus Journal of Pure and Applied Mathematics

https://www.mtjpamjournal.com

MTJPAM

ISSN: 2687-4814

A study of sound numbers derived from iterated products of matrices in $\mathbb{Z}_2$

Hyeon Kyeong Lee [a], Daeyeoul Kim [b]

[a] *Department of Mathematics and Institute of Pure and Applied Mathematics, Jeonbuk National University 567 Baekje-daero, Deokjin-gu, Jeonju-si, Jeonbuk-do 54896, South Korea*

[b] *Department of Mathematics and Institute of Pure and Applied Mathematics, Jeonbuk National University 567 Baekje-daero, Deokjin-gu, Jeonju-si, Jeonbuk-do 54896, South Korea*

Abstract

In this article, we first investigate the various variations of the 512 matrix sequences generated by iterating over matrices defined in $\mathbb{Z}_2$. Along the way, we define and find the stable, amicable, and sociable matrix sequences for the iterated matrices. We will utilize the results obtained for matrices to convert them to sounds and to obtain results for matrices with equal leading and trailing numbers.

Keywords: Iteration, matrices, amicable pair, sociable, palindromic, anti-palindromic number, reverse sound

2020 MSC: 11A25, 11B50, 11C20

1. Introduction

A palindromic sound is one in which the front and back sounds are the same. There are a lot of interesting results on the internet about palindromic, but here is just one. See [12]. If we convert these sounds into integers, we can define a palindromic number as one that has the same number of preceding and following sounds. In this article, we will use matrices to think about palindromic numbers and translate them into palindromic sounds. We start with a set of matrices whose simplest form is the set $\{0, 1\}$. In order to do this, we first introduce some basic terminology.

Let $\mathbb{Z}_t$ be the ring of residue classes modulo t, $M_2(\mathbb{Z}_t) := \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} | \alpha, \beta, \gamma, \delta \in \mathbb{Z}_t \right\}$ and $\mathbb{N}$ be the set of positive integers. Here, $t \in \mathbb{N} - \{1\}$.

In elementary number theory, it is very interesting that the result of iterations of an arithmetical function can be repeated, divergent, or a few values appear over and over again. There is a great deal of literature concerning the iteration of the aliquot function, much of it concerned with whether the iterated values eventually terminate at zero (cycle) or become unbounded, depending on the value. See [3, pp. 92–93]. It is natural to ask what happens when you iterate a value of the fixed arithmetical function. In [1, 6, 7], the notions and properties of sociable, amicable pair and stable for iterations of restricted divisor functions are discussed. The notions of sociable, amicable pair, and stable were introduced in [2, 5] for matrix sequences of iterated multiplications defined over $M_2(\mathbb{Z}_4)$.

†*Article ID:* MTJPAM-D-24-00146

Email addresses: reveuse0831@naver.com (Hyeon Kyeong Lee), kdaeyeoul@jbnu.ac.kr (Daeyeoul Kim)

Received:9 September 2024, Accepted:25 January 2025, Published:6 February 2026

⋆*Corresponding Author:* Daeyeoul Kim

For 2×2 matrix, $L = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the characteristic polynomial [8, p.200] is given by $p(X) := X^2 - (a+d)X + (ad-bc)$, so the Cayley-Hamilton theorem states that

$$p(L) := L^2 - (a+d)L + (ad-bc)I = O. \tag{1.1}$$

Here, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $L^t := \underbrace{L \cdots L}_{t \text{ times}}$ and $L^0 = O$.

This article is organized as follows: In section 2, we introduce the 16 matrices used in this article and give definitions for stable, amicable, sociable, order, period, etc. We classified the 512 matrix sequences derived from the 16 matrices into stable, amicable, and sociable. In section 3, we classify the 512 matrix sequences considered in section 2 by investigating whether the matrices that make up each sequence are singular or not. In section 4, we converted {0, 1} to the sound {boom, clap}. We use the matrices studied in sections 2 and 3 to represent the sounds and reverse sounds and implement them directly as sounds. Section 5 presents a study on classifying the sounds of the 512 matrices transformed in section 4. In particular, the sequences of sounds whose first and last sounds are the same or opposite are investigated. In addition, we implement and introduce sounds that are either the same or completely reversed in the front and back of the 512 matrix sequences. In section 6, we introduce the subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the various images of the subgroups obtained by considering the linear transformations of the matrix sequences obtained in section 2 from the defining domain to the space domain of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Finally, the appendix contains the specific matrix and subgroup sequences obtained in sections 2 and 6. In [16], we suggest all of the matrix sequences that represented by two colors.

2. Iterated Sequences of 2 × 2 matrices over $\mathbb{Z}_2$

For convenience in this article, we use the following 16 matrices, i.e.,

$$A_1 := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_3 := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_4 := \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

$$A_5 := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_6 := \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_7 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_8 := \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

$$A_9 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{10} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_{11} := \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_{12} := \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

$$A_{13} := \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_{14} := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_{15} := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_{16} := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

This is not difficult, but to make the results of this article easier to understand, we compute A_iA_j for all $1 \le i, j \le 16$. See the following Table 1.

$\cdot$	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}
A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1
A_2	A_1	A_2	A_3	A_4	A_1	A_2	A_3	A_4	A_1	A_2	A_3	A_4	A_1	A_2	A_3	A_4
A_3	A_1	A_1	A_1	A_1	A_2	A_2	A_2	A_2	A_3	A_3	A_3	A_3	A_4	A_4	A_4	A_4
A_4	A_1	A_2	A_3	A_4	A_2	A_1	A_4	A_3	A_3	A_4	A_1	A_2	A_4	A_3	A_2	A_1
A_5	A_1	A_5	A_9	A_{13}	A_1	A_5	A_9	A_{13}	A_1	A_5	A_9	A_{13}	A_1	A_5	A_9	A_{13}
A_6	A_1	A_6	A_{11}	A_{16}	A_1	A_6	A_{11}	A_{16}	A_1	A_6	A_{11}	A_{16}	A_1	A_6	A_{11}	A_{16}
A_7	A_1	A_5	A_9	A_{13}	A_2	A_6	A_{10}	A_{14}	A_3	A_7	A_{11}	A_{15}	A_4	A_8	A_{12}	A_{16}
A_8	A_1	A_6	A_{11}	A_{16}	A_2	A_5	A_{12}	A_{15}	A_3	A_8	A_9	A_{14}	A_4	A_7	A_{10}	A_{13}
A_9	A_1	A_1	A_1	A_1	A_5	A_5	A_5	A_5	A_9	A_9	A_9	A_9	A_{13}	A_{13}	A_{13}	A_{13}
A_{10}	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}
A_{11}	A_1	A_1	A_1	A_1	A_6	A_6	A_6	A_6	A_{11}	A_{11}	A_{11}	A_{11}	A_{16}	A_{16}	A_{16}	A_{16}
A_{12}	A_1	A_2	A_3	A_4	A_6	A_5	A_8	A_7	A_{11}	A_{12}	A_9	A_{10}	A_{16}	A_{15}	A_{14}	A_{13}
A_{13}	A_1	A_5	A_9	A_{13}	A_5	A_1	A_{13}	A_9	A_9	A_{13}	A_1	A_5	A_{13}	A_9	A_5	A_1
A_{14}	A_1	A_6	A_{11}	A_{16}	A_5	A_2	A_{15}	A_{12}	A_9	A_{14}	A_3	A_8	A_{13}	A_{10}	A_7	A_4
A_{15}	A_1	A_5	A_9	A_{13}	A_6	A_2	A_{14}	A_{10}	A_{11}	A_{15}	A_3	A_7	A_{16}	A_{12}	A_8	A_4
A_{16}	A_1	A_6	A_{11}	A_{16}	A_6	A_1	A_{16}	A_{11}	A_{11}	A_{16}	A_1	A_6	A_{16}	A_{11}	A_6	A_1

Table 1. Values of A_iA_j $(1 \le i, j \le 16)$

In order to investigate the variation of different forms of matrices by iterations of the product of matrices, it is necessary to categorize the sets as follows. Let $U := \{A_i \mid 1 \leq i \leq 16\}$, $U_1 := \{A_i \mid A_i = A_i^2\}$, $U_2 := \{A_i \mid A_i^2 = A_1, A_i \neq A_1\}$, $U_3 := \{A_i \mid A_i^2 = A_{10},\ A_i \neq A_{10}\}$ and $U_4 := \{A_i \mid A_i^3 = A_{10}, A_i^2 \neq A_{10}\}$. By Table 1, we obtain $U_1 = \{A_1, A_2, A_4, A_6, A_9, A_{10}, A_{11}, A_{13}\}$, $U_2 = \{A_3, A_5, A_{16}\}$, $U_3 = \{A_7, A_{12}, A_{14}\}$, $U_4 = \{A_8, A_{15}\}$ and $U = \cup_{i=1}^{4} U_i$.

Depending on whether the matrix is being multiplied from the left or right, the following notations are introduced. For $n \in \mathbb{N}$ and $1 \leq i, j \leq 16$, let $(A_{i,j})^{(n)} := (A_i)(A_i + A_j)(A_i + 2A_j)\cdots(A_i + (n-1)A_j)$ and ${}^{(n)}(A_{i,j}) := (A_i + (n-1)A_j)(A_i + (n-2)A_j)\cdots(A_i + A_j)(A_i)$.

Let $\mathfrak{R}_{i,j} : (A_{i,j})^{(1)} \to (A_{i,j})^{(2)} \to (A_{i,j})^{(3)} \to \cdots$ be the right matrix sequence derived from $A_{i,j}$ and $\mathfrak{L}_{i.j} : {}^{(1)}(A_{i,j}) \to {}^{(2)}(A_{i,j}) \to {}^{(3)}(A_{i,j}) \to \cdots$ be the left matrix sequence derived from $A_{i,j}$. Matrix sequences $\mathfrak{R}_{i,j}$ and $\mathfrak{L}_{i.j}$ can be found in detail in [13]. The fact that $A + 2B = 2B + A = A$ is easy to obtain. Using this, we get Lemma 2.1.

Lemma 2.1. *Let* $1 \leq i, j \leq 16$. *Then*

(a) *If* $x \in \mathbb{N}$ *then* $2xA_i = A_1$.

(b) $(A_{i,j})^{(n)} = \begin{cases} (A_i(A_i + A_j))^{\frac{n-1}{2}} A_i & \text{if } n \text{ is odd,} \\ (A_i(A_i + A_j))^{\frac{n}{2}} & \text{if } n \text{ is even.} \end{cases}$

(c) ${}^{(n)}(A_{i,j}) = \begin{cases} A_i((A_i + A_j)A_i)^{\frac{n-1}{2}} & \text{if } n \text{ is odd,} \\ ((A_i + A_j)A_i)^{\frac{n}{2}} & \text{if } n \text{ is even.} \end{cases}$

If $\mathfrak{R}_{i,j}$ and $\mathfrak{L}_{i,j}$ are matrix sequences with the same matrices are repeated over and over;

$$\mathfrak{R}_{i,j} : \cdots \to (A_{i,j})^{(m)} \to \cdots \to (A_{i,j})^{(m+l)} \to (A_{i,j})^{(m)} \to \cdots \to (A_{i,j})^{(m+l)} \to \cdots$$

and

$$\mathfrak{L}_{i,j} : \cdots \to {}^{(m)}(A_{i,j}) \to \cdots \to {}^{(m+l)}(A_{i,j}) \to {}^{(m)}(A_{i,j}) \to \cdots \to {}^{(m+l)}(A_{i,j}) \to \cdots$$

with $l \geq 2$, then $\mathfrak{R}_{i,j}$ (resp., $\mathfrak{L}_{i,j}$) is a right (resp., left) sociable matrix sequence derived from $A_{i,j}$. If $l = 1$, that is,

$$\mathfrak{R}_{i,j} : \cdots \to (A_{i,j})^{(m)} \to (A_{i,j})^{(m+1)} \to (A_{i,j})^{(m)} \to (A_{i,j})^{(m+1)} \to \cdots$$

and

$$\mathfrak{L}_{i,j} : \cdots \to {}^{(m)}(A_{i,j}) \to {}^{(m+1)}(A_{i,j}) \to {}^{(m)}(A_{i,j}) \to {}^{(m+1)}(A_{i,j}) \to \cdots,$$

then $\mathfrak{R}_{i,j}$ (resp., $\mathfrak{L}_{i,j}$) is a right (resp., left) amicable pair (shortly, amicable) matrix sequence derived from $A_{i,j}$. If $l = 0$, that is,

$$\mathfrak{R}_{i,j} : \cdots \to (A_{i,j})^{(m-1)} \to (A_{i,j})^{(m)} \to (A_{i,j})^{(m)} \to (A_{i,j})^{(m)} \to \cdots$$

and

$$\mathfrak{L}_{i,j} : \cdots \to {}^{(m-1)}(A_{i,j}) \to {}^{(m)}(A_{i,j}) \to {}^{(m)}(A_{i,j}) \to {}^{(m)}(A_{i,j}) \to \cdots,$$

then $\mathfrak{R}_{i,j}$ (resp., $\mathfrak{L}_{i,j}$) is a right (resp., left) stable matrix sequence derived from $A_{i,j}$. Similar notions are in [5]-[7].

Throughout this article, let $\mathfrak{T}_{i,j} = \mathfrak{L}_{i,j}$ or $\mathfrak{R}_{i,j}$. In convenience, if $\mathfrak{T}_{i,j} : \cdots \to A_s \to A_k \to A_k \to A_k \to ...$ is a stable matrix sequence then we write $\mathfrak{T}_{i,j} : \cdots \to A_s \to A_k \circlearrowleft$, and if

$$\mathfrak{T}_{i,j} : \cdots \to A_l \to A_t \to A_l \to A_t \to A_l \to \cdots$$

is an amicable matrix sequence then we write $\mathfrak{T}_{i,j} : \cdots \to A_l \rightleftarrows A_t$ and if

$$\mathfrak{T}_{i,j} : \cdots \to A_{m_1} \to A_{m_2} \to ... \to A_{m_s} \to A_{m_1} \to ... \to A_{m_s} \to A_{m_1} \to \cdots$$

is a sociable matrix sequence then we write $\mathfrak{T}_{i,j} : \cdots \to \underleftarrow{A_{m_1} \to A_{m_2} \to \cdots \to A_{m_s}}$.

Let m be the order of $\mathfrak{T}_{i,j}$, denoted by $Ord(\mathfrak{T}_{i,j})$ and let $l + 1$ be the period length of $\mathfrak{T}_{i,j}$, denoted by $Per(\mathfrak{T}_{i,j})$. If $\mathfrak{R}_{i,j}$ (resp., $\mathfrak{L}_{i,j}$) is a right (resp., left) stable matrix sequence, then we call $(A_{i,j})^{(m)}$ (resp., ${}^{(m)}(A_{i,j})$) is the right(resp., left) stable point of $\mathfrak{R}_{i,j}$ (resp., $\mathfrak{L}_{i,j}$) denoted by $StP(\mathfrak{R}_{i,j})$ (resp., $StP(\mathfrak{L}_{i,j})$. Similar notations are in [5]-[7]. It is easily checked that $AO = OA = O$. So we get Lemma 2.2.

Lemma 2.2. *If* $1 \le j \le 16$ *then* $\mathfrak{T}_{1,j} : A_1 \circlearrowleft$ *is a stable matrix sequence with* $Ord(\mathfrak{T}_{1,j}) = 1$.

The following Lemma 2.3 gives us an upper bound on the order for all $\mathfrak{T}_{i,j}$.

Lemma 2.3. *The following inequality holds true:*

$$Ord(\mathfrak{T}_{i,j}) \le \begin{cases} 4 & \text{if } \mathfrak{T}_{i,j} \text{ is a stable matrix sequence,} \\ 2 & \text{if } \mathfrak{T}_{i,j} \text{ is an amicable matrix sequence,} \\ 1 & \text{if } \mathfrak{T}_{i,j} \text{ is a sociable matrix sequence.} \end{cases}$$

Proof. Assume that $\mathfrak{T}_{i,j} = \mathfrak{R}_{i,j}$ and $(A_{i,j})^{(2)} = A_k$. Since $U = \cup_{i=1}^{4} U_i$, we consider the four cases of $A_k \in U_i (1 \le i \le 4)$ in turn below.

First, assume that $A_k \in U_1$. Then $A_k = {A_k}^2$ and $A_k A_i = {A_k}^2 A_i$. It follows that $(A_{i,j})^{(2)} = (A_{i,j})^{(2n)}$ and $(A_{i,j})^{(3)} = (A_{i,j})^{(2n+1)}$ for all $n \in \mathbb{N}$. From $A_k \in U_1$, we obtain

$$\mathfrak{R}_{i,j} : A_i \to A_k \to A_k A_i \to A_k \to A_k A_i \to A_k \to A_k A_i \to \cdots .$$

Hence if $A_i = A_k$ then $\mathfrak{R}_{i,j}$ is a right stable matrix sequence with $Ord(\mathfrak{R}_{i,j}) = 1$ and $\mathfrak{R}_{i,j} : A_k \circlearrowleft$. If $A_i \ne A_k$ and $A_i = A_k A_i$ then $\mathfrak{R}_{i,j}$ is a right amicable sequence with $Ord(\mathfrak{R}_{i,j}) = 1$ and $\mathfrak{R}_{i,j} : A_i \rightleftarrows A_k$. And if $A_i \ne A_k$ and $A_i \ne A_k A_i$ then $Ord(\mathfrak{R}_{i,j}) = 2$ and $\mathfrak{R}_{i,j} : A_i \to A_k \rightleftarrows A_k A_i$.

In addition, if $A_k = A_k A_i$ then $\mathfrak{R}_{i,j}$ is a right stable matrix sequence and if $A_k \ne A_k A_i$ then $\mathfrak{R}_{i,j}$ is a right amicable matrix sequence.

Second, suppose that $A_k \in U_2$. Then

$$\mathfrak{R}_{i,j} : A_i \to A_k \to A_k A_i \to A_1 \to A_1 \to \cdots .$$

So $\mathfrak{R}_{i,j}$ is a right stable matrix sequence with $Ord(\mathfrak{R}_{i,j}) = 3$ if $(A_{i,j})^{(3)} = A_1$ and $Ord(\mathfrak{R}_{i,j}) = 4$ if $(A_{i,j})^{(3)} \ne A_1$. Since $A_k \ne A_1$, $Ord(\mathfrak{R}_{i,j}) \le 2$ is impossible.

Third, suppose that $A_k \in U_3$. Then ${A_k}^2 = A_{10}$. So $(A_{i,j})^{(5)} = {A_k}^2 A_i = A_i = (A_{i,j})^{(1)}$, $(A_{i,j})^{(6)} = {A_k}^3 = A_k = (A_{i,j})^{(2)}$, $(A_{i,j})^{(7)} = {A_k}^3 A_i = A_k A_i = (A_{i,j})^{(3)}$ and $(A_{i,j})^{(8)} = {A_k}^4 = I = (A_{i,j})^{(4)}$. From $A_k \ne I$, we obtain

$$\mathfrak{R}_{i,j} : A_i \to A_k \to A_k A_i \to A_{10} \to A_i \to A_k \to A_k A_i \to A_{10} \to \cdots .$$

Hence $\mathfrak{R}_{i,j}$ is a right sociable matrix sequence with $Ord(\mathfrak{R}_{i,j}) = 1$ and $Per(\mathfrak{R}_{i,j}) = 4$.

Lastly, suppose that $A_k \in U_4$. Since $(A_{i,j})^{(6)} = {A_k}^3 = A_{10}$, we get $(A_{i,j})^{(7)} = A_i$. Thus A_k, ${A_k}^2$ and ${A_k}^3$ are all distinct. Because if $A_k = {A_k}^2$ then ${A_k}^3 = A_k {A_k}^2 = A_k A_k = {A_k}^2 \ne A_{10}$ and if $A_k = {A_k}^3 = A_{10}$ then ${A_k}^2 = A_{10}$. These are contradictory to ${A_k}^2 \ne A_{10}$ and ${A_k}^3 = A_{10}$. Hence, $\mathfrak{R}_{i,j}$ is a right sociable matrix sequence with $Ord(\mathfrak{R}_{i,j}) = 1$ and $Per(\mathfrak{R}_{i,j}) = 3$ or 6. In fact,

$$\mathfrak{R}_{i,j} \;:\; \underleftarrow{A_i \to A_k \to A_{10}} \tag{2.1}$$

or

$$\mathfrak{R}_{i,j} \;:\; \underleftarrow{A_i \to A_k \to A_k A_i \to {A_k}^2 \to {A_k}^2 A_i \to A_{10}}. \tag{2.2}$$

The case $\mathfrak{T}_{i,j} = \mathfrak{L}_{i,j}$ can be proved using almost exactly the same methods used in the proof of $\mathfrak{T}_{i,j} = \mathfrak{R}_{i,j}$. □

As a consequence of Lemma 2.3, we obtain the following corollary.

Corollary 2.4. *Let*

$$A_k = \begin{cases} (A_{i,j})^{(2)} & \text{if } \mathfrak{T}_{i,j} = \mathfrak{R}_{i,j}, \\ {}^{(2)}(A_{i,j}) & \text{if } \mathfrak{T}_{i,j} = \mathfrak{L}_{i,j}. \end{cases}$$

Then

- **(a)** *If* $\mathfrak{T}_{i,j}$ *is a stable matrix sequence with* $Ord(\mathfrak{T}_{i,j}) = 1$ *or* 2 *then* $A_k \in U_1$.
- **(b)** *If* $\mathfrak{T}_{i,j}$ *is a stable matrix sequence with* $Ord(\mathfrak{T}_{i,j}) = 3$ *or* 4 *then* $A_k \in U_2$.

(c) *If $\mathfrak{T}_{i,j}$ is an amicable matrix sequence then $A_k \in U_1$.*

(d) *If $\mathfrak{T}_{i,j}$ is a sociable matrix sequence with $Per(\mathfrak{T}_{i,j}) = 4$ then $A_k \in U_3$.*

(e) *If $\mathfrak{T}_{i,j}$ is a sociable matrix sequence with $Per(\mathfrak{T}_{i,j}) = 3$ or 6 then $A_k \in U_4$.*

Proof. The proof follows naturally from the proof of Lemma 2.3. □

Lemma 2.5. *Let $\mathfrak{R}_{i,j}$ be a right stable matrix sequence and $A_k = A_i(A_i + A_j)$. Then*

(a) *$Ord(\mathfrak{R}_{i,j}) = 1$ if and only if $A_iA_j = A_1$ with $A_i \in U_1$.*

(b) *$Ord(\mathfrak{R}_{i,j}) = 2$ if and only if $A_i \neq A_k$ and $A_k = A_kA_i$ with $A_k \in U_1$.*

(c) *$Ord(\mathfrak{R}_{i,j}) = 3$ if and only if $A_kA_i = A_1$ with $A_k \in U_2$.*

(d) *$Ord(\mathfrak{R}_{i,j}) = 4$ if and only if $A_kA_i \neq A_1$ with $A_k \in U_2$.*

Proof. (a) If $\mathfrak{R}_{i,j}$ is a right stable matrix sequence with $Ord(\mathfrak{R}_{i,j}) = 1$ then $A_k \in U_1$ by Corollary 2.4 (a). Since $Ord(\mathfrak{R}_{i,j}) = 1$, we obtain $A_i = A_k$ and $A_i \in U_1$. So $A_i = A_i(A_i + A_j) = A_i{}^2 + A_iA_j = A_i + A_iA_j$. Therefore, $A_iA_j = A_1$.

Conversely, if $A_iA_j = A_1$ with $A_i \in U_1$, then $A_i(A_i + A_j) = A_i{}^2 + A_iA_j = A_i{}^2 = A_i$. This is the first case in the proof of Lemma 2.3, so $\mathfrak{R}_{i,j}$ is a right stable matrix sequence with $Ord(\mathfrak{R}_{i,j}) = 1$.

(b) If $\mathfrak{R}_{i,j}$ is a right stable matrix sequence with $Ord(\mathfrak{R}_{i,j}) = 2$ then $A_k \in U_1$ by Corollary 2.4 (a). Thus $A_i \neq A_k$ and $A_k = A_kA_i$.

Conversely, it is proved in the first case of the proof of Lemma 2.3.

(c) If $\mathfrak{R}_{i,j}$ is a right stable matrix sequence with $Ord(\mathfrak{R}_{i,j}) = 3$ then $A_k \in U_2$ and by Corollary 2.4 (b) and $A_k{}^2 = A_1$. Hence, $StP(\mathfrak{R}_{i,j}) = A_kA_i = A_k{}^2 = A_1$.

Conversely, if $A_k \in U_2$ then $\mathfrak{R}_{i,j}$ is a right stable matrix sequence with $Ord(\mathfrak{R}_{i,j}) = 3$ or 4 by the second case in the prof of Lemma 2.3. Since $A_kA_i = A_1$ and $A_k{}^2 = A_1$, we have $A_kA_i = A_k{}^2$. Therefore, $Ord(\mathfrak{R}_{i,j}) = 3$.

(d) If $\mathfrak{R}_{i,j}$ is a right stable matrix sequence with $Ord(\mathfrak{R}_{i,j}) = 4$ then $A_k \in U_2$ by Corollary 2.4 (b). Since $StP(\mathfrak{R}_{i,j}) = A_k{}^2 = A_1$, we find that $A_kA_i \neq A_1$.

Conversely, if $A_k \in U_2$ then $\mathfrak{R}_{i,j}$ is a right stable matrix sequence with $Ord(\mathfrak{R}_{i,j}) = 3$ or 4 by Lemma 2.3. By assumption, $A_kA_i \neq A_1$ and $A_k{}^2 = A_1$. This implies that $A_kA_i \neq A_k{}^2$. Therefore, $Ord(\mathfrak{R}_{i,j}) = 4$. □

Lemma 2.6. *Let $\mathfrak{L}_{i,j}$ be a left stable matrix sequence and $A_k = (A_i + A_j)A_i$. Then*

(a) *$Ord(\mathfrak{L}_{i,j}) = 1$ if and only if $A_jA_i = A_1$ with $A_i \in U_1$.*

(b) *$Ord(\mathfrak{L}_{i,j}) = 2$ if and only if $A_i \neq A_k$ and $A_k = A_iA_k$ with $A_k \in U_1$.*

(c) *$Ord(\mathfrak{L}_{i,j}) = 3$ if and only if $A_iA_k = A_1$ with $A_k \in U_2$.*

(d) *$Ord(\mathfrak{L}_{i,j}) = 4$ if and only if $A_iA_k \neq A_1$ with $A_k \in U_2$.*

Proof. It is proved similarly to Lemma 2.5. □

Remark 2.7. If $\mathfrak{T}_{i,j}$ is a stable matrix sequence with $Ord(\mathfrak{T}_{i,j}) = 2$ and $A_k = StP(\mathfrak{T}_{i,j}) \neq A_1$ then $A_i \in U_3 \cup \{A_{10}\}$ and A_k is a singular matrix. We can check Remark 2.7 by Appendix A.2 and B.2.

By Appendix A.3 and B.3 (resp., Appendix A.4 and B.4), we get the following Remark 2.8 (a) (resp., (b)).

Remark 2.8. Let $\mathfrak{T}_{i,j}$ be a stable matrix sequence and

$$A_k = \begin{cases} (A_{i,j})^{(2)} & \text{if } \mathfrak{T}_{i,j} = \mathfrak{R}_{i,j}, \\ {}^{(2)}(A_{i,j}) & \text{if } \mathfrak{T}_{i,j} = \mathfrak{L}_{i,j}. \end{cases}$$

Then

(a) $Ord(\mathfrak{T}_{i,j}) = 3$ if and only if A_i is a singular matrix with $A_k \in U_2$.

(b) $Ord(\mathfrak{T}_{i,j}) = 4$ if and only if A_i is an invertible matrix with $A_k \in U_2$.

Lemma 2.9. *Let $\mathfrak{R}_{i,j}$ be a right amicable matrix sequence and $A_k = A_i(A_i + A_j)$. Then*

(a) $Ord(\mathfrak{R}_{i,j}) = 1$ *if and only if $A_i \neq A_k$ and $A_i = A_k A_i$ with $A_k \in U_1$.*

(b) $Ord(\mathfrak{R}_{i,j}) = 2$ *if and only if $A_i \neq A_k$, $A_k \neq A_k A_i$ and $A_k A_i \neq A_i$ with $A_k \in U_1$.*

Proof. This is proved in the first case of proof of Lemma 2.3 and Corollary 2.4. □

Lemma 2.10. *Let $\mathfrak{L}_{i,j}$ be a left amicable matrix sequence and $(A_i + A_j)A_i = A_k$. Then*

(a) $Ord(\mathfrak{L}_{i,j}) = 1$ *if and only if $A_i \neq A_k$ and $A_i = A_i A_k$ with $A_k \in U_1$.*

(b) $Ord(\mathfrak{L}_{i,j}) = 2$ *if and only if $A_i \neq A_k$, $A_k \neq A_i A_k$ and $A_i A_k \neq A_i$ with $A_k \in U_1$.*

Proof. It is proved similarly to Lemma 2.9. □

Using Appendix A.5.1 and B.5.1, we get the following Remark 2.11.

Remark 2.11. Let $\mathfrak{T}_{i,j}$ be an amicable matrix sequence. Then

(a) If $\mathfrak{T}_{i,j} : A_i \rightleftarrows A_k$ and A_i is an invertible matrix then $A_k = A_{10}$.

(b) If $\mathfrak{T}_{i,j} : A_i \to A_p \rightleftarrows A_q$ then A_i is an invertible matrix and A_p and A_q are singular matrices.

Lemma 2.12. *Let $\mathfrak{T}_{i,j}$ be a sociable matrix sequence with $Ord(\mathfrak{T}_{i,j}) = 1$ and*

$$A_k = \begin{cases} (A_{i,j})^{(2)} & \text{if } \mathfrak{T}_{i,j} = \mathfrak{R}_{i,j}, \\ {}^{(2)}(A_{i,j}) & \text{if } \mathfrak{T}_{i,j} = \mathfrak{L}_{i,j}. \end{cases}$$

Then

(a) $Per(\mathfrak{T}_{i,j}) = 3$ *if and only if $A_i \in U_4$ and $j = 1$.*

(b) $Per(\mathfrak{T}_{i,j}) = 4$ *if and only if $A_k \in U_3$.*

(c) $Per(\mathfrak{T}_{i,j}) = 6$ *if and only if $A_k \in U_4$ and $j \neq 1$.*

Proof. The case $\mathfrak{T}_{i,j} = \mathfrak{L}_{i,j}$ can be proved using almost exactly the same method used in the proof of $\mathfrak{T}_{i,j} = \mathfrak{R}_{i,j}$. For this reason, we will only consider the case $\mathfrak{T}_{i,j} = \mathfrak{R}_{i,j}$.

(a) Assume that $Per(\mathfrak{R}_{i,j}) = 3$. We note that $A_k \in U_4$ by Corollary 2.4 (e). Let

$$\mathfrak{R}_{i,j} \ : \ A_i \to A_k \to A_k A_i (= A_{10}) \to A_i (= {A_k}^2) \to A_k (= {A_k}^2 A_i) \to A_{10} (= {A_k}^3) \to \cdots$$

be a matrix sequence. Then we obtain $A_i = {A_k}^2 = {A_k}^{-1}$ and $A_k = {A_i}^2$. From ${A_i}^2 = A_k = A_i(A_i + A_j) = {A_i}^2 + A_i A_j$, we get $A_i A_j = A_1$. Using this, we obtain $A_j = {A_i}^{-1} A_i A_j = {A_i}^{-1} A_1 = A_1$.

Conversely, assume that $A_i \in U_4 = \{A_8, A_{15}\}$. Then $A_k = A_i(A_i + A_1) = {A_i}^2$ and $(A_{i,1})^{(n)} = {A_i}^n$. Therefore, we have

$$\mathfrak{R}_{8,1} \quad : \quad A_8 \underleftarrow{\to {A_8}^2 (= A_{15}) \to {A_8}^3 (= A_{10})}$$

and

$$\mathfrak{R}_{15,1} \quad : \quad A_{15} \underleftarrow{\to {A_{15}}^2 (= A_8) \to {A_{15}}^3 (= A_{10})}.$$

This completes the proof of (a).

(b) If $\mathfrak{R}_{i,j}$ is a right sociable matrix sequence with $Ord(\mathfrak{R}_{i,j}) = 1$ and $Per(\mathfrak{R}_{i,j}) = 4$ then $A_k \in U_3$ by Corollary 2.4 (d).

Conversely, if $A_k \in U_3$ then $\mathfrak{R}_{i,j}$ is a right sociable matrix sequence with $Ord(\mathfrak{R}_{i,j}) = 1$ and $Per(\mathfrak{R}_{i,j}) = 4$ by Lemma 2.3.

(c) If $\mathfrak{R}_{i,j}$ is a right sociable matrix sequence with $Ord(\mathfrak{R}_{i,j}) = 1$ and $Per(\mathfrak{R}_{i,j}) = 6$ then $A_k \in U_4$ by Corollary 2.4 (e). By the last case in proof of Lemma 2.3, $\mathfrak{R}_{i,j}$ is a right sociable matrix sequence with $Ord(\mathfrak{R}_{i,j}) = 1$ and $Per(\mathfrak{R}_{i,j}) = 3$ or 6 for $A_k \in U_4$. In conclusion, from (a), $j \neq 1$.

Conversely, if $A_k \in U_4$ then $\mathfrak{R}_{i,j}$ is a right sociable matrix sequence with $Per(\mathfrak{R}_{i,j}) = 3$ or 6 by Lemma 2.3. From (a), $j = 1$ if and only if $Per(\mathfrak{R}_{i,j}) = 3$. Therefore, $Per(\mathfrak{R}_{i,j}) = 6$ with $j \neq 1$. □

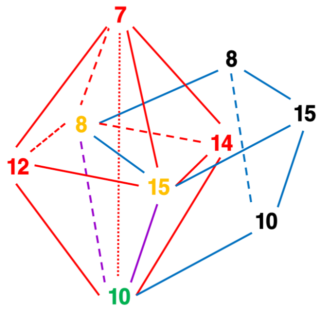

Figure 1. Right stable matrix sequence with period 6

Remark 2.13. Figure 1 is a structure of all right stable matrix sequences with period 6. Another structure of all stable matrix sequences is in [15].

Remark 2.14. If $\mathfrak{T}_{i,j} : \underleftarrow{A_i \to \cdots \to A_n}$ is a sociable matrix sequence then $A_n = A_{10}$. This is easily obtained by the third and fourth parts of the proof of Lemma 2.3.

From Lemmas 2.2–2.12, we get the following theorem.

Theorem 2.15. *The followings hold true:*

- **(a)** *All stable matrix sequences in $\mathfrak{T}_{i,j}$ are given in Tables* 2–4.
- **(b)** *All amicable matrix sequences in $\mathfrak{T}_{i,j}$ are given in Table* 5.
- **(c)** *Table* 6 *classifies all the sociable matrix sequences in $\mathfrak{T}_{i,j}$.*

$StP(\mathfrak{T}_{i,j})$	$\mathfrak{T}_{i,j}$	$\#\mathfrak{T}_{i,j}$
A_1	$\mathfrak{T}_{1,j}$ $(1 \leq j \leq 16)$	32
A_2	$\mathfrak{T}_{2,1}$, $\mathfrak{T}_{2,9}$, $\mathfrak{R}_{2,5}$, $\mathfrak{R}_{2,13}$, $\mathfrak{L}_{2,3}$, $\mathfrak{L}_{2,11}$	8
A_4	$\mathfrak{T}_{4,1}$, $\mathfrak{T}_{4,11}$, $\mathfrak{R}_{4,6}$, $\mathfrak{R}_{4,16}$, $\mathfrak{L}_{4,3}$, $\mathfrak{L}_{4,9}$	8
A_6	$\mathfrak{T}_{6,1}$, $\mathfrak{T}_{6,13}$, $\mathfrak{R}_{6,5}$, $\mathfrak{R}_{6,9}$, $\mathfrak{L}_{6,4}$, $\mathfrak{L}_{6,16}$	8
A_9	$\mathfrak{T}_{9,1}$, $\mathfrak{T}_{9,2}$, $\mathfrak{R}_{9,3}$, $\mathfrak{R}_{9,4}$, $\mathfrak{L}_{9,5}$, $\mathfrak{L}_{9,6}$	8
A_{10}	$\mathfrak{T}_{10,1}$	2
A_{11}	$\mathfrak{T}_{11,1}$, $\mathfrak{T}_{11,4}$, $\mathfrak{R}_{11,2}$, $\mathfrak{R}_{11,3}$, $\mathfrak{L}_{11,13}$, $\mathfrak{L}_{11,16}$	8
A_{13}	$\mathfrak{T}_{13,1}$, $\mathfrak{T}_{13,6}$, $\mathfrak{R}_{13,11}$, $\mathfrak{R}_{13,16}$, $\mathfrak{L}_{13,2}$, $\mathfrak{L}_{13,5}$	8
Total		82

Table 2. Stable matrix sequences with $Ord(\mathfrak{T}_{i,j}) = 1$

$StP(\mathfrak{T}_{i,j})$	$\mathfrak{T}_{i,j}$	$\#\mathfrak{T}_{i,j}$
A_1	$\mathfrak{T}_{i,i}$ $(2 \leq i \leq 16)$, $\mathfrak{T}_{2,10}$, $\mathfrak{T}_{3,1}$, $\mathfrak{T}_{4,10}$, $\mathfrak{T}_{5,1}$, $\mathfrak{T}_{6,10}$, $\mathfrak{T}_{9,10}$, $\mathfrak{T}_{11,10}$, $\mathfrak{T}_{13,10}$, $\mathfrak{T}_{16,1}$, $\mathfrak{R}_{2,6}$, $\mathfrak{R}_{2,14}$, $\mathfrak{R}_{3,2}$, $\mathfrak{R}_{3,4}$, $\mathfrak{R}_{4,7}$, $\mathfrak{R}_{4,13}$, $\mathfrak{R}_{5,9}$, $\mathfrak{R}_{5,13}$, $\mathfrak{R}_{6,2}$, $\mathfrak{R}_{6,14}$, $\mathfrak{R}_{9,11}$, $\mathfrak{R}_{9,12}$, $\mathfrak{R}_{11,9}$, $\mathfrak{R}_{11,12}$, $\mathfrak{R}_{13,4}$, $\mathfrak{R}_{13,7}$, $\mathfrak{R}_{16,6}$, $\mathfrak{R}_{16,11}$, $\mathfrak{L}_{2,4}$, $\mathfrak{L}_{2,12}$, $\mathfrak{L}_{3,9}$, $\mathfrak{L}_{3,11}$, $\mathfrak{L}_{4,2}$, $\mathfrak{L}_{4,12}$, $\mathfrak{L}_{5,2}$, $\mathfrak{L}_{5,6}$, $\mathfrak{L}_{6,7}$, $\mathfrak{L}_{6,11}$, $\mathfrak{L}_{9,13}$, $\mathfrak{L}_{9,14}$, $\mathfrak{L}_{11,6}$, $\mathfrak{L}_{11,7}$, $\mathfrak{L}_{13,9}$, $\mathfrak{L}_{13,14}$, $\mathfrak{L}_{16,4}$, $\mathfrak{L}_{16,13}$	84
A_2	$\mathfrak{T}_{10,9}$, $\mathfrak{R}_{14,9}$, $\mathfrak{L}_{12,9}$	4
A_4	$\mathfrak{T}_{10,11}$, $\mathfrak{R}_{7,11}$, $\mathfrak{L}_{12,11}$	4
A_6	$\mathfrak{T}_{10,13}$, $\mathfrak{R}_{14,13}$, $\mathfrak{L}_{7,13}$	4
A_9	$\mathfrak{T}_{10,2}$, $\mathfrak{R}_{12,2}$, $\mathfrak{L}_{14,2}$	4
A_{11}	$\mathfrak{T}_{10,4}$, $\mathfrak{R}_{12,4}$, $\mathfrak{L}_{7,4}$	4
A_{13}	$\mathfrak{T}_{10,6}$, $\mathfrak{R}_{7,6}$, $\mathfrak{L}_{14,6}$	4
Total		108

Table 3. Stable matrix sequences with $Ord(\mathfrak{T}_{i,j}) = 2$

$Ord(\mathfrak{T}_{i,j})$	$\mathfrak{T}_{i,j}$ with $StP(\mathfrak{T}_{i,j}) = A_1$	$\#\mathfrak{T}_{i,j}$
3	$\mathfrak{T}_{2,8}$, $\mathfrak{T}_{2,16}$, $\mathfrak{T}_{3,10}$, $\mathfrak{T}_{3,12}$, $\mathfrak{T}_{4,5}$, $\mathfrak{T}_{4,15}$, $\mathfrak{T}_{5,10}$, $\mathfrak{T}_{5,14}$, $\mathfrak{T}_{6,3}$, $\mathfrak{T}_{6,15}$, $\mathfrak{T}_{9,15}$, $\mathfrak{T}_{9,16}$, $\mathfrak{T}_{11,5}$, $\mathfrak{T}_{11,8}$, $\mathfrak{T}_{13,3}$, $\mathfrak{T}_{13,8}$, $\mathfrak{T}_{16,7}$, $\mathfrak{T}_{16,10}$, $\mathfrak{R}_{2,4}$, $\mathfrak{R}_{2,12}$, $\mathfrak{R}_{3,9}$, $\mathfrak{R}_{3,11}$, $\mathfrak{R}_{4,2}$, $\mathfrak{R}_{4,12}$, $\mathfrak{R}_{5,2}$, $\mathfrak{R}_{5,6}$, $\mathfrak{R}_{6,7}$, $\mathfrak{R}_{6,11}$, $\mathfrak{R}_{9,13}$, $\mathfrak{R}_{9,14}$, $\mathfrak{R}_{11,6}$, $\mathfrak{R}_{11,7}$, $\mathfrak{R}_{13,9}$, $\mathfrak{R}_{13,14}$, $\mathfrak{R}_{16,4}$, $\mathfrak{R}_{16,13}$, $\mathfrak{L}_{2,6}$, $\mathfrak{L}_{2,14}$, $\mathfrak{L}_{3,2}$, $\mathfrak{L}_{3,4}$, $\mathfrak{L}_{4,7}$, $\mathfrak{L}_{4,13}$, $\mathfrak{L}_{5,9}$, $\mathfrak{L}_{5,13}$, $\mathfrak{L}_{6,2}$, $\mathfrak{L}_{6,14}$, $\mathfrak{L}_{9,11}$, $\mathfrak{L}_{9,12}$, $\mathfrak{L}_{11,9}$, $\mathfrak{L}_{11,12}$, $\mathfrak{L}_{13,4}$, $\mathfrak{L}_{13,7}$, $\mathfrak{L}_{16,6}$, $\mathfrak{L}_{16,11}$	72
4	$\mathfrak{T}_{7,8}$, $\mathfrak{T}_{7,10}$, $\mathfrak{T}_{7,15}$, $\mathfrak{T}_{8,3}$, $\mathfrak{T}_{8,5}$, $\mathfrak{T}_{8,16}$, $\mathfrak{T}_{10,7}$, $\mathfrak{T}_{10,12}$, $\mathfrak{T}_{10,14}$, $\mathfrak{T}_{12,8}$, $\mathfrak{T}_{12,10}$, $\mathfrak{T}_{12,15}$, $\mathfrak{T}_{14,8}$, $\mathfrak{T}_{14,10}$, $\mathfrak{T}_{14,15}$, $\mathfrak{T}_{15,3}$, $\mathfrak{T}_{15,5}$, $\mathfrak{T}_{15,16}$	36
Total		108

Table 4. Stable matrix sequences with $Ord(\mathfrak{T}_{i,j}) = 3, 4$

$Ord(\mathfrak{T}_{i,j})$	$\mathfrak{T}_{i,j}$	$\#\mathfrak{T}_{i,j}$
1	$\mathfrak{T}_{2,7}$, $\mathfrak{T}_{2,15}$, $\mathfrak{T}_{3,5}$, $\mathfrak{T}_{3,6}$, $\mathfrak{T}_{3,7}$, $\mathfrak{T}_{3,8}$, $\mathfrak{T}_{3,13}$, $\mathfrak{T}_{3,14}$, $\mathfrak{T}_{3,15}$, $\mathfrak{T}_{3,16}$, $\mathfrak{T}_{4,8}$, $\mathfrak{T}_{4,14}$, $\mathfrak{T}_{5,3}$, $\mathfrak{T}_{5,4}$, $\mathfrak{T}_{5,7}$, $\mathfrak{T}_{5,8}$, $\mathfrak{T}_{5,11}$, $\mathfrak{T}_{5,12}$, $\mathfrak{T}_{5,15}$, $\mathfrak{T}_{5,16}$, $\mathfrak{T}_{6,8}$, $\mathfrak{T}_{6,12}$, $\mathfrak{T}_{7,1}$, $\mathfrak{T}_{8,10}$, $\mathfrak{T}_{9,7}$, $\mathfrak{T}_{9,8}$, $\mathfrak{T}_{11,14}$, $\mathfrak{T}_{11,15}$, $\mathfrak{T}_{12,1}$, $\mathfrak{T}_{13,12}$, $\mathfrak{T}_{13,15}$, $\mathfrak{T}_{14,1}$, $\mathfrak{T}_{15,10}$, $\mathfrak{T}_{16,2}$, $\mathfrak{T}_{16,3}$, $\mathfrak{T}_{16,5}$, $\mathfrak{T}_{16,8}$, $\mathfrak{T}_{16,9}$, $\mathfrak{T}_{16,12}$, $\mathfrak{T}_{16,14}$, $\mathfrak{T}_{16,15}$, $\mathfrak{R}_{2,3}$, $\mathfrak{R}_{2,11}$, $\mathfrak{R}_{4,3}$, $\mathfrak{R}_{4,9}$, $\mathfrak{R}_{6,4}$, $\mathfrak{R}_{6,16}$, $\mathfrak{R}_{9,5}$, $\mathfrak{R}_{9,6}$, $\mathfrak{R}_{11,13}$, $\mathfrak{R}_{11,16}$, $\mathfrak{R}_{13,2}$, $\mathfrak{R}_{13,5}$, $\mathfrak{L}_{2,5}$, $\mathfrak{L}_{2,13}$, $\mathfrak{L}_{4,6}$, $\mathfrak{L}_{4,16}$, $\mathfrak{L}_{6,5}$, $\mathfrak{L}_{6,9}$, $\mathfrak{L}_{9,3}$, $\mathfrak{L}_{9,4}$, $\mathfrak{L}_{11,2}$, $\mathfrak{L}_{11,3}$, $\mathfrak{L}_{13,11}$, $\mathfrak{L}_{13,16}$,	106
2	$\mathfrak{T}_{7,3}$, $\mathfrak{T}_{7,5}$, $\mathfrak{T}_{7,13}$, $\mathfrak{T}_{8,4}$, $\mathfrak{T}_{8,6}$, $\mathfrak{T}_{8,7}$, $\mathfrak{T}_{8,9}$, $\mathfrak{T}_{8,12}$, $\mathfrak{T}_{8,14}$, $\mathfrak{T}_{12,5}$, $\mathfrak{T}_{12,16}$, $\mathfrak{T}_{14,3}$, $\mathfrak{T}_{14,16}$, $\mathfrak{T}_{15,2}$, $\mathfrak{T}_{15,7}$, $\mathfrak{T}_{15,11}$, $\mathfrak{T}_{15,12}$, $\mathfrak{T}_{15,13}$, $\mathfrak{T}_{15,14}$, $\mathfrak{R}_{7,4}$, $\mathfrak{R}_{7,13}$, $\mathfrak{R}_{12,9}$, $\mathfrak{R}_{12,11}$, $\mathfrak{R}_{14,2}$, $\mathfrak{R}_{14,6}$, $\mathfrak{L}_{7,6}$, $\mathfrak{L}_{7,11}$, $\mathfrak{L}_{12,1}$, $\mathfrak{L}_{12,4}$, $\mathfrak{L}_{14,9}$, $\mathfrak{L}_{14,13}$	48
Total		154

Table 5. Amicable matrix sequences

$Per(\mathfrak{T}_{i,j})$	$\mathfrak{T}_{i,j}$ with $Ord(\mathfrak{T}_{i,j}) = 1$	$\#\mathfrak{T}_{i,j}$
3	$\mathfrak{T}_{8,1}$, $\mathfrak{T}_{15,1}$	4
4	$\mathfrak{T}_{7,2}$, $\mathfrak{T}_{7,9}$, $\mathfrak{T}_{7,16}$, $\mathfrak{T}_{8,2}$, $\mathfrak{T}_{8,11}$, $\mathfrak{T}_{8,13}$, $\mathfrak{T}_{10,3}$, $\mathfrak{T}_{10,5}$, $\mathfrak{T}_{10,16}$, $\mathfrak{T}_{12,3}$, $\mathfrak{T}_{12,6}$, $\mathfrak{T}_{12,13}$, $\mathfrak{T}_{14,4}$, $\mathfrak{T}_{14,5}$, $\mathfrak{T}_{14,11}$, $\mathfrak{T}_{15,4}$, $\mathfrak{T}_{15,6}$, $\mathfrak{T}_{15,9}$	36
6	$\mathfrak{T}_{7,12}$, $\mathfrak{T}_{7,14}$, $\mathfrak{T}_{8,15}$, $\mathfrak{T}_{10,8}$, $\mathfrak{T}_{10,15}$, $\mathfrak{T}_{12,7}$, $\mathfrak{T}_{12,14}$, $\mathfrak{T}_{14,7}$, $\mathfrak{T}_{14,12}$, $\mathfrak{T}_{15,8}$	20
Total		60

Table 6. Sociable matrix sequences

Corollary 2.16. *Let $\mathfrak{A}_i = \mathfrak{T}_{i,1} : A_i \to {A_i}^2 \to {A_i}^3 \to \cdots$ be a matrix sequence. Then $\mathfrak{A}_i$ has no social matrix sequence. In fact, Table 7 shows the classification of each $\mathfrak{A}_i$ into stable and amicable.*

Sequences	$\mathfrak{A}_i$	$\#\mathfrak{A}_i$
Stable with $Ord(\mathfrak{A}_i) = 1$	$\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_4, \mathfrak{A}_6, \mathfrak{A}_9, \mathfrak{A}_{10}, \mathfrak{A}_{11}, \mathfrak{A}_{13}$	8
Stable with $Ord(\mathfrak{A}_i) = 2$ and $StP(\mathfrak{A}_i) = A_1$	$\mathfrak{A}_3, \mathfrak{A}_5, \mathfrak{A}_{16}$	3
Amicable with $Ord(\mathfrak{A}_i) = 1$	$\mathfrak{A}_7, \mathfrak{A}_{12}, \mathfrak{A}_{14}$	3
Amicable with $Ord(\mathfrak{A}_i) = 2$	$\mathfrak{A}_8, \mathfrak{A}_{15}$	2
Total		16

Table 7. All matrix sequences in $\mathfrak{A}_i$

3. Singular, regular and middle sets

Let $S(\mathfrak{R}_{i,j}) := \{(A_{i,j})^{(n)} \mid n \in \mathbb{N}\}$ and $S(\mathfrak{L}_{i,j}) := \{^{(n)}(A_{i,j}) \mid n \in \mathbb{N}\}$.

For every natural number n, $S(\mathfrak{R}_{i,j})$ is called a singular (resp., regular) set if $(A_{i,j})^{(n)}$ are singular (resp., regular) matrices. When $S(\mathfrak{R}_{i,j})$ is neither singular nor regular, we will call $S(\mathfrak{R}_{i,j})$ a middle-singular. We call $\mathfrak{R}_{i,j}$ a singular (resp., middle-singular, regular) matrix sequence if $S(\mathfrak{R}_{i,j})$ is a singular (resp., middle-singular, regular) set. We use the same notations for $S(\mathfrak{L}_{i,j})$ as for $S(\mathfrak{R}_{i,j})$.

It is well known that if A_i is a singular matrix, then for every B, A_iB are singular matrices. If

$$i \in \{1, 2, 3, 4, 5, 6, 9, 11, 13, 16\},$$

then A_i and $S(\mathfrak{T}_{i,j})$ are singular with $1 \le j \le 16$. On the other hand, if A_i is regular, that is, $i \in \{7, 8, 10, 12, 14, 15\}$, then $S(\mathfrak{T}_{i,j})$ is regular or middle-singular. Therefore, we obtain the following Lemma 3.1.

Lemma 3.1. *Let $S_s(\mathfrak{T}) := \{\mathfrak{T}_{i,j} | S(\mathfrak{T}_{i,j})$ is singular$\}$, $S_r(\mathfrak{T}) := \{\mathfrak{T}_{i,j} | S(\mathfrak{T}_{i,j})$ is regular$\}$ and $S_m(\mathfrak{T}) := \{\mathfrak{T}_{i,j} | S(\mathfrak{T}_{i,j})$ is middle-singular $\}$. Then $|S_s(\mathfrak{T})| = 160$, $|S_r(\mathfrak{T})| = 36$ and $|S_m(\mathfrak{T})| = 60$. Table 8 details 256 matrix sequences.*

$\mathfrak{T}_{i,j}$	(i,j)	#(i,j)
$S_r(\mathfrak{T})$	(7,1), (7,2), (7,9), (7,12), (7,14), (7,16), (8,1), (8,2), (8,10), (8,11), (8,13), (8,15), (10,1), (10,3), (10,5), (10,8), (10,15), (10,16), (12,1), (12,3), (12,6), (12,7), (12,13), (12,14), (14,1), (14,4), (14,5), (14,7), (14,11), (14,12), (15,1), (15,4), (15,6), (15,8), (15,9), (15,10)	36
$S_s(\mathfrak{T})$	(1,j), (2,j), (3,j), (4,j), (5,j), (6,j), (9,j), (11,j), (13,j), (16,j), $1 \le j \le 16$	160
$S_m(\mathfrak{T})$	(7,3), (7,4), (7,5), (7,6), (7,7), (7,8), (7,10), (7,11), (7,13), (7,15), (8,3), (8,4), (8,5), (8,6), (8,7), (8,8), (8,9), (8,12), (8,14), (8,16), (10,2), (10,4), (10,6), (10,7), (10,9), (10,10), (10,11), (10,12), (10,13), (10,14), (12,2), (12,4), (12,5), (12,8), (12,9), (12,10), (12,11), (12,12), (12,15), (12,16), (14,2), (14,3), (14,6), (14,8), (14,9), (14,10), (14,14), (14,14), (14,15), (14,16), (15,2), (15,3), (15,5), (15,7), (15,11), (15,12), (15,13), (15,14), (15,15), (15,16)	60

Table 8. $S(\mathfrak{T}_{i,j})$

4. Sounds derived from matrix sequences

If 0 is represented by boom (Korean, Kung) and 1 by clap (Korean, Jjag), the matrix $A_k (1 \le k \le 16)$ discussed in this article can be converted to sound as follows.

α_i	Sound	Wave
α_1 = 0000	Boom Boom Boom Boom	_ _ _ _
α_2 = 0001	Boom Boom Boom Clap	_ _ _ ⊓
α_3 = 0010	Boom Boom Clap Boom	_ _ ⊓ _
α_4 = 0011	Boom Boom Clap Clap	_ _ ⊓ ⊓
α_5 = 0100	Boom Clap Boom Boom	_ ⊓ _ _
α_6 = 0101	Boom Clap Boom Clap	_ ⊓ _ ⊓
α_7 = 0110	Boom Clap Clap Boom	_ ⊓ ⊓ _
α_8 = 0111	Boom Clap Clap Clap	_ ⊓ ⊓ ⊓
α_9 = 1000	Clap Boom Boom Boom	⊓ _ _ _
α_{10} = 1001	Clap Boom Boom Clap	⊓ _ _ ⊓
α_{11} = 1010	Clap Boom Clap Boom	⊓ _ ⊓ _
α_{12} = 1011	Clap Boom Clap Clap	⊓ _ ⊓ ⊓
α_{13} = 1100	Clap Clap Boom Boom	⊓ ⊓ _ _
α_{14} = 1101	Clap Clap Boom Clap	⊓ ⊓ _ ⊓
α_{15}1110	Clap Clap Clap Boom	⊓ ⊓ ⊓ _
α_{16} = 1111	Clap Clap Clap Clap	⊓ ⊓ ⊓ ⊓

Table 9. Sounds derived from matrix

We can write $\mathfrak{T}_{i,j} \ : \ A_{r_1} \to \cdots \to A_{r_s}\underleftarrow{\to \cdots \to \mathrm{A}_{\mathrm{r_t}}}$ in binary code using Table 9. See [14] for a more accurate sounding conversion of matrix sequences.

Let α_i be the binary code (or elements) of A_i, that is, if $A_i = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ then $\alpha_i := a_1a_2a_3a_4$. If $\alpha_i = a_1a_2a_3a_4$ and $\alpha_j = a_4a_3a_2a_1$ then we say that A_i (resp., A_j) is a reverse sound matrix of A_j (resp., A_i). The reverse sound matrices for the 16 matrices are given in Table 10.

A_i	α_i	α_j	A_j
A_1	0000	0000	A_1
A_2	0001	1000	A_9
A_3	0010	0100	A_5
A_4	0011	1100	A_{13}
A_5	0100	0010	A_3
A_6	0101	1010	A_{11}
A_7	0110	0110	A_7
A_8	0111	1110	A_{15}
A_9	1000	0001	A_2
A_{10}	1001	1001	A_{10}
A_{11}	1010	0101	A_6
A_{12}	1011	1101	A_{14}
A_{13}	1100	0011	A_4
A_{14}	1101	1011	A_{12}
A_{15}	1110	0111	A_8
A_{16}	1111	1111	A_{16}

Table 10. Reverse sound matrices of A_i $(1 \le i \le 16)$

Lemmas 4.1–4.4 are easy to obtain using Table 1.

Lemma 4.1. *Let* $A_iA_j = A_k$ *with* $1 \le i, j, k \le 16$. *Then*

(a) *If* $i = 1$ *then* $k = 1$ *for all* $1 \leq j \leq 16$.

(b) *If* $i \in \{2, 3, 4\}$ *then* $k \in \{1, 2, 3, 4\}$ *for all* $1 \leq j \leq 16$.

(c) *If* $i \in \{5, 9, 13\}$ *then* $k \in \{1, 5, 9, 13\}$ *for all* $1 \leq j \leq 16$.

(d) *If* $i \in \{6, 11, 16\}$ *then* $k \in \{1, 6, 11, 16\}$ *for all* $1 \leq j \leq 16$.

(e) *If* A_i *is an invertible matrix then* $1 \leq k \leq 16$.

Lemma 4.2. *Let* $A_jA_i = A_k$ *with* $1 \leq i, j, k \leq 16$. *Then*

(a) *If* $i = 1$ *then* $k = 1$ *for all* $1 \leq j \leq 16$.

(b) *If* $i \in \{2, 5, 6\}$ *then* $k \in \{1, 2, 5, 6\}$ *for all* $1 \leq j \leq 16$.

(c) *If* $i \in \{3, 9, 11\}$ *then* $k \in \{1, 3, 9, 11\}$ *for all* $1 \leq j \leq 16$.

(d) *If* $i \in \{4, 13, 16\}$ *then* $k \in \{1, 4, 13, 16\}$ *for all* $1 \leq j \leq 16$.

(e) *If* A_i *is an invertible matrix then* $1 \leq k \leq 16$.

A_i	A_jA_i, $1 \leq j \leq 16$	A_iA_j, $1 \leq j \leq 16$
A_1	A_1	A_1
A_2	A_1, A_2, A_5, A_6	A_1, A_2, A_3, A_4
A_3	A_1, A_3, A_9, A_{11}	A_1, A_2, A_3, A_4
A_4	A_1, A_4, A_{13}, A_{16}	A_1, A_2, A_3, A_4
A_5	A_1, A_2, A_5, A_6	A_1, A_5, A_9, A_{13}
A_6	A_1, A_2, A_5, A_6	A_1, A_6, A_{11}, A_{16}
A_7	A_k $(1 \leq k \leq 16)$	A_k $(1 \leq k \leq 16)$
A_8	A_k $(1 \leq k \leq 16)$	A_k $(1 \leq k \leq 16)$
A_9	A_1, A_3, A_9, A_{11}	A_1, A_5, A_9, A_{13}
A_{10}	A_k $(1 \leq k \leq 16)$	A_k $(1 \leq k \leq 16)$
A_{11}	A_1, A_3, A_9, A_{11}	A_1, A_6, A_{11}, A_{16}
A_{12}	A_k $(1 \leq k \leq 16)$	A_k $(1 \leq k \leq 16)$
A_{13}	A_1, A_4, A_{13}, A_{16}	A_1, A_5, A_9, A_{13}
A_{14}	A_k $(1 \leq k \leq 16)$	A_k $(1 \leq k \leq 16)$
A_{15}	A_k $(1 \leq k \leq 16)$	A_k $(1 \leq k \leq 16)$
A_{16}	A_1, A_4, A_{13}, A_{16}	A_1, A_6, A_{11}, A_{16}

Table 11. Given A_i, possible A_jA_i and A_iA_j for all $1 \leq j \leq 16$

Remark 4.3. We use the same i, k used in Lemma 4.1 and Lemma 4.2.

(a) If $A_i (i \neq 1)$ is a singular matrix then each four k appears four times.

(b) If A_i is an invertible matrix then each k appears one time.

From Lemma 4.1 and Lemma 4.2 we can observe the above facts.

Lemma 4.4 below can be easily obtained using the determinant properties of matrices.

Lemma 4.4. *A reverse sound matrix of singular(resp., invertible) matrix is a singular(resp., invertible) matrix.*

Let $\mathfrak{R}_{i,j}$ or $\mathfrak{L}_{i,j}$: $A_{r_1} \to \cdots \to \underleftarrow{A_{r_p} \to \cdots \to A_{r_q}}$ be a matrix sequence with $r_1 = i$. We define $\mathfrak{r}_{i,j}$(resp., $\mathfrak{l}_{i,j}$) $= \alpha_{r_1} \cdots \alpha_{r_p} \cdots \alpha_{r_q}$ to be a totally sound matrix sequence code of $\mathfrak{R}_{i,j}$(resp., $\mathfrak{L}_{i,j}$). And we call $\hat{\mathfrak{r}}_{i,j}$ (resp., $\hat{\mathfrak{l}}_{i,j}$) $= \alpha_{r_p} \cdots \alpha_{r_q}$ a partially sound matrix sequence code of $\mathfrak{R}_{i,j}$(resp., $\mathfrak{L}_{i,j}$).

If $\mathfrak{r}_{i,j}$ (resp., $\hat{\mathfrak{r}}_{i,j}$) is $a_1a_2a_3 \cdots a_n$ and $\mathfrak{l}_{i\prime,j\prime}$ (resp., $\hat{\mathfrak{l}}_{i\prime,j\prime}$) is $a_n \cdots a_3a_2a_1$ then we call $\mathfrak{l}_{i\prime,j\prime}$ (resp., $\hat{\mathfrak{l}}_{i\prime,j\prime}$) a totally (resp., partially) reverse sound matrix sequence code of $\mathfrak{r}_{i,j}$ (resp., $\hat{\mathfrak{r}}_{i,j}$).

Lemma 4.5. *Let $\mathfrak{T}_{i,j}$ be a stable matrix sequence. Then*

- **(a)** *If $Ord(\mathfrak{T}_{i,j}) = 1$ then $\mathfrak{t}_{i,j}$ is a totally reverse sound matrix sequence code for some $\mathfrak{l}_{i',j'}$.*
- **(b)** *If $Ord(\mathfrak{T}_{i,j}) \geq 2$ then $\mathfrak{t}_{i,j}$ is not a totally reverse sound matrix sequence code.*

Proof. Let $\mathfrak{T}_{i,j} = \mathfrak{R}_{i,j}$.

(a) If $\mathfrak{R}_{i,j}$ is a stable matrix sequence with $Ord(\mathfrak{R}_{i,j}) = 1$ then $A_i \in U_1$ by Lemma 2.5 (a). From Table 10, the reverse sound matrix of $A_i(\in U_1)$ is also in U_1. There exists at least one (i, j) such that $StP(\mathfrak{L}_{i,j}) \in U_1$ for all element in U_1. Since $\mathfrak{r}_{i,j} = \alpha_i$, the proof of (a) is completed.

(b) We divide it into two cases by whether $StP(\mathfrak{R}_{i,j}) = A_1$ or not.

First, suppose that $\mathfrak{R}_{i,j}$ is a stable matrix sequence with $Ord(\mathfrak{R}_{i,j}) \geq 2$ and $StP(\mathfrak{R}_{i,j}) = A_1$. Then

$$\mathfrak{r}_{i,j} = a_1a_2a_3a_4 \cdots 0000$$

with $a_1a_2a_3a_4 \neq 0000$. So, the totally reverse sound sequence code of this does not exist.

Second, suppose that $\mathfrak{R}_{i,j} : A_i \to A_k \circlearrowleft$, where $A_k \neq A_1$. Then A_i is an invertible matrix and A_k is a singular matrix by Remark 2.7. Suppose that $\mathfrak{r}_{i,j}$ has a totally reverse sound matrix sequence code $\mathfrak{l}_{i',j'}$. By Lemma 4.4, $A_{i'}$ is a singular matrix and A_p is an invertible matrix, where $\mathfrak{L}_{i',j'} : A_{i'} \to A_p$ or $A_{i'} \rightleftarrows A_p$. Hence, there is no totally reverse sound matrix sequence code.

The case of $\mathfrak{L}_{i,j}$ can be proved using the same method in the proof of $\mathfrak{R}_{i,j}$. □

$\mathfrak{R}_{i,j}$	$\mathfrak{r}_{i,j}$	$\mathfrak{l}_{i\prime,j\prime}$	$\mathfrak{L}_{i\prime,j\prime}$
$\mathfrak{R}_{1,j}$ $(1 \leq j \leq 16)$	0000	0000	$\mathfrak{L}_{1,j\prime}$ $(1 \leq j\prime \leq 16)$
$\mathfrak{R}_{2,1}, \mathfrak{R}_{2,5}, \mathfrak{R}_{2,9}, \mathfrak{R}_{2,13}$	0001	1000	$\mathfrak{L}_{9,1}, \mathfrak{L}_{9,2}, \mathfrak{L}_{9,5}, \mathfrak{L}_{9,6}$
$\mathfrak{R}_{4,1}, \mathfrak{R}_{4,6}, \mathfrak{R}_{4,11}, \mathfrak{R}_{4,16}$	0011	1100	$\mathfrak{L}_{13,1}, \mathfrak{L}_{13,2}, \mathfrak{L}_{13,5}, \mathfrak{L}_{13,6}$
$\mathfrak{R}_{6,1}, \mathfrak{R}_{6,5}, \mathfrak{R}_{6,9}, \mathfrak{R}_{6,13}$	0101	1010	$\mathfrak{L}_{11,1}, \mathfrak{L}_{11,4}, \mathfrak{L}_{11,13}, \mathfrak{L}_{11,16}$
$\mathfrak{R}_{9,1}, \mathfrak{R}_{9,2}, \mathfrak{R}_{9,3}, \mathfrak{R}_{9,4}$	1000	0001	$\mathfrak{L}_{2,1}, \mathfrak{L}_{2,3}, \mathfrak{L}_{2,9}, \mathfrak{L}_{2,11}$
$\mathfrak{R}_{10,1}$	1001	1001	$\mathfrak{L}_{10,1}$
$\mathfrak{R}_{11,1}, \mathfrak{R}_{11,2}, \mathfrak{R}_{11,3}, \mathfrak{R}_{11,4}$	1010	0101	$\mathfrak{L}_{6,1}, \mathfrak{L}_{6,4}, \mathfrak{L}_{6,13}, \mathfrak{L}_{6,16}$
$\mathfrak{R}_{13,1}, \mathfrak{R}_{13,6}, \mathfrak{R}_{13,11}, \mathfrak{R}_{13,16}$	1100	0011	$\mathfrak{L}_{4,1}, \mathfrak{L}_{4,3}, \mathfrak{L}_{4,9}, \mathfrak{L}_{4,11}$

Table 12. Totally reverse sound stable matrix sequences

Lemma 4.6. *If $\mathfrak{T}_{i,j}$ is an amicable matrix sequence then $\mathfrak{t}_{i,j}$ is not a totally reverse sound matrix sequence code.*

Proof. Let $\mathfrak{T}_{i,j} = \mathfrak{R}_{i,j}$. By Lemma 2.9, we prove it by dividing it into two cases according to $Ord(\mathfrak{R}_{i,j})$.

Case 1. Consider $\mathfrak{R}_{i,j} : A_i \rightleftarrows A_k$. In this case, we can divide into two parts depending on A_i, that is, A_i is invertible or not.

First, let A_i be an invertible matrix. By Remark 2.8, we find that $\mathfrak{R}_{i,j} : A_i \rightleftarrows A_{10}$ and $\mathfrak{r}_{i,j} = \alpha_i 1001$. If $\mathfrak{r}_{i,j}$ has a totally reverse sound matrix sequence code $\mathfrak{l}_{i',j'}$, then $\mathfrak{L}_{i',j'} : A_{10} \rightleftarrows A_s$ or $A_{10} \to A_s \circlearrowleft$, where A_s is the reverse sound matrix for A_i. Since A_s is the reverse sound matrix of an invertible matrix A_i, A_s is an invertible matrix by Lemma 4.4. Suppose that $\mathfrak{L}_{i',j'} : A_{10} \rightleftarrows A_s$. Since A_{10} is an invertible matrix, $A_s = A_{10}$ by Remark 2.11. But it is not an amicable matrix sequence. Next, suppose that $\mathfrak{L}_{i',j'} : A_{10} \to A_s \circlearrowleft$. By Lemma 2.5, $A_s \in U_1$ and $A_s \neq A_{10}$. Then A_s is a singular matrix. It is a contradiction.

Now, let A_i be a singular matrix. Suppose that $\mathfrak{r}_{i,j}$ has a reverse sound matrix sequence code $\mathfrak{l}_{i',j'}$ and $\mathfrak{L}_{i',j'}$: $A_{i'} \rightleftarrows A_q$ or $A_{i'} \to A_q \circlearrowleft$, where $A_{i'}$ and A_q are reverse sound matrices for A_k and A_i, respectively. A singular matrix A_i is in $\{A_2, A_3, A_4, A_5, A_6, A_9, A_{11}, A_{13}, A_{16}\}$. If $A_i = A_2$ then $A_k \in \{A_3, A_4\}$ by Table 11. Then $A_{i'} \in \{A_5, A_{13}\}$ and $A_q = A_9$ by Table 10. By the definition of $\mathfrak{L}_{i',j'}$, $A_9 = (A_{i'} + A_{j'})A_{i'}$. Using Table 10, there is no matrix $A_{j'}$ such that $A_9 = (A_{i'} + A_{j'})A_{i'}$. Therefore, there is no totally reverse sound sequence code.

We can check in the same way for the other singular matrices A_i.

Case 2. Consider $\mathfrak{R}_{i,j} : A_i \to A_m \rightleftarrows A_n$. By Remark 2.11, A_i is an invertible matrix but A_m and A_n are singular matrices. If there is a totally reverse sound sequence code $\mathfrak{l}_{i',j'}$ of $\mathfrak{r}_{i,j}$ then $\mathfrak{L}_{i',j'} : A_{i'} \to A_s \rightleftarrows A_t$, or $A_{i'} \to A_s \to A_t \circlearrowleft$ or $\underleftarrow{A_{i'} \to A_s \to A_t}$, where $A_{i'}$, A_s and A_t are reverse sound matrices of A_n, A_m, and A_i, respectively. Then A_t is an invertible matrix and $A_{i'}$ and A_s are singular matrices by Lemma 4.4. But it is impossible because $A_t = A_{i'}A_s$.

The case of $\mathfrak{T}_{i,j} = \mathfrak{L}_{i,j}$ can be proved by the same method in proof of the case $\mathfrak{T}_{i,j} = \mathfrak{R}_{i,j}$ □

Lemma 4.7. *Let $\mathfrak{R}_{i,j}$ and $\mathfrak{L}_{i',j'}$ be sociable matrix sequences. If $\mathfrak{r}_{i,j}$ has a totally reverse sound matrix sequence code $\mathfrak{l}_{i',j'}$ then $\mathfrak{r}_{i',j'}$ has a totally reverse sound matrix sequence code $\mathfrak{l}_{i,j}$.*

Proof. Suppose that $\mathfrak{r}_{i,j}$ has a totally reverse sound matrix sequence code $\mathfrak{l}_{i',j'}$. By Remark 2.14, $\mathfrak{r}_{i,j} = \cdots 1001$. So $\mathfrak{l}_{i',j'} = 1001\cdots$. Since $\mathfrak{L}_{i',j'}$ is also sociable matrix sequence, $\mathfrak{l}_{i',j'} = 1001\cdots 1001$. Then $\mathfrak{r}_{i,j} = 1001\cdots 1001$ because it is a reverse sound matrix sequence of $\mathfrak{l}_{i',j'}$. Hence, $i = i' = 10$. Since A_{10} is the identity matrix, $\mathfrak{R}_{10,k} = \mathfrak{L}_{10,k}$ for all $1 \le k \le 16$, i.e., $\mathfrak{r}_{10,k} = \mathfrak{l}_{10,k}$. Hence $\mathfrak{r}_{i,j}$ has a reverse sound matrix sequence code $\mathfrak{l}_{i',j'}$ if and only if $\mathfrak{r}_{i',j'}$ has a totally reverse sound matrix sequence code $\mathfrak{l}_{i,j}$. □

Using Lemma 4.7, we obtain the following theorem.

Theorem 4.8. *Let $\mathfrak{T}_{i,j}$ be a sociable matrix sequence. If $\mathfrak{t}_{i,j}$ has a reverse sound matrix sequence code then $(i, j) \in \{(10,3),(10,5),(10,8),(10,15),(10,16)\}$. See Table 13 for details.*

$\mathfrak{R}_{i,j}$	$\mathfrak{r}_{i,j}$	$\mathfrak{l}_{i',j'}$	$\mathfrak{L}_{i',j'}$
$\mathfrak{R}_{10,3}$	1001101110111001	1001110111011001	$\mathfrak{L}_{10,5}$
$\mathfrak{R}_{10,5}$	1001110111011001	1001101110111001	$\mathfrak{L}_{10,3}$
$\mathfrak{R}_{10,16}$	1001011001101001	1001011001101001	$\mathfrak{L}_{10,16}$
$\mathfrak{R}_{10,8}$	100111101110011101111001	100111101110011101111001	$\mathfrak{L}_{10,8}$
$\mathfrak{R}_{10,15}$	100101110111111011101001	100101110111111011101001	$\mathfrak{L}_{10,15}$

Table 13. Totally reverse sound sociable matrix sequences

Lemma 4.9. *If $\mathfrak{T}_{i,j}$ is a stable matrix sequence then $\hat{\mathfrak{t}}_{i,j}$ has a partial reverse sound matrix sequence code.*

Proof. Let $\mathfrak{T}_{i,j}$ be a stable matrix sequence. By tables in Theorem 10, $StP(\mathfrak{T}_{i,j}) \in S = \{A_1, A_2, A_4, A_6, A_9, A_{11}, A_{13}\}$. Therefore, $\hat{\mathfrak{t}}_{i,j} = \alpha_i$, where $i \in \{1, 2, 4, 6, 9, 11, 13\}$. Using Table 10, the reverse sound matrices of $A_i \in S$ are also in S. For example, the reverse sound matrix of A_4 is A_{13}. Hence, we can give the partial reverse sound matrix sequence code of $\hat{\mathfrak{t}}_{i,j}$. □

5. Palindromic sound matrix sequence codes

Let A_m and A_n be a palindromic sound matrix pair if $\alpha_m = abcd$ and $\alpha_n = dcba$. And let A_m and A_n be anti-palindromic sound matrix pair if $\alpha_m = abcd$ and $\alpha_n = (1-d)(1-c)(1-b)(1-a)$. Table 14 and 15 present all of the palindromic sound matrix pair and anti-palindromic sound matrix pair, respectively.

A_m	α_m	α_n	A_n
A_1	0000	0000	A_1
A_2	0001	1000	A_9
A_3	0010	0100	A_5
A_4	0011	1100	A_{13}
A_5	0100	0010	A_3
A_6	0101	1010	A_{11}
A_7	0110	0110	A_7
A_8	0111	1110	A_{15}
A_9	1000	0001	A_2
A_{10}	1001	1001	A_{10}
A_{11}	1010	0101	A_6
A_{12}	1011	1101	A_{14}
A_{13}	1100	0011	A_4
A_{14}	1101	1011	A_{12}
A_{15}	1110	0111	A_8
A_{16}	1111	1111	A_{16}

Table 14. Palindromic sound matrix pair

Remark 5.1. If A_m and A_n is a palindromic sound matrix pair then both A_m and A_n are either invertible matrices or singular matrices. See Table 14.

A_m	α_m	α_n	A_n
A_1	0000	1111	A_{16}
A_2	0001	0111	A_8
A_3	0010	1011	A_{12}
A_4	0011	0011	A_4
A_5	0100	1101	A_{14}
A_6	0101	0101	A_6
A_7	0110	1001	A_{10}
A_8	0111	0001	A_2
A_9	1000	1110	A_{15}
A_{10}	1001	0110	A_7
A_{11}	1010	1010	A_{11}
A_{12}	1011	0010	A_3
A_{13}	1100	1100	A_{13}
A_{14}	1101	0100	A_5
A_{15}	1110	1000	A_9
A_{16}	1111	0000	A_1

Table 15. Anti-palindromic sound matrix pair

$\mathfrak{T}_{i,j} : A_{i_0} \to A_{i_1} \to \cdots \to A_{i_m} \to \cdots \to A_{i_n} \to A_{i_m} \to \cdots \to A_{i_n} \to \cdots$. $\mathfrak{t}_{i,j}$ is a totally palindromic(resp., anti-palindromic) sound matrix sequence code if and only if A_{i_t} and $A_{i_{n-t}}$ is a palindromic(resp., anti-palindromic) sound matrix pair for $0 \leq t \leq n$. Similarly, $\hat{\mathfrak{t}}_{i,j}$ is a partially palindromic(resp., anti-palindromic) sound matrix sequence code if and only if $A_{i_{m+t}}$ and $A_{i_{n-t}}$ is a palindromic(resp., anti-palindromic) sound matrix pair for $0 \leq t \leq n-m$.

Lemma 5.2. *Let $\mathfrak{T}_{i,j}$ be a stable matrix sequence with $Ord(\mathfrak{T}_{i,j}) = 1$. Then*

- **(a)** $\mathfrak{t}_{i,j}$ *is a totally palindromic sound matrix sequence code if and only if $i \in \{1, 10\}$.*

(b) $\mathfrak{t}_{i,j}$ *is a totally anti-palindromic sound matrix sequence code if and only if* $i \in \{4, 6, 11, 13\}$.

Proof. Let $\mathfrak{T}_{i,j} = \mathfrak{R}_{i,j}$.

(a) Suppose that $\mathfrak{r}_{i,j}$ is a totally palindromic sound matrix sequence code. Then A_i is a palindromic sound matrix pair itself. By Table 14,

$$A_i \in \{A_1, A_7, A_{10}, A_{16}\}. \tag{5.1}$$

Since $\mathfrak{R}_{i,j}$ is a stable matrix sequence with $Ord(\mathfrak{R}_{i,j}) = 1$,

$$A_i \in U_1 \tag{5.2}$$

by Lemma 2.5 (a). Combining (5.1) and (5.2), $A_i \in \{A_1, A_7, A_{10}, A_{16}\} \cap U_1 = \{A_1, A_{10}\}$. Hence $i \in \{1, 10\}$.

Conversely, it is obvious by Table 14.

(b) Suppose that $\mathfrak{r}_{i,j}$ is a totally anti-palindromic sound matrix sequence code. Then A_i is an anti-palindromic sound matrix pair itself. By Table 15,

$$A_i \in \{A_4, A_6, A_{11}, A_{13}\}. \tag{5.3}$$

Since $\mathfrak{R}_{i,j}$ is a stable matrix sequence with $Ord(\mathfrak{R}_{i,j}) = 1$, we have

$$A_i \in U_1 \tag{5.4}$$

by Lemma 2.5 (a). Therefore, $A_i \in \{A_4, A_6, A_{11}, A_{13}\} \cap U_1 = \{A_4, A_6, A_{11}, A_{13}\}$. Hence $i \in \{4, 6, 11, 13\}$.

Conversely, it is trivial by Table 15.

The case of $\mathfrak{T}_{i,j} = \mathfrak{L}_{i,j}$ can be proved by the same method in the proof of $\mathfrak{T}_{i,j} = \mathfrak{R}_{i,j}$. □

Lemma 5.3. *Let* $\mathfrak{T}_{i,j}$ *be a stable matrix sequence with* $Ord(\mathfrak{T}_{i,j}) = 2$. *Then*

(a) $\mathfrak{t}_{i,j}$ *is not a totally palindromic sound matrix sequence code.*

(b) $\hat{\mathfrak{t}}_{i,j}$ *is a partially palindromic sound matrix sequence code if and only if* $StP(\mathfrak{T}_{i,j}) = A_1$.

(c) $\mathfrak{t}_{i,j}$ *is a totally anti-palindromic sound matrix sequence code if and only if* $A_i = A_{16}$.

(d) $\hat{\mathfrak{t}}_{i,j}$ *is a partially anti-palindromic sound matrix sequence code if and only if* $StP(\mathfrak{T}_{i,j}) \in \{A_4, A_6, A_{11}, A_{13}\}$.

Proof. Let $\mathfrak{T}_{i,j} = \mathfrak{R}_{i,j}$.

(a) If $Ord(\mathfrak{R}_{i,j}) = 2$ then A_i is an invertible matrix and $A_k = A_i(A_i + A_j)$ is a singular matrix by Remark 2.7. Hence, A_i and A_k is not a palindromic sound matrix pair by Remark 5.1.

(b) Suppose that $\mathfrak{r}_{i,j}$ is a partially palindromic sound matrix sequence code with $StP(\mathfrak{R}_{i,j}) = A_k$. Then A_k and A_k is a palindromic sound matrix pair itself. From Table 14, we obtain

$$A_k \in \{A_1, A_7, A_{10}, A_{16}\}. \tag{5.5}$$

By Table 3, we checked that

$$A_k \in \{A_1, A_2, A_4, A_6, A_9, A_{11}, A_{13}\}. \tag{5.6}$$

Combining (5.5) and (5.6), $A_k \in \{A_1, A_7, A_{10}, A_{16}\} \cap \{A_1, A_2, A_4, A_6, A_9, A_{11}, A_{13}\} = \{A_1\}$. Hence $A_k = A_1$.

Conversely, it is easily obtained by Table 14.

(c) Suppose that $\mathfrak{r}_{i,j}$ is a totally anti-palindromic sound matrix sequence code. Since $\mathfrak{R}_{i,j}$ is a stable matrix sequence with $Ord(\mathfrak{R}_{i,j}) = 2$, we get $A_k = A_i(A_i + A_j) \in U_1$ by Lemma 2.5 (b). Divide U_1 into three sets $U_{11} = \{A_4, A_6, A_{11}, A_{13}\}$, $U_{12} = \{A_2, A_9\}$ and $U_{13} = \{A_1\}$.

First, suppose that $A_k \in U_{11}$. Then A_k is an anti-palindromic sound matrix pair itself. So it is impossible. Second, suppose that $A_k \in U_{12}$. By Table 15, $A_i \in \{A_8, A_{15}\}$. On the other hand, $A_i \in U_3 \cup \{A_{10}\}$ by Remark 2.7 because $\mathfrak{R}_{i,j}$

is a stable matrix sequence with $Ord(\mathfrak{R}_{i,j}) = 2$. Therefore, it is not possible. Last, suppose that $A_k \in U_{13}$, that is, $A_k = A_1$. Then $A_i = A_{16}$ by Table 15.

Conversely, suppose that $A_i = A_{16}$. By Table 3, we get $A_k = A_i(A_i + A_j) = A_1$. A_1 and A_{16} is an anti-palindromic sound matrix pair by Table 15. So $\mathfrak{r}_{i,j}$ is a totally anti-palindromic sound matrix sequence code.

(d) It is proved similar to Lemma 5.2 (b).

The case of $\mathfrak{T}_{i,j} = \mathfrak{L}_{i,j}$ can be proved by the similar way in the proof of $\mathfrak{T}_{i,j} = \mathfrak{R}_{i,j}$. □

Lemma 5.4. *Let $\mathfrak{T}_{i,j}$ be a stable matrix sequence with $Ord(\mathfrak{T}_{i,j}) \geq 3$. Then*

- **(a)** $\mathfrak{t}_{i,j}$ *is not a totally palindromic sound matrix sequence code.*
- **(b)** $\hat{\mathfrak{t}}_{i,j}$ *is a partially palindromic sound matrix sequence code if and only if $StP(\mathfrak{T}_{i,j}) = A_1$.*
- **(c)** $\mathfrak{t}_{i,j}$ *is not a totally anti-palindromic sound matrix sequence code.*
- **(d)** $\hat{\mathfrak{t}}_{i,j}$ *is not a partially anti-palindromic sound matrix sequence code.*

Proof. Let $\mathfrak{T}_{i,j} = \mathfrak{R}_{i,j}$.

(a) By Table 3, we assume that $StP(\mathfrak{R}_{i,j}) = A_1$. If $\mathfrak{r}_{i,j}$ is a totally palindromic sound matrix sequence code then $A_i = A_1$. Because A_1 and A_1 is a palindromic sound matrix pair itself by Table 14. But if $A_i = A_1$ then $\mathfrak{R}_{i,j}$ is a stable matrix sequence with $Ord(\mathfrak{R}_{i,j}) = 1$ by Lemma 2.2.

(b) It is proved similar to Lemma 5.3 (b).

(c) Suppose that $Ord(\mathfrak{R}_{i,j}) = 3$. By Table 3, we put $StP(\mathfrak{R}_{i,j}) = A_1$. If $\mathfrak{r}_{i,j}$ is a totally anti-palindromic sound matrix sequence code then $A_i = A_{16}$ and $A_k = A_i(A_i + A_j)$ is an anti-palindromic sound matrix pair itself. By Lemma 2.5 (c),

$$A_k = A_i(A_i + A_k) \in U_2. \tag{5.7}$$

Since $A_i = A_{16}$, we obtain

$$A_k \in \{A_1, A_6, A_{11}, A_{16}\} \tag{5.8}$$

by Lemma 4.1 (d). Combining (5.7) and (5.8), $A_k \in U_2 \cap \{A_1, A_6, A_{11}, A_{16}\} = \{A_{16}\}$. By Table 15 $A_k = A_{16}$ is not an anti-palindromic sound matrix pair itself. Next, suppose that $Ord(\mathfrak{R}_{i,j}) = 4$. Then A_i is an invertible matrix by Remark 2.8 (b).

If $\mathfrak{r}_{i,j}$ is a totally anti-palindromic sound matrix sequence code then $A_i = A_{16}$ because $StP(\mathfrak{R}_{i,j}) = A_1$ by Table 3. But A_{16} is not an invertible matrix.

(d) $StP(\mathfrak{R}_{i,j}) = A_1$. By Table 15, A_1 and A_1 is not an anti-palindromic sound matrix pair itself. Therefore, $\mathfrak{r}_{i,j}$ is not a partially anti-palindromic sound matrix sequence code.

The case of $\mathfrak{T}_{i,j} = \mathfrak{L}_{i,j}$ can be proved by the same method in the proof of the case $\mathfrak{T}_{i,j} = \mathfrak{R}_{i,j}$. □

Lemma 5.5. *Let $\mathfrak{T}_{i,j}$ be an amicable matrix sequence with $Ord(\mathfrak{T}_{i,j}) = 1$. Then*

- **(a)** $\mathfrak{r}_{i,j}$ *is a totally palindromic sound matrix sequence code if and only if $\mathfrak{R}_{i,j} : A_6 \rightleftarrows A_{11}$ or $A_{11} \rightleftarrows A_6$.*
- **(b)** $\mathfrak{l}_{i,j}$ *is a totally palindromic sound matrix sequence code if and only if $\mathfrak{L}_{i,j} : A_4 \rightleftarrows A_{13}$ or $A_{13} \rightleftarrows A_4$.*
- **(c)** $\mathfrak{t}_{i,j}$ *is a totally anti-palindromic sound matrix sequence code if and only if $i = 7$.*

Proof. (a) First, suppose that A_i is an invertible matrix. Then $\mathfrak{R}_{i,j} : A_i \rightleftarrows A_{10}$ by Remark 2.11 (a). Using Table 14, A_{10} and A_{10} is a palindromic sound matrix pair itself. Hence, $A_i = A_{10}$ for a totally palindromic sound matrix sequence code $\mathfrak{r}_{i,j}$. But $A_{10} \rightleftarrows A_{10}$ is a stable matrix sequence $A_{10} \circlearrowleft$. Therefore, there is no invertible matrix A_i such that $\mathfrak{r}_{i,j}$ is a totally(=partially) palindromic sound matrix sequence code. Second, suppose that A_i is a singular matrix. Using Table 14, A_2 and A_9, A_3 and A_5, A_4 and A_{13}, A_6 and A_{11} are palindromic sound matrix pairs, respectively. And A_{16} and A_{16} is a palindromic sound matrix pair itself. By Lemma 4.1, A_2 and A_9, A_3 and A_5, A_4 and A_{13} can not make same matrix sequence starting with singular matrix. If $A_i = A_{16}$ then $\mathfrak{R}_{i,j} : A_{16} \rightleftarrows A_{16}$. But it is not an amicable matrix sequence. Hence, $\mathfrak{R}_{i,j} : A_6 \rightleftarrows A_{11}$ or $A_{11} \rightleftarrows A_6$.

Conversely, it is obvious by Table 14.

(b) It can be proved by the same method in (a).

(c) Let $\mathfrak{T}_{i,j} = \mathfrak{R}_{i,j}$. First, suppose that A_i is an invertible matrix. Then we obtain $\mathfrak{R}_{i,j} : A_i \rightleftarrows A_{10}$ by Remark 2.11 (a). By Table 15, A_7 and A_{10} is an anti-palindromic sound matrix pair. So $i = 7$.

Second, suppose that A_i is a singular matrix. If the anti-palindromic sound matrix pair A_k of a singular matrix A_i is different from A_i then we can not find $\mathfrak{T}_{i,j} : A_i \rightleftarrows A_k$ by Table 15. If $A_k = A_i$ then $\mathfrak{T}_{i,j} : A_i \rightleftarrows A_k$ is not an amicable matrix sequence. Hence, there is no singular matrix A_i that satisfies $\mathfrak{t}_{i,j}$ is a totally anti-palindromic sound matrix sequence code.

Conversely, by Remark 2.11 (a), $\mathfrak{R}_{i,j} = A_7 \rightleftarrows A_{10}$. Then it is done by Table 15.

We can prove $\mathfrak{T}_{i,j} = \mathfrak{R}_{i,j}$ by the same method of the case $\mathfrak{T}_{i,j} = \mathfrak{R}_{i,j}$. □

Lemma 5.6. *Let $\mathfrak{T}_{i,j}$ be an amicable matrix sequence with $Ord(\mathfrak{T}_{i,j}) = 2$. Then*

- **(a)** $\mathfrak{t}_{i,j}$ *is not a totally palindromic sound matrix sequence code.*
- **(b)** $\hat{\mathfrak{r}}_{i,j}$ *is a partially palindromic sound matrix sequence code if and only if $\mathfrak{R}_{i,j} : A_i \to A_6 \rightleftarrows A_{11}$ or $\mathfrak{R}_{i,j} : A_i \to A_{11} \rightleftarrows A_6$.*
- **(c)** $\hat{\mathfrak{l}}_{i,j}$ *is a partially palindromic sound matrix sequence code if and only if $\mathfrak{R}_{i,j} : A_i \to A_4 \rightleftarrows A_{13}$ or $\mathfrak{R}_{i,j} : A_i \to A_{13} \rightleftarrows A_4$.*
- **(d)** $\mathfrak{t}_{i,j}$ *is not a totally anti-palindromic sound matrix sequence code.*
- **(e)** $\hat{\mathfrak{l}}_{i,j}$ *is not a partially anti-palindromic sound matrix sequence code.*

Proof. Let $\mathfrak{T}_{i,j} = \mathfrak{R}_{i,j}$.

(a) By Remark 2.11 (b), A_i is an invertible matrix and $A_q = A_i(A_i + A_j)A_i$ is a singular matrix. This proof is done by Remark 5.1.

(b) Suppose that $\hat{\mathfrak{r}}_{i,j} = \alpha_p\alpha_q$ is a partially palindromic matrix sequence code, where $\mathfrak{R}_{i,j} : A_i \to A_p \rightleftarrows A_q$. By Remark 2.11, A_p and A_q are singular matrices. The rest of the proof is almost identical to Lemma 5.5 (a).

Conversely, it is obvious by Table 14.

(c) It can be proved by the same method in (b).

(d) By Remark 2.11 (b), A_i is an invertible matrix, A_p and A_q are singular matrices, where $\mathfrak{R}_{i,j} : A_i \to A_p \rightleftarrows A_q$. Then it is almost same with Lemma 5.5 (c). But in this case, $A_p = A_7$ and $A_q = A_{10}$ is impossible because A_p and A_q are singular matrices. Hence, there is no partially anti-palindromic sound matrix sequence code.

(e) By Remark 2.11 (b), A_i is an invertible matrix, A_p and A_q are singular matrices, where $\mathfrak{R}_{i,j} : A_i \to A_p \rightleftarrows A_q$. Then the next process of this proof is similar to Lemma 5.5 (c).

The case of $\mathfrak{T}_{i,j} = \mathfrak{L}_{i,j}$ can be proved by the same method by the similar method of the case $\mathfrak{T}_{i,j} = \mathfrak{R}_{i,j}$. □

All of the totally palindromic matrix sound sequence codes $\mathfrak{t}_{i,j}$ of stable and amicable matrix sequences are given in Table 16. And all partially palindromic sound matrix sequence coeds $\hat{\mathfrak{t}}_{i,j}$ of stable and amicable matrix sequences except $Ord(\mathfrak{T}_{i,j}) = 1$ are given in Table 17.

$\mathfrak{t}_{i,j}$	Binary code	Wave
$\mathfrak{t}_{1,j}$, $1 \le j \le 16$	0000	_ _ _ _
$\mathfrak{t}_{10,1}$	1001	⊓ _ _ ⊓
$\mathfrak{r}_{6,4}$, $\mathfrak{r}_{6,8}$, $\mathfrak{r}_{6,12}$, $\mathfrak{r}_{6,16}$	01011010	_ ⊓ _ ⊓ ⊓ _ ⊓ _
$\mathfrak{r}_{11,13}$, $\mathfrak{r}_{11,14}$, $\mathfrak{r}_{11,15}$, $\mathfrak{r}_{11,16}$	10100101	⊓ _ ⊓ _ _ ⊓ _ ⊓
$\mathfrak{l}_{4,6}$, $\mathfrak{l}_{4,8}$, $\mathfrak{l}_{4,14}$, $\mathfrak{l}_{4,16}$	00111100	_ _ ⊓ ⊓ ⊓ ⊓ _ _
$\mathfrak{l}_{13,11}$, $\mathfrak{l}_{13,12}$, $\mathfrak{l}_{13,15}$, $\mathfrak{l}_{13,16}$	11000011	⊓ ⊓ _ _ _ _ ⊓ ⊓

Table 16. Totally palindromic sound matrix sequence codes of stable and amicable matrix sequences

$\hat{\mathfrak{t}}_{i,j}$	Binary code	Wave
$\hat{\mathfrak{t}}_{i,i}$ $(2 \leq i \leq 16)$, $\hat{\mathfrak{t}}_{2,10}$, $\hat{\mathfrak{t}}_{3,1}$, $\hat{\mathfrak{t}}_{4,10}$, $\hat{\mathfrak{t}}_{5,1}$, $\hat{\mathfrak{t}}_{6,10}$, $\mathfrak{t}_{9,10}$, $\hat{\mathfrak{t}}_{11,10}$, $\hat{\mathfrak{t}}_{13,10}$, $\hat{\mathfrak{t}}_{16,1}$, $\hat{\mathfrak{t}}_{2,8}$, $\hat{\mathfrak{t}}_{2,16}$, $\hat{\mathfrak{t}}_{3,12}$, $\hat{\mathfrak{t}}_{4,5}$, $\hat{\mathfrak{t}}_{4,15}$, $\hat{\mathfrak{t}}_{5,14}$, $\hat{\mathfrak{t}}_{6,3}$, $\hat{\mathfrak{t}}_{6,15}$, $\hat{\mathfrak{t}}_{9,15}$, $\hat{\mathfrak{t}}_{9,16}$, $\hat{\mathfrak{t}}_{11,5}$, $\hat{\mathfrak{t}}_{11,8}$, $\hat{\mathfrak{t}}_{13,3}$, $\hat{\mathfrak{t}}_{13,8}$, $\hat{\mathfrak{t}}_{16,7}$, $\hat{\mathfrak{t}}_{7,8}$, $\hat{\mathfrak{t}}_{7,10}$, $\hat{\mathfrak{t}}_{7,15}$, $\hat{\mathfrak{t}}_{8,3}$, $\hat{\mathfrak{t}}_{8,5}$, $\hat{\mathfrak{t}}_{8,16}$, $\hat{\mathfrak{t}}_{10,7}$, $\hat{\mathfrak{t}}_{10,12}$, $\hat{\mathfrak{t}}_{10,14}$, $\hat{\mathfrak{t}}_{12,8}$, $\hat{\mathfrak{t}}_{12,10}$, $\hat{\mathfrak{t}}_{12,15}$, $\hat{\mathfrak{t}}_{14,8}$, $\hat{\mathfrak{t}}_{14,10}$, $\hat{\mathfrak{t}}_{14,15}$, $\hat{\mathfrak{t}}_{15,3}$, $\hat{\mathfrak{t}}_{15,5}$, $\hat{\mathfrak{t}}_{15,16}$, $\hat{\mathfrak{r}}_{2,6}$, $\hat{\mathfrak{r}}_{2,14}$, $\hat{\mathfrak{r}}_{3,2}$, $\hat{\mathfrak{r}}_{3,4}$, $\hat{\mathfrak{r}}_{4,7}$, $\hat{\mathfrak{r}}_{4,13}$, $\hat{\mathfrak{r}}_{5,9}$, $\hat{\mathfrak{r}}_{5,13}$, $\hat{\mathfrak{r}}_{6,2}$, $\hat{\mathfrak{r}}_{6,14}$, $\hat{\mathfrak{r}}_{9,11}$, $\hat{\mathfrak{r}}_{9,12}$, $\hat{\mathfrak{r}}_{11,9}$, $\hat{\mathfrak{r}}_{11,12}$, $\hat{\mathfrak{r}}_{13,4}$, $\hat{\mathfrak{r}}_{13,7}$, $\hat{\mathfrak{r}}_{16,6}$, $\hat{\mathfrak{r}}_{16,11}$, $\hat{\mathfrak{r}}_{2,4}$, $\hat{\mathfrak{r}}_{2,12}$, $\hat{\mathfrak{r}}_{3,9}$, $\hat{\mathfrak{r}}_{3,10}$, $\hat{\mathfrak{r}}_{3,11}$, $\hat{\mathfrak{r}}_{4,2}$, $\hat{\mathfrak{r}}_{4,12}$, $\hat{\mathfrak{r}}_{5,2}$, $\hat{\mathfrak{r}}_{5,6}$, $\hat{\mathfrak{r}}_{5,10}$, $\hat{\mathfrak{r}}_{6,7}$, $\hat{\mathfrak{r}}_{6,11}$, $\hat{\mathfrak{r}}_{9,13}$, $\hat{\mathfrak{r}}_{9,14}$, $\hat{\mathfrak{r}}_{11,6}$, $\hat{\mathfrak{r}}_{11,7}$, $\hat{\mathfrak{r}}_{13,9}$, $\hat{\mathfrak{r}}_{13,14}$, $\hat{\mathfrak{r}}_{16,4}$, $\hat{\mathfrak{r}}_{16,10}$, $\hat{\mathfrak{r}}_{16,13}$, $\hat{\mathfrak{l}}_{2,4}$, $\hat{\mathfrak{l}}_{2,12}$, $\hat{\mathfrak{l}}_{3,9}$, $\hat{\mathfrak{l}}_{3,11}$, $\hat{\mathfrak{l}}_{4,2}$, $\hat{\mathfrak{l}}_{4,12}$, $\hat{\mathfrak{l}}_{5,2}$, $\hat{\mathfrak{l}}_{5,6}$, $\hat{\mathfrak{l}}_{6,7}$, $\hat{\mathfrak{l}}_{6,11}$, $\hat{\mathfrak{l}}_{9,13}$, $\hat{\mathfrak{l}}_{9,14}$, $\hat{\mathfrak{l}}_{11,6}$, $\hat{\mathfrak{l}}_{11,7}$, $\hat{\mathfrak{l}}_{13,9}$, $\hat{\mathfrak{l}}_{13,14}$, $\hat{\mathfrak{l}}_{16,4}$, $\hat{\mathfrak{l}}_{16,13}$, $\hat{\mathfrak{l}}_{2,6}$, $\hat{\mathfrak{l}}_{2,14}$, $\hat{\mathfrak{l}}_{3,2}$, $\hat{\mathfrak{l}}_{3,4}$, $\hat{\mathfrak{l}}_{3,10}$, $\hat{\mathfrak{l}}_{4,7}$, $\hat{\mathfrak{l}}_{4,13}$, $\hat{\mathfrak{l}}_{5,9}$, $\hat{\mathfrak{l}}_{5,10}$, $\hat{\mathfrak{l}}_{5,13}$, $\hat{\mathfrak{l}}_{6,2}$, $\hat{\mathfrak{l}}_{6,14}$, $\hat{\mathfrak{l}}_{9,11}$, $\hat{\mathfrak{l}}_{9,12}$, $\hat{\mathfrak{l}}_{11,9}$, $\hat{\mathfrak{l}}_{11,12}$, $\hat{\mathfrak{l}}_{13,,4}$, $\hat{\mathfrak{l}}_{13,7}$, $\hat{\mathfrak{l}}_{16,6}$, $\hat{\mathfrak{l}}_{16,10}$, $\hat{\mathfrak{l}}_{16,11}$	0000	_ _ _ _
$\hat{\mathfrak{r}}_{7,4}$, $\hat{\mathfrak{r}}_{15,11}$	01011010	_ ⊓ _ ⊓ ⊓ _ ⊓ _
$\hat{\mathfrak{r}}_{7,13}$, $\hat{\mathfrak{r}}_{8,6}$	10100101	⊓ _ ⊓ _ _ ⊓ _ ⊓
$\hat{\mathfrak{l}}_{7,6}$, $\hat{\mathfrak{l}}_{15,13}$	00111100	_ _ ⊓ ⊓ ⊓ ⊓ _ _
$\hat{\mathfrak{l}}_{7,11}$, $\hat{\mathfrak{l}}_{8,4}$	11000011	⊓ ⊓ _ _ _ _ ⊓ ⊓

Table 17. Partially palindromic sound matrix sequence code

All of totally anti-palindromic matrix sound sequence codes $\mathfrak{t}_{i,j}$ are given in Table 18. And all partially anti-palindromic sound matrix sequence coeds $\hat{\mathfrak{t}}_{i,j}$ except $Ord(\mathfrak{T}_{i,j}) = 1$ are given in Table 19.

$\mathfrak{t}_{i,j}$	Binary code	Wave
$\mathfrak{t}_{4,1}$, $\mathfrak{t}_{4,11}$, $\mathfrak{r}_{4,6}$, $\mathfrak{r}_{4,16}$, $\mathfrak{l}_{4,3}$, $\mathfrak{l}_{4,9}$	0011	_ _ ⊓ ⊓
$\mathfrak{t}_{6,1}$, $\mathfrak{t}_{6,13}$, $\mathfrak{r}_{6,5}$, $\mathfrak{r}_{6,9}$, $\mathfrak{l}_{6,4}$, $\mathfrak{l}_{6,16}$	0101	_ ⊓ _ ⊓
$\mathfrak{t}_{11,1}$, $\mathfrak{t}_{11,4}$, $\mathfrak{r}_{11,2}$, $\mathfrak{r}_{11,3}$, $\mathfrak{l}_{11,13}$, $\mathfrak{l}_{11,16}$	1010	⊓ _ ⊓ _
$\mathfrak{t}_{13,1}$, $\mathfrak{t}_{13,6}$, $\mathfrak{r}_{13,11}$, $\mathfrak{r}_{13,16}$, $\mathfrak{l}_{13,2}$, $\mathfrak{l}_{13,5}$	1100	⊓ ⊓ _ _
$\mathfrak{t}_{7,1}$	01101001	_ ⊓ ⊓ _ ⊓ _ _ ⊓
$\mathfrak{t}_{16,1}$, $\mathfrak{t}_{16,16}$, $\mathfrak{r}_{16,6}$, $\mathfrak{r}_{16,11}$, $\mathfrak{l}_{16,4}$, $\mathfrak{l}_{16,13}$	11110000	⊓ ⊓ ⊓ ⊓ _ _ _ _
$\mathfrak{t}_{7,16}$	0110011010011001	_⊓⊓_ _⊓⊓_⊓_ _⊓⊓_ _⊓

Table 18. Totally anti-palindromic sound matrix sequences code

$\hat{\mathfrak{t}}_{i,j}$	Binary code	Wave
$\hat{\mathfrak{r}}_{7,11}$, $\hat{\mathfrak{l}}_{12,11}$	0011	_ _ ⊓ ⊓
$\hat{\mathfrak{t}}_{10,13}$, $\hat{\mathfrak{r}}_{14,13}$, $\hat{\mathfrak{l}}_{7,13}$	0101	_ ⊓ _ ⊓
$\hat{\mathfrak{t}}_{10,4}$, $\hat{\mathfrak{r}}_{12,4}$, $\hat{\mathfrak{l}}_{7,4}$	1010	⊓ _ ⊓ _
$\hat{\mathfrak{t}}_{10,6}$, $\hat{\mathfrak{r}}_{7,6}$, $\hat{\mathfrak{l}}_{14,6}$	1100	⊓ ⊓ _ _

Table 19. Partially anti-palindromic sound matrix sequence code

Lemma 5.7. *Let $\mathfrak{T}_{i,j}$ be a sociable matrix sequence. Then*

(a) $\mathfrak{t}_{i,j}$ *is a totally palindromic matrix sequence code if and only if* $(i, j) \in \{(10, 8), (10, 15), (10, 16)\}$.

(b) $\mathfrak{t}_{i,j}$ *is not a totally anti-palindromic matrix sequence code.*

All of the totally palindromic matrix sequence code $\mathfrak{t}_{i,j}$ of a sociable matrix sequence are given in Table 20.

$\mathfrak{t}_{i,j}$	Binary code	Wave
$\mathfrak{t}_{10,16}$	1001011001101001	⊓ _ _ ⊓ _ ⊓ ⊓ _ ⊓ _ _ ⊓ ⊓ _ _ ⊓
$\mathfrak{t}_{10,8}$	1001111011100111011111001	⊓___⊓⊓⊓⊓_⊓⊓⊓___⊓⊓⊓_⊓⊓⊓⊓___⊓
$\mathfrak{t}_{10,15}$	1001011101111111011101001	⊓___⊓_⊓⊓⊓_⊓⊓⊓⊓⊓⊓_⊓⊓⊓_⊓___⊓

Table 20. Totally palindromic sound matrix sequences of sociable matrix sequence code

6. Subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$

Let E/F be an elliptic curve over a number field F and $E[2] := \{R \in E(\overline{F}) | 2R = O\}$. Here, $\overline{F}$ is the algebraic closure of F and O is the point at infinity. If $\mathrm{Char}(F) \nmid 2$, then $E[2]$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as an abstract group, [10, p.70]. The group $Gal(\overline{F}/F)$ acts on $E[2]$, including a basis for $E[2]$, a group homomorphism

$$\rho_{E,2} : Gal(\overline{F}/F) \to Aut(E[2]) \cong GL_2(\mathbb{Z}/2\mathbb{Z})$$

called the mod 2 Galois representation associated to E. $E(F)$ has a point of order 2 if and only if, up to conjugation,

$$\rho_{E,2} \subseteq \left\{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \in GL_2(\mathbb{Z}/2\mathbb{Z}) \right\}.$$

Since $E[2] \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, the study in this section is helpful in studying $E[2]$.

Subgoups of $K := K_{4,1} = \mathbb{Z}_2 \times \mathbb{Z}_2$ are $K_{1,1} := \{(0,0)\}$, $K_{2,1} := \{(0,0),(0,1)\}$, $K_{2,2} := \{(0,0),(1,0)\}$, $K_{2,3} := \{(0,0),(1,1)\}$.

Let $\gamma_i : K \to K$ be the mapping defined by $\gamma_i(x,y) = A_i(x,y)$ with $1 \le i \le 16$. Put $I_{0,i} := K$, $I_{1,i} := Im(\gamma_i(K))$, $I_{k,i} := Im(\gamma_i(I_{k-1,i}))$. In this section, we consider the following sequence

$$\mathfrak{K}_i : I_{0,i} \to I_{1,i} \to I_{2,i} \to \cdots .$$

See 7 for a detailed list of sequences for $\mathfrak{K}_i$. Similarly as defined for $\mathfrak{T}_{i,j}$, we define the notions of stable, amicable, sociable, period length and order of $\mathfrak{K}_i$. In addition, let $St(\mathfrak{K}_i) := \{P = (x,y) \in K | \gamma_i(P) = P\}$ be the stable set of $\mathfrak{K}_i$ and $deg(\mathfrak{K}_i) := \#\{P \in K | \gamma_i(P) = (0,0)\}$ be the degree of $\mathfrak{K}_i$.

We get the following result without difficulty, so we omit the proof.

Lemma 6.1. *The followings hold true:*

(a) $\mathfrak{K}_1$ *is a stable mapping sequence with* $Ord(\mathfrak{K}_1) = 2$ *and* $St(\mathfrak{K}_1) = K_{1,1}$.

(b) $\mathfrak{K}_{10}$ *is a stable mapping sequence with* $Ord(\mathfrak{K}_1) = 1$ *with* $St(\mathfrak{K}_1) = K$.

Lemma 6.2. *The followings hold true:*

(a) $I_{1,i} = K$ *if and only if* A_i *is an invertible matrix.*

(b) $I_{1,i} = K_{1,1}$ *if and only if* $i = 1$.

(c) $I_{1,i} = K_{2,1}$ *if and only if* $i \in \{2,3,4\}$.

(d) $I_{1,i} = K_{2,2}$ *if and only if* $i \in \{5,9,13\}$.

(e) $I_{1,i} = K_{2,3}$ *if and only if* $i \in \{6,11,16\}$.

Proof. Let $A_i = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\gamma_i(0,0) = (0,0)$, $\gamma_i(0,1) = (b,d)$, $\gamma_i(1,0) = (a,c)$, and $\gamma_i(1,1) = (a+b, c+d)$.

(a) Suppose that $I_{1,i} = K$. We can get following condition; $(0,0)$, (b,d), (a,c), and $(a+b, c+d)$ are all distinct. If the first column of A_i is zero, that is, $A_i = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}$ then $(0+b, 0+d) = (b,d)$. Similarly, if the second column of A_i is zero then $(a+0, c+0) = (a,c)$. So there is no zero column. Now, we divide three case. First, suppose that $(a,c) = (1,1)$. Because of the condition, b and d are distinct. Then $det(A_i) = d - b = 1$. Second, let $(b,d) = (1,1)$. It is similar to the first case, $a \neq c$ and $det(A_i) = a - c = 1$. Lastly, suppose that $(a+b, c+d) = (1,1)$. Since $a \neq b$ and $c \neq d$, we obtain $a = d, b = c$ or $a = c, b = d$. It follows that $det(A_i) = ad - bc = 1$.

Conversely, suppose that A_i is an invertible matrix. Then either $(a,d) = (1,1)$ or $(b,c) = (1,1)$ because $det(A_i) = ad - bc = 1$. Let $(a,d) = (1,1)$. Then $\gamma_i(0,0) = (0,0)$, $\gamma_i(0,1) = (b,1)$, $\gamma_i(1,0) = (1,c)$, and $\gamma_i(1,1) = (b+1, c+1)$. Therefore, we can see that $(0,0)$, $(b,1)$, $(1,c)$, and $(b+1, c+1)$ are all different for $(b,c) \in K \backslash \{(0,0)\}$. Hence, $I_{1,i} = K$.

(b) It's trivial.

(c) Suppose that $I_{1,i} = K_{2,1}$, that is, $\gamma_i(x,y) = (0,z)$ for all $(x,y) \in K$. If $(c,d) = (0,0)$ then $I_{1,i} = K_{1,1}$. We can easily obtain $a = b = 0$ and $(c,d) \neq (0,0)$. Hence $i \in \{2,3,4\}$.

Conversely, if $i \in \{2,3,4\}$ then $A_i = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}$ and $(c,d) \neq (0,0)$. Therefore, $\gamma_i(x,y) = (0,z)$ for all $(x,y) \in K$. $\gamma_i(0,0) = (0,0)$ and since $(c,d) \neq (0,0)$, there exists at least one $(x,y) \in K$ such that $\gamma_i(x,y) = (0,1)$. Hence $I_{1,i} = K_{2,1}$.

(d) Suppose that $I_{1,i} = K_{2,2}$, that is, $\gamma_i(x,y) = (z,0)$ for all $(x,y) \in K$. if $(c,d) = (0,0)$ then $I_{1,i} = K_{1,1}$. We can easily obtain $c = d = 0$ and $(a,b) \neq (0,0)$. Hence $i \in \{5,9,13\}$.

Conversely, if $i \in \{5,9,13\}$ then $A_i = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ and $(a,b) \neq (0,0)$. Therefore, $\gamma_i(x,y) = (z,0)$ for all $(x,y) \in K - \{(0,0)\}$. Since $(a,b) \neq (0,0)$, there exists at least one $(x,y) \in K$ such that $\gamma_i(x,y) = (1,0)$. Hence $I_{1,i} = K_{2,2}$.

(e) Suppose that $I_{1,i} = K_{2,3}$, that is, $\gamma_i(x,y) = (z,z)$ for all $(x,y) \in K$. If $a = b = c = d = 0$ then $I_{1,i} = K_{1,1}$ by (b). We can easily obtain $a = c$, $b = d$ and not all zero. Hence, $i \in \{6,11,16\}$.

Conversely, if $i \in \{6,11,16\}$ then $A_i = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$ and $(a,b) \neq (0,0)$. Therefore, $\gamma_i(x,y) = (z,z)$ for all $(x,y) \in K - \{(0,0)\}$. Since $(a,b) \neq (0,0)$, there exists at least one $(x,y) \in K$ such that $\gamma_i(x,y) = (1,1)$. Hence $I_{1,i} = K_{2,3}$. □

Lemma 6.3. *Let $A_i = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an invertible matrix except A_{10}. Then*

- **(a)** *$\Re_i$ is an amicable mapping sequence with $Ord(\Re_i) = 1$ if and only if $a + d = 0$.*
- **(b)** *$\Re_i$ is a sociable mapping sequence with $Ord(\Re_i) = 1$ and $Per(\Re_i) = 3$ if and only if $a + d = 1$.*

Proof. Note that $A_i(0,0) = (0,0)$, $A_i(0,1) = (b,d)$, $A_i(1,0) = (a,c)$, $A_i(1,1) = (a+b, c+d)$.

(a) Suppose that $\Re_i$ is an amicable mapping sequence with $Ord(\Re_i) = 1$. Since $(0,0) \in St(\Re_i)$, one of the other three points is in $St(\Re_i)$. In other words, two points change each other. If $(0,1) \in St(\Re_i)$ then $b = 0$, $d = 1$, $(a,c) = (1,1)$, and $(a+b, c+d) = (1,0)$, that is, $a = 1$ and $c = 1$. Therefore, $a + d = 1 + 1 = 0$. We can check in the same way for the other cases $(1,0) \in St(\Re_i)$ and $(1,1) \in St(\Re_i)$.

Conversely, suppose that $a + d = 0$. We divide it two cases by the value of ad. First, we assme that $ad = 0$, that is, $a = d = 0$. Then we get $b = c = 1$. Therefore, $A_i(0,1) = (1,0)$, $A_i(1,0) = (0,1)$ and $(0,0), (1,1) \in St(\Re_i)$. It is an amicable mapping sequence with $Ord(\Re_i) = 1$. Next, we assume that $ad = 1$. Then we get $a = d = 1$ and $bc = 0$ by $ad - bc = 1$. Since $A_i \neq A_{10}$, we get $(b,c) \neq (0,0)$, that is, (b,c) can be $(0,1)$ or $(1,0)$. If $(b,c) = (0,1)$ then $A_i(1,0) = (1,1)$, $A_i(1,1) = (1,0)$ and $(0,0), (0,1) \in St(\Re_i)$. If $(b,c) = (1,0)$ then $A_i(0,1) = (1,1)$, $A_i(1,1) = (0,1)$ and $(0,0), (1,0) \in St(\Re_i)$. These are amicable mapping sequences with $Ord(\Re_i) = 1$.

(b) Suppose that $\Re_i$ is a sociable mapping sequence with $Ord(\Re_i) = 1$ and $Per(\Re_i) = 3$. Since $(0,0) \in St(\Re_i)$, the other three points must change each other. If $A_i(0,1) = (1,0)$ then $A_i(1,0) = (1,1)$ and $A_i(1,1) = (0,1)$. So,we derive that $a = 1$, $b = 1$, $c = 1$, $d = 0$ and $a + d = 1$. If $A_i(0,1) = (1,1)$ then $A_i(1,0) = (0,1)$ and $A_i(1,1) = (1,0)$. Then $a = 0$, $b = 1$, $c = 1$, $d = 1$ and then $a + d = 1$.

Conversely, suppose that $a + d = 1$. Since $ad - bc = 1$ and $a \neq d$, we get $ad = 0$ and $bc = 1$. Naturally, we obtain that $b = c = 1$, $A_i(0,0) = (0,0)$, $A_i(0,1) = (1,d)$, $A_i(1,0) = (a,1)$, and $A_i(1,1) = (a+1, d+1)$. Thus we have $(0,1) \neq (1,d)$, $(1,0) \neq (a,1)$ and $(1,1) \neq (a+1, d+1)$ by $(a,d) \neq (0,0)$. Hence $\Re_i$ is a sociable mapping sequence with $Ord(\Re_i) = 1$ and $Per(\Re_i) = 3$. □

Lemma 6.4. *Let* $A_i = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be a singular matrix except* A_1. *Then*

- **(a)** *If* $a \neq d$ *then* $\Re_i$ *is a stable mapping sequence with* $Ord(\Re_i) = 2$.
- **(b)** *If* $a = d$ *then* $\Re_i$ *is a stable mapping sequence with* $Ord(\Re_i) = 3$.

Proof. Note that $A_i(0,0) = (0,0)$, $A_i(0,1) = (b,d)$, $A_i(1,0) = (a,c)$, $A_i(1,1) = (a+b, c+d)$ and $ad = bc$.

(a) If $a \neq d$ then $ad = bc = 0$. Consider three cases depending on the values of b and c. Let $b = c = 0$. Then $I_{1,i} = \{(0,0),(0,1)\}$ or $I_{1,i} = \{(0,0),(1,0)\}$ because $A_i(0,0) = (0,0)$, $A_i(0,1) = (0,d)$, $A_i(1,0) = (a,0)$, $A_i(1,1) = (a,d)$. And we can easily know that $I_{1,i} = I_{k,i}$ for all $k \geq 1$. The other two cases can be shown by the same method of the first case. Each of cases has two possible A_i, so there exist six matrices A_i satisfying that $\Re_i$ are stable mapping sequences with $Ord(\Re_i) = 2$.

(b) We consider two cases, that is, $a = d = 1$ and $a = d = 0$. First, let $a = d = 1$. Since $ad = bc = 1$, we have $b = c = 1$. Then we have $I_{1,i} = \{(0,0),(1,1)\}$ and $I_{2,i} = \{(0,0)\}$ by $A_i(0,0) = (0,0)$, $A_i(0,1) = (1,1)$, $A_i(1,0) = (1,1)$ and $A_i(1,1) = (0,0)$. Next, let $a = d = 0$. By the fact that $A_i \neq A_1$, we get $bc = 0$ and $b \neq c$. Putting $a = d = 0$, we have $A_i(0,0) = (0,0)$, $A_i(0,1) = (b,0)$, $A_i(1,0) = (0,c)$ and $A_i(1,1) = (b,c)$. One of $(b,0)$ and $(0,c)$ is $(0,0)$, and the other is (b,c). Therefore, $I_{1,i} = \{(0,0),(0,1)\}$ or $I_{1,i} = \{(0,0),(1,0)\}$. And then $I_{2,i} = \{(0,0)\}$. Hence $\Re_i$ is a stable mapping sequence with $Ord(\Re_i) = 3$. □

7. Conclusion

There is a lot of research on the computation of many and varied matrices. The research on $\mathbb{Z}_2$ is also diverse and applied. We started out studying matrix sequences and turned them into sounds, so we consider that's kind of significant. These findings would be significant if generalized, and could have a huge impact if used to analyze the sounds of the world.

Acknowledgments

This paper is dedicated to Professor Abdelmejid Bayad on the occasion of his 61st anniversary.

Author Contributions: All authors are equally contributed.

Conflict of Interest: There are no conflict of interest.

Funding (Financial Disclosure): The authors did not receive any financial support from any organization for this study.

References

[1] G. L. Cohen and H. J. J. te Riele, *Iterating the sum-of-divisors function,* Exp. Math. **5**, 91–100, 1996.
[2] Y. Fan, C. Y. Ahn and D. Kim, *Iterations of endomorphisms for* 4*-torsion subgroups of elliptic curve,* Submitted.
[3] R. K. Guy, *Unsolved problems in number theory*, Springer, 2004.
[4] D. Jeon, C. Kim and Y. Lee, *Families of elliptic curves over quadratic number fields with prescribed torsion subgroups,* Math. Comp. **80**, 2395–2410, 2011.
[5] D. Kim, *Iteration of* 2×2 *matrices in* $\mathbb{Z}_4$ *and their four color expressions (I),* J. Appl. & Pure Math. **6**, 37–45, 2024.
[6] D. Kim and A. Bayad, *Polygon numbers associated with the sum of odd divisors function,* Exp. Math. **26**, 287–297, 2017.
[7] D. Kim, A. Bayad and S. Park, *Iterated stable numbers and iterated amicable pairs,* Math. Methods Appl. Sci. **44**, 12479–12499, 2021.
[8] S. Lang, *Linear algebra*, Springer-Verlag, New York, 1987.

[9] G. A. Miller, *Determination of all the groups of order* 96, Anal. Math. **31**, 163–168, 1930.
[10] L. C. Washington, *Elliptic curves number theory and cryptography*, Chapman and Hall CRC Press, 2008.
[11] F. Zhang, *Matrix theory*, Springer, New York, 1999.
[12] Avaliable at https://digitalcommons.butler.edu/cgi/viewcontent.cgi?article=3062&context=wordways, (Accession date: 22 Jan. 2025).
[13] Avaliable at https://bit.ly/matrixsequences, (Accession date: 22 Jan. 2025).
[14] Avaliable at https://drive.google.com/drive/folders/1OlFDFbn8iQDxrcDiH0ashZJmoQMXYQMH, (Accession date: 22 Jan. 2025).
[15] Avaliable at https://url.kr/q91eqv, (Accession date: 22 Jan. 2025).
[16] Avaliable at https://url.kr/g2it6q, (Accession date: 22 Jan. 2025).

How to cite this article: H. K. Lee and D. Kim, *A study of sound numbers derived from iterated products of matrices in* $\mathbb{Z}_2$, Montes Taurus J. Pure Appl. Math. **8 (1)**, 1–28, 2026; Article ID: MTJPAM-D-24-00146.

Appendix

A. Sequences of $\Re_{i,j}$

A.1. Right stable with $Ord(\Re_{i,j}) = 1$

A.1.1. $StP(\Re_{i,j}) = A_1$
$\Re_{1,1}$: A_1 ⟳
$\Re_{1,2}$: A_1 ⟳
$\Re_{1,3}$: A_1 ⟳
$\Re_{1,4}$: A_1 ⟳
$\Re_{1,5}$: A_1 ⟳
$\Re_{1,6}$: A_1 ⟳
$\Re_{1,7}$: A_1 ⟳
$\Re_{1,8}$: A_1 ⟳
$\Re_{1,9}$: A_1 ⟳
$\Re_{1,10}$: A_1 ⟳
$\Re_{1,11}$: A_1 ⟳
$\Re_{1,12}$: A_1 ⟳
$\Re_{1,13}$: A_1 ⟳
$\Re_{1,14}$: A_1 ⟳
$\Re_{1,15}$: A_1 ⟳
$\Re_{1,16}$: A_1 ⟳

A.1.2. $StP(\Re_{i,j}) = A_2$
$\Re_{2,1}$: A_2 ⟳
$\Re_{2,5}$: A_2 ⟳
$\Re_{2,9}$: A_2 ⟳
$\Re_{2,13}$: A_2 ⟳

A.1.3. $StP(\Re_{i,j}) = A_4$
$\Re_{4,1}$: A_4 ⟳
$\Re_{4,6}$: A_4 ⟳
$\Re_{4,11}$: A_4 ⟳
$\Re_{4,16}$: A_4 ⟳

A.1.4. $StP(\Re_{i,j}) = A_6$
$\Re_{6,1}$: A_6 ⟳
$\Re_{6,5}$: A_6 ⟳
$\Re_{6,9}$: A_6 ⟳
$\Re_{6,13}$: A_6 ⟳

A.1.5. $StP(\Re_{i,j}) = A_9$
$\Re_{9,1}$: A_9 ⟳
$\Re_{9,2}$: A_9 ⟳
$\Re_{9,3}$: A_9 ⟳
$\Re_{9,4}$: A_9 ⟳

A.1.6. $StP(\Re_{i,j}) = A_{10}$
$\Re_{10,1}$: A_{10} ⟳

A.1.7. $StP(\Re_{i,j}) = A_{11}$
$\Re_{11,1}$: A_{11} ⟳
$\Re_{11,2}$: A_{11} ⟳
$\Re_{11,3}$: A_{11} ⟳
$\Re_{11,4}$: A_{11} ⟳

A.1.8. $StP(\Re_{i,j}) = A_{13}$
$\Re_{13,1}$: A_{13} ⟳
$\Re_{13,6}$: A_{13} ⟳
$\Re_{13,11}$: A_{13} ⟳
$\Re_{13,16}$: A_{13} ⟳

A.2. Right stable with $Ord(\Re_{i,j}) = 2$

A.2.1. $StP(\Re_{i,j}) = A_1$
$\Re_{2,2}$: $A_2 \to A_1$ ⟳
$\Re_{2,6}$: $A_2 \to A_1$ ⟳
$\Re_{2,10}$: $A_2 \to A_1$ ⟳
$\Re_{2,14}$: $A_2 \to A_1$ ⟳

$\Re_{3,1} : A_3 \to A_1 \circlearrowleft$
$\Re_{3,2} : A_3 \to A_1 \circlearrowleft$
$\Re_{3,3} : A_3 \to A_1 \circlearrowleft$
$\Re_{3,4} : A_3 \to A_1 \circlearrowleft$
$\Re_{4,4} : A_4 \to A_1 \circlearrowleft$
$\Re_{4,7} : A_4 \to A_1 \circlearrowleft$
$\Re_{4,10} : A_4 \to A_1 \circlearrowleft$
$\Re_{4,13} : A_4 \to A_1 \circlearrowleft$
$\Re_{5,1} : A_5 \to A_1 \circlearrowleft$
$\Re_{5,5} : A_5 \to A_1 \circlearrowleft$
$\Re_{5,9} : A_5 \to A_1 \circlearrowleft$
$\Re_{5,13} : A_5 \to A_1 \circlearrowleft$
$\Re_{6,2} : A_6 \to A_1 \circlearrowleft$
$\Re_{6,6} : A_6 \to A_1 \circlearrowleft$
$\Re_{6,10} : A_6 \to A_1 \circlearrowleft$
$\Re_{6,14} : A_6 \to A_1 \circlearrowleft$
$\Re_{7,7} : A_7 \to A_1 \circlearrowleft$
$\Re_{8,8} : A_8 \to A_1 \circlearrowleft$
$\Re_{9,9} : A_9 \to A_1 \circlearrowleft$
$\Re_{9,10} : A_9 \to A_1 \circlearrowleft$
$\Re_{9,11} : A_9 \to A_1 \circlearrowleft$
$\Re_{9,12} : A_9 \to A_1 \circlearrowleft$
$\Re_{10,10} : A_{10} \to A_1 \circlearrowleft$
$\Re_{11,9} : A_{11} \to A_1 \circlearrowleft$
$\Re_{11,10} : A_{11} \to A_1 \circlearrowleft$
$\Re_{11,11} : A_{11} \to A_1 \circlearrowleft$
$\Re_{11,12} : A_{11} \to A_1 \circlearrowleft$
$\Re_{12,12} : A_{12} \to A_1 \circlearrowleft$
$\Re_{13,4} : A_{13} \to A_1 \circlearrowleft$
$\Re_{13,7} : A_{13} \to A_1 \circlearrowleft$
$\Re_{13,10} : A_{13} \to A_1 \circlearrowleft$
$\Re_{13,13} : A_{13} \to A_1 \circlearrowleft$
$\Re_{14,14} : A_{14} \to A_1 \circlearrowleft$
$\Re_{15,15} : A_{15} \to A_1 \circlearrowleft$
$\Re_{16,1} : A_{16} \to A_1 \circlearrowleft$
$\Re_{16,6} : A_{16} \to A_1 \circlearrowleft$
$\Re_{16,11} : A_{16} \to A_1 \circlearrowleft$
$\Re_{16,16} : A_{16} \to A_1 \circlearrowleft$

A.2.2. $StP(\Re_{i,j}) = A_2$
$\Re_{10,9} : A_{10} \to A_2 \circlearrowleft$
$\Re_{14,9} : A_{14} \to A_2 \circlearrowleft$

A.2.3. $StP(\Re_{i,j}) = A_4$
$\Re_{7,11} : A_7 \to A_4 \circlearrowleft$
$\Re_{10,11} : A_{10} \to A_4 \circlearrowleft$

A.2.4. $StP(\Re_{i,j}) = A_6$
$\Re_{10,13} : A_{10} \to A_6 \circlearrowleft$
$\Re_{14,13} : A_{14} \to A_6 \circlearrowleft$

A.2.5. $StP(\Re_{i,j}) = A_9$
$\Re_{10,2} : A_{10} \to A_9 \circlearrowleft$
$\Re_{12,2} : A_{12} \to A_9 \circlearrowleft$

A.2.6. $StP(\Re_{i,j}) = A_{11}$
$\Re_{10,4} : A_{10} \to A_{11} \circlearrowleft$
$\Re_{12,4} : A_{12} \to A_{11} \circlearrowleft$

A.2.7. $StP(\Re_{i,j}) = A_{13}$
$\Re_{7,6} : A_7 \to A_{13} \circlearrowleft$
$\Re_{10,6} : A_{10} \to A_{13} \circlearrowleft$

A.3. Right stable with $Ord(\Re_{i,j}) = 3$

A.3.1. $StP(\Re_{i,j}) = A_1$
$\Re_{2,4} : A_2 \to A_3 \to A_1 \circlearrowleft$
$\Re_{2,8} : A_2 \to A_3 \to A_1 \circlearrowleft$
$\Re_{2,12} : A_2 \to A_3 \to A_1 \circlearrowleft$
$\Re_{2,16} : A_2 \to A_3 \to A_1 \circlearrowleft$
$\Re_{3,9} : A_3 \to A_3 \to A_1 \circlearrowleft$
$\Re_{3,10} : A_3 \to A_3 \to A_1 \circlearrowleft$
$\Re_{3,11} : A_3 \to A_3 \to A_1 \circlearrowleft$
$\Re_{3,12} : A_3 \to A_3 \to A_1 \circlearrowleft$
$\Re_{4,2} : A_4 \to A_3 \to A_1 \circlearrowleft$
$\Re_{4,5} : A_4 \to A_3 \to A_1 \circlearrowleft$
$\Re_{4,12} : A_4 \to A_3 \to A_1 \circlearrowleft$
$\Re_{4,15} : A_4 \to A_3 \to A_1 \circlearrowleft$
$\Re_{5,2} : A_5 \to A_5 \to A_1 \circlearrowleft$
$\Re_{5,6} : A_5 \to A_5 \to A_1 \circlearrowleft$
$\Re_{5,10} : A_5 \to A_5 \to A_1 \circlearrowleft$
$\Re_{5,14} : A_5 \to A_5 \to A_1 \circlearrowleft$
$\Re_{6,3} : A_6 \to A_{16} \to A_1 \circlearrowleft$
$\Re_{6,7} : A_6 \to A_{16} \to A_1 \circlearrowleft$
$\Re_{6,11} : A_6 \to A_{16} \to A_1 \circlearrowleft$
$\Re_{6,15} : A_6 \to A_{16} \to A_1 \circlearrowleft$
$\Re_{9,13} : A_9 \to A_5 \to A_1 \circlearrowleft$
$\Re_{9,14} : A_9 \to A_5 \to A_1 \circlearrowleft$
$\Re_{9,15} : A_9 \to A_5 \to A_1 \circlearrowleft$
$\Re_{9,16} : A_9 \to A_5 \to A_1 \circlearrowleft$
$\Re_{11,5} : A_{11} \to A_{16} \to A_1 \circlearrowleft$
$\Re_{11,6} : A_{11} \to A_{16} \to A_1 \circlearrowleft$
$\Re_{11,7} : A_{11} \to A_{16} \to A_1 \circlearrowleft$
$\Re_{11,8} : A_{11} \to A_{16} \to A_1 \circlearrowleft$
$\Re_{13,3} : A_{13} \to A_5 \to A_1 \circlearrowleft$
$\Re_{13,8} : A_{13} \to A_5 \to A_1 \circlearrowleft$
$\Re_{13,9} : A_{13} \to A_5 \to A_1 \circlearrowleft$
$\Re_{13,14} : A_{13} \to A_5 \to A_1 \circlearrowleft$
$\Re_{16,4} : A_{16} \to A_{16} \to A_1 \circlearrowleft$
$\Re_{16,7} : A_{16} \to A_{16} \to A_1 \circlearrowleft$
$\Re_{16,10} : A_{16} \to A_{16} \to A_1 \circlearrowleft$
$\Re_{16,13} : A_{16} \to A_{16} \to A_1 \circlearrowleft$

A.4. Right stable with $Ord(\Re_{i,j}) = 4$

A.4.1 $StP(\Re_{i,j}) = A_1$
$\Re_{7,8}$: $A_7 \to A_5 \to A_9 \to A_1$ ↺
$\Re_{7,10}$: $A_7 \to A_{16} \to A_{16} \to A_1$ ↺
$\Re_{7,15}$: $A_7 \to A_3 \to A_2 \to A_1$ ↺
$\Re_{8,3}$: $A_8 \to A_5 \to A_{13} \to A_1$ ↺
$\Re_{8,5}$: $A_8 \to A_{16} \to A_{11} \to A_1$ ↺
$\Re_{8,16}$: $A_8 \to A_3 \to A_2 \to A_1$ ↺
$\Re_{10,7}$: $A_{10} \to A_{16} \to A_{16} \to A_1$ ↺
$\Re_{10,12}$: $A_{10} \to A_3 \to A_3 \to A_1$ ↺
$\Re_{10,14}$: $A_{10} \to A_5 \to A_5 \to A_1$ ↺
$\Re_{12,8}$: $A_{12} \to A_{16} \to A_6 \to A_1$ ↺
$\Re_{12,10}$: $A_{12} \to A_3 \to A_3 \to A_1$ ↺
$\Re_{12,15}$: $A_{12} \to A_5 \to A_{13} \to A_1$ ↺
$\Re_{14,8}$: $A_{14} \to A_3 \to A_4 \to A_1$ ↺
$\Re_{14,10}$: $A_{14} \to A_5 \to A_5 \to A_1$ ↺
$\Re_{14,15}$: $A_{14} \to A_{16} \to A_{11} \to A_1$ ↺
$\Re_{15,3}$: $A_{15} \to A_{16} \to A_6 \to A_1$ ↺
$\Re_{15,5}$: $A_{15} \to A_3 \to A_4 \to A_1$ ↺
$\Re_{15,16}$: $A_{15} \to A_5 \to A_9 \to A_1$ ↺

A.5. Right amicable

A.5.1. $Ord(\Re_{i,j}) = 1$
$\Re_{2,3}$: $A_2 \rightleftarrows A_4$
$\Re_{2,7}$: $A_2 \rightleftarrows A_4$
$\Re_{2,11}$: $A_2 \rightleftarrows A_4$
$\Re_{2,15}$: $A_2 \rightleftarrows A_4$
$\Re_{3,5}$: $A_3 \rightleftarrows A_2$
$\Re_{3,6}$: $A_3 \rightleftarrows A_2$
$\Re_{3,7}$: $A_3 \rightleftarrows A_2$
$\Re_{3,8}$: $A_3 \rightleftarrows A_2$
$\Re_{3,13}$: $A_3 \rightleftarrows A_4$
$\Re_{3,14}$: $A_3 \rightleftarrows A_4$
$\Re_{3,15}$: $A_3 \rightleftarrows A_4$
$\Re_{3,16}$: $A_3 \rightleftarrows A_4$
$\Re_{4,3}$: $A_4 \rightleftarrows A_2$
$\Re_{4,8}$: $A_4 \rightleftarrows A_2$
$\Re_{4,9}$: $A_4 \rightleftarrows A_2$
$\Re_{4,14}$: $A_4 \rightleftarrows A_2$
$\Re_{5,3}$: $A_5 \rightleftarrows A_9$
$\Re_{5,4}$: $A_5 \rightleftarrows A_{13}$
$\Re_{5,7}$: $A_5 \rightleftarrows A_9$
$\Re_{5,8}$: $A_5 \rightleftarrows A_{13}$
$\Re_{5,11}$: $A_5 \rightleftarrows A_9$
$\Re_{5,12}$: $A_5 \rightleftarrows A_{13}$
$\Re_{5,15}$: $A_5 \rightleftarrows A_9$
$\Re_{5,16}$: $A_5 \rightleftarrows A_{13}$
$\Re_{6,4}$: $A_6 \rightleftarrows A_{11}$
$\Re_{6,8}$: $A_6 \rightleftarrows A_{11}$
$\Re_{6,12}$: $A_6 \rightleftarrows A_{11}$
$\Re_{6,16}$: $A_6 \rightleftarrows A_{11}$
$\Re_{7,1}$: $A_7 \rightleftarrows A_{10}$
$\Re_{8,10}$: $A_8 \rightleftarrows A_{10}$
$\Re_{9,5}$: $A_9 \rightleftarrows A_{13}$
$\Re_{9,6}$: $A_9 \rightleftarrows A_{13}$
$\Re_{9,7}$: $A_9 \rightleftarrows A_{13}$
$\Re_{9,8}$: $A_9 \rightleftarrows A_{13}$
$\Re_{11,13}$: $A_{11} \rightleftarrows A_6$
$\Re_{11,14}$: $A_{11} \rightleftarrows A_6$
$\Re_{11,15}$: $A_{11} \rightleftarrows A_6$
$\Re_{11,16}$: $A_{11} \rightleftarrows A_6$
$\Re_{12,1}$: $A_{12} \rightleftarrows A_{10}$
$\Re_{13,2}$: $A_{13} \rightleftarrows A_9$
$\Re_{13,5}$: $A_{13} \rightleftarrows A_9$
$\Re_{13,12}$: $A_{13} \rightleftarrows A_9$
$\Re_{13,15}$: $A_{13} \rightleftarrows A_9$
$\Re_{14,1}$: $A_{14} \rightleftarrows A_{10}$
$\Re_{15,10}$: $A_{15} \rightleftarrows A_{10}$
$\Re_{16,2}$: $A_{16} \rightleftarrows A_6$
$\Re_{16,3}$: $A_{16} \rightleftarrows A_{11}$
$\Re_{16,5}$: $A_{16} \rightleftarrows A_6$
$\Re_{16,8}$: $A_{16} \rightleftarrows A_{11}$
$\Re_{16,9}$: $A_{16} \rightleftarrows A_{11}$
$\Re_{16,12}$: $A_{16} \rightleftarrows A_6$
$\Re_{16,14}$: $A_{16} \rightleftarrows A_{11}$
$\Re_{16,15}$: $A_{16} \rightleftarrows A_6$

A.5.2. $Ord(\Re_{i,j}) = 2$
$\Re_{7,3}$: $A_7 \to A_2 \rightleftarrows A_3$
$\Re_{7,4}$: $A_7 \to A_6 \rightleftarrows A_{11}$
$\Re_{7,5}$: $A_7 \to A_9 \rightleftarrows A_5$
$\Re_{7,13}$: $A_7 \to A_{11} \rightleftarrows A_6$
$\Re_{8,4}$: $A_8 \to A_2 \rightleftarrows A_4$
$\Re_{8,6}$: $A_8 \to A_{11} \rightleftarrows A_6$
$\Re_{8,7}$: $A_8 \to A_6 \rightleftarrows A_{16}$
$\Re_{8,9}$: $A_8 \to A_{13} \rightleftarrows A_9$
$\Re_{8,12}$: $A_8 \to A_4 \rightleftarrows A_3$
$\Re_{8,14}$: $A_8 \to A_9 \rightleftarrows A_5$
$\Re_{12,5}$: $A_{12} \to A_{13} \rightleftarrows A_5$
$\Re_{12,9}$: $A_{12} \to A_4 \rightleftarrows A_2$
$\Re_{12,11}$: $A_{12} \to A_2 \rightleftarrows A_4$
$\Re_{12,16}$: $A_{12} \to A_6 \rightleftarrows A_{16}$
$\Re_{14,2}$: $A_{14} \to A_{13} \rightleftarrows A_9$
$\Re_{14,3}$: $A_{14} \to A_4 \rightleftarrows A_3$
$\Re_{14,6}$: $A_{14} \to A_9 \rightleftarrows A_{13}$
$\Re_{14,16}$: $A_{14} \to A_{11} \rightleftarrows A_{16}$
$\Re_{15,2}$: $A_{15} \to A_4 \rightleftarrows A_2$
$\Re_{15,7}$: $A_{15} \to A_{11} \rightleftarrows A_{16}$
$\Re_{15,11}$: $A_{15} \to A_6 \rightleftarrows A_{11}$
$\Re_{15,12}$: $A_{15} \to A_2 \rightleftarrows A_3$
$\Re_{15,13}$: $A_{15} \to A_9 \rightleftarrows A_{13}$
$\Re_{15,14}$: $A_{15} \to A_{13} \rightleftarrows A_5$

A.6. Right sociable

A.6.1. $Ord(\mathfrak{R}_{i,j}) = 1$ and $Per(\mathfrak{R}_{i,j}) = 3$

$\mathfrak{R}_{8,1} : A_8 \underleftarrow{\to A_{15} \to A_{10}}$

$\mathfrak{R}_{15,1} : A_{15} \underleftarrow{\to A_8 \to A_{10}}$

A.6.2. $Ord(\mathfrak{R}_{i,j}) = 1$ and $Per(\mathfrak{R}_{i,j}) = 4$

$\mathfrak{R}_{7,2} : A_7 \underleftarrow{\to A_{14} \to A_{15} \to A_{10}}$

$\mathfrak{R}_{7,9} : A_7 \underleftarrow{\to A_{12} \to A_8 \to A_{10}}$

$\mathfrak{R}_{7,16} : A_7 \underleftarrow{\to A_7 \to A_{10} \to A_{10}}$

$\mathfrak{R}_{8,2} : A_8 \underleftarrow{\to A_{12} \to A_7 \to A_{10}}$

$\mathfrak{R}_{8,11} : A_8 \underleftarrow{\to A_7 \to A_{14} \to A_{10}}$

$\mathfrak{R}_{8,13} : A_8 \underleftarrow{\to A_{14} \to A_{12} \to A_{10}}$

$\mathfrak{R}_{10,3} : A_{10} \underleftarrow{\to A_{12} \to A_{12} \to A_{10}}$

$\mathfrak{R}_{10,5} : A_{10} \underleftarrow{\to A_{14} \to A_{14} \to A_{10}}$

$\mathfrak{R}_{10,16} : A_{10} \underleftarrow{\to A_7 \to A_7 \to A_{10}}$

$\mathfrak{R}_{12,3} : A_{12} \underleftarrow{\to A_{12} \to A_{10} \to A_{10}}$

$\mathfrak{R}_{12,6} : A_{12} \underleftarrow{\to A_{14} \to A_8 \to A_{10}}$

$\mathfrak{R}_{12,13} : A_{12} \underleftarrow{\to A_7 \to A_{15} \to A_{10}}$

$\mathfrak{R}_{14,4} : A_{14} \underleftarrow{\to A_7 \to A_8 \to A_{10}}$

$\mathfrak{R}_{14,5} : A_{14} \underleftarrow{\to A_{14} \to A_{10} \to A_{10}}$

$\mathfrak{R}_{14,11} : A_{14} \underleftarrow{\to A_{12} \to A_{15} \to A_{10}}$

$\mathfrak{R}_{15,4} : A_{15} \underleftarrow{\to A_{12} \to A_{14} \to A_{10}}$

$\mathfrak{R}_{15,6} : A_{15} \underleftarrow{\to A_7 \to A_{12} \to A_{10}}$

$\mathfrak{R}_{15,9} : A_{15} \underleftarrow{\to A_{14} \to A_7 \to A_{10}}$

A.6.3. $Ord(\mathfrak{R}_{i,j}) = 1$ and $Per(\mathfrak{R}_{i,j}) = 6$

$\mathfrak{R}_{7,12} : A_7 \underleftarrow{\to A_8 \to A_{12} \to A_{15} \to A_{14} \to A_{10}}$

$\mathfrak{A}_{7,14} : A_7 \underleftarrow{\to A_{15} \to A_{14} \to A_8 \to A_{12} \to A_{10}}$

$\mathfrak{R}_{8,15} : A_8 \underleftarrow{\to A_8 \to A_{15} \to A_{15} \to A_{10} \to A_{10}}$

$\mathfrak{R}_{10,8} : A_{10} \underleftarrow{\to A_{15} \to A_{15} \to A_8 \to A_8 \to A_{10}}$

$\mathfrak{R}_{10,15} : A_{10} \underleftarrow{\to A_8 \to A_8 \to A_{15} \to A_{15} \to A_{10}}$

$\mathfrak{R}_{12,7} : A_{12} \underleftarrow{\to A_{15} \to A_7 \to A_8 \to A_{14} \to A_{10}}$

$\mathfrak{R}_{12,14} : A_{12} \underleftarrow{\to A_8 \to A_{14} \to A_{15} \to A_7 \to A_{10}}$

$\mathfrak{R}_{14,7} : A_{14} \underleftarrow{\to A_8 \to A_7 \to A_{15} \to A_{12} \to A_{10}}$

$\mathfrak{R}_{14,12} : A_{14} \underleftarrow{\to A_{15} \to A_{12} \to A_8 \to A_7 \to A_{10}}$

$\mathfrak{R}_{15,8} : A_{15} \underleftarrow{\to A_{15} \to A_8 \to A_8 \to A_{10} \to A_{10}}$

B. Sequences of $\mathfrak{L}_{i,j}$

B.1. Left stable with $Ord(\mathfrak{L}_{i,j}) = 1$

B.1.1. $StP(\mathfrak{L}_{i,j}) = A_1$

$\mathfrak{L}_{1,1} : A_1 \circlearrowleft$
$\mathfrak{L}_{1,2} : A_1 \circlearrowleft$
$\mathfrak{L}_{1,3} : A_1 \circlearrowleft$
$\mathfrak{L}_{1,4} : A_1 \circlearrowleft$
$\mathfrak{L}_{1,5} : A_1 \circlearrowleft$
$\mathfrak{L}_{1,6} : A_1 \circlearrowleft$
$\mathfrak{L}_{1,7} : A_1 \circlearrowleft$
$\mathfrak{L}_{1,8} : A_1 \circlearrowleft$
$\mathfrak{L}_{1,9} : A_1 \circlearrowleft$
$\mathfrak{L}_{1,10} : A_1 \circlearrowleft$
$\mathfrak{L}_{1,11} : A_1 \circlearrowleft$
$\mathfrak{L}_{1,12} : A_1 \circlearrowleft$
$\mathfrak{L}_{1,13} : A_1 \circlearrowleft$
$\mathfrak{L}_{1,14} : A_1 \circlearrowleft$
$\mathfrak{L}_{1,15} : A_1 \circlearrowleft$
$\mathfrak{L}_{1,16} : A_1 \circlearrowleft$

B.1.2. $StP(\mathfrak{L}_{i,j}) = A_2$

$\mathfrak{L}_{2,1} : A_2 \circlearrowleft$
$\mathfrak{L}_{2,3} : A_2 \circlearrowleft$
$\mathfrak{L}_{2,9} : A_2 \circlearrowleft$
$\mathfrak{L}_{2,11} : A_2 \circlearrowleft$

B.1.3. $StP(\mathfrak{L}_{i,j}) = A_4$

$\mathfrak{L}_{4,1} : A_4 \circlearrowleft$
$\mathfrak{L}_{4,3} : A_4 \circlearrowleft$
$\mathfrak{L}_{4,9} : A_4 \circlearrowleft$
$\mathfrak{L}_{4,11} : A_4 \circlearrowleft$

B.1.4. $StP(\mathfrak{L}_{i,j}) = A_6$

$\mathfrak{L}_{6,1} : A_6 \circlearrowleft$
$\mathfrak{L}_{6,4} : A_6 \circlearrowleft$
$\mathfrak{L}_{6,13} : A_6 \circlearrowleft$
$\mathfrak{L}_{6,16} : A_6 \circlearrowleft$

B.1.5. $StP(\mathfrak{L}_{i,j}) = A_9$

$\mathfrak{L}_{9,1} : A_9 \circlearrowleft$
$\mathfrak{L}_{9,2} : A_9 \circlearrowleft$
$\mathfrak{L}_{9,5} : A_9 \circlearrowleft$
$\mathfrak{L}_{9,6} : A_9 \circlearrowleft$

B.1.6. $StP(\mathfrak{L}_{i,j}) = A_{10}$

$\mathfrak{L}_{10,1} : A_{10} \circlearrowleft$

B.1.7. $StP(\mathfrak{L}_{i,j}) = A_{11}$

$\mathfrak{L}_{11,1}$: A_{11} $\circlearrowleft$
$\mathfrak{L}_{11,4}$: A_{11} $\circlearrowleft$
$\mathfrak{L}_{11,13}$: A_{11} $\circlearrowleft$
$\mathfrak{L}_{11,16}$: A_{11} $\circlearrowleft$

B.1.8. $StP(\mathfrak{L}_{i,j}) = A_{!3}$
$\mathfrak{L}_{13,1}$: A_{13} $\circlearrowleft$
$\mathfrak{L}_{13,2}$: A_{13} $\circlearrowleft$
$\mathfrak{L}_{13,5}$: A_{13} $\circlearrowleft$
$\mathfrak{L}_{13,6}$: A_{13} $\circlearrowleft$

B.2. Left stable with $Ord(\mathfrak{L}_{i,j}) = 2$

B.2.1. $StP(\mathfrak{L}_{i,j}) = A_1$
$\mathfrak{L}_{2,2}$: $A_2 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{2,4}$: $A_2 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{2,10}$: $A_2 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{2,12}$: $A_2 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{3,1}$: $A_3 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{3,3}$: $A_3 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{3,9}$: $A_3 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{3,11}$: $A_3 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{4,2}$: $A_4 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{4,4}$: $A_4 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{4,10}$: $A_4 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{4,12}$: $A_4 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{5,1}$: $A_5 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{5,2}$: $A_5 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{5,5}$: $A_5 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{5,6}$: $A_5 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{6,6}$: $A_6 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{6,7}$: $A_6 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{6,10}$: $A_6 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{6,11}$: $A_6 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{7,7}$: $A_7 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{8,8}$: $A_8 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{9,9}$: $A_9 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{9,10}$: $A_9 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{9,13}$: $A_9 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{9,14}$: $A_9 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{10,10}$: $A_{10} \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{11,6}$: $A_{11} \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{11,7}$: $A_{11} \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{11,10}$: $A_{11} \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{11,11}$: $A_{11} \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{12,12}$: $A_{12} \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{13,9}$: $A_{13} \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{13,10}$: $A_{13} \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{13,13}$: $A_{13} \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{13,14}$: $A_{13} \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{14,14}$: $A_{14} \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{15,15}$: $A_{15} \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{16,1}$: $A_{16} \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{16,4}$: $A_{16} \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{16,13}$: $A_{16} \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{16,16}$: $A_{16} \to A_1$ $\circlearrowleft$

B.2.2. $StP(\mathfrak{L}_{i,j}) = A_2$
$\mathfrak{L}_{10,9}$: $A_{10} \to A_2$ $\circlearrowleft$
$\mathfrak{L}_{12,9}$: $A_{12} \to A_2$ $\circlearrowleft$

B.2.3. $StP(\mathfrak{L}_{i,j}) = A_4$
$\mathfrak{L}_{10,11}$: $A_{10} \to A_4$ $\circlearrowleft$
$\mathfrak{L}_{12,11}$: $A_{12} \to A_4$ $\circlearrowleft$

B.2.4. $StP(\mathfrak{L}_{i,j}) = A_6$
$\mathfrak{L}_{7,13}$: $A_7 \to A_6$ $\circlearrowleft$
$\mathfrak{L}_{10,13}$: $A_{10} \to A_6$ $\circlearrowleft$

B.2.5. $StP(\mathfrak{L}_{i,j}) = A_9$
$\mathfrak{L}_{10,2}$: $A_{10} \to A_9$ $\circlearrowleft$
$\mathfrak{L}_{14,2}$: $A_{14} \to A_9$ $\circlearrowleft$

B.2.6. $StP(\mathfrak{L}_{i,j}) = A_{11}$
$\mathfrak{L}_{7,4}$: $A_7 \to A_{11}$ $\circlearrowleft$
$\mathfrak{L}_{10,4}$: $A_{10} \to A_{11}$ $\circlearrowleft$

B.2.7. $StP(\mathfrak{L}_{i,j}) = A_{13}$
$\mathfrak{L}_{10,6}$: $A_{10} \to A_{13}$ $\circlearrowleft$
$\mathfrak{L}_{14,6}$: $A_{14} \to A_{13}$ $\circlearrowleft$

B.3. Left stable with $Ord(\mathfrak{L}_{i,j}) = 3$

B.3.1. $StP(\mathfrak{L}_{i,j}) = A_1$
$\mathfrak{L}_{2,6}$: $A_2 \to A_5 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{2,8}$: $A_2 \to A_5 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{2,14}$: $A_2 \to A_5 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{2,16}$: $A_2 \to A_5 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{3,2}$: $A_3 \to A_3 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{3,4}$: $A_3 \to A_3 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{3,10}$: $A_3 \to A_3 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{3,12}$: $A_3 \to A_3 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{4,5}$: $A_4 \to A_{16} \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{4,7}$: $A_4 \to A_{16} \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{4,13}$: $A_4 \to A_{16} \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{4,15}$: $A_4 \to A_{16} \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{5,9}$: $A_5 \to A_5 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{5,10}$: $A_5 \to A_5 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{5,13}$: $A_5 \to A_5 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{5,14}$: $A_5 \to A_5 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{6,2}$: $A_6 \to A_5 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{6,3}$: $A_6 \to A_5 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{6,14}$: $A_6 \to A_5 \to A_1$ $\circlearrowleft$
$\mathfrak{L}_{6,15}$: $A_6 \to A_5 \to A_1$ $\circlearrowleft$

$\mathfrak{L}_{9,11} : A_9 \to A_3 \to A_1 \circlearrowleft$
$\mathfrak{L}_{9,12} : A_9 \to A_3 \to A_1 \circlearrowleft$
$\mathfrak{L}_{9,15} : A_9 \to A_3 \to A_1 \circlearrowleft$
$\mathfrak{L}_{9,16} : A_9 \to A_3 \to A_1 \circlearrowleft$
$\mathfrak{L}_{11,5} : A_{11} \to A_3 \to A_1 \circlearrowleft$
$\mathfrak{L}_{11,8} : A_{11} \to A_3 \to A_1 \circlearrowleft$
$\mathfrak{L}_{11,9} : A_{11} \to A_3 \to A_1 \circlearrowleft$
$\mathfrak{L}_{11,12} : A_{11} \to A_3 \to A_1 \circlearrowleft$
$\mathfrak{L}_{13,3} : A_{13} \to A_{16} \to A_1 \circlearrowleft$
$\mathfrak{L}_{13,4} : A_{13} \to A_{16} \to A_1 \circlearrowleft$
$\mathfrak{L}_{13,7} : A_{13} \to A_{16} \to A_1 \circlearrowleft$
$\mathfrak{L}_{13,8} : A_{13} \to A_{16} \to A_1 \circlearrowleft$
$\mathfrak{L}_{16,6} : A_{16} \to A_{16} \to A_1 \circlearrowleft$
$\mathfrak{L}_{16,7} : A_{16} \to A_{16} \to A_1 \circlearrowleft$
$\mathfrak{L}_{16,10} : A_{16} \to A_{16} \to A_1 \circlearrowleft$
$\mathfrak{L}_{16,11} : A_{16} \to A_{16} \to A_1 \circlearrowleft$

B.4. Left stable with $Ord(\mathfrak{L}_{i,j}) = 4$

B.4.1 $StP(\mathfrak{L}_{i,j}) = A_1$
$\mathfrak{L}_{7,8} : A_7 \to A_3 \to A_9 \to A_1 \circlearrowleft$
$\mathfrak{L}_{7,10} : A_7 \to A_{16} \to A_{16} \to A_1 \circlearrowleft$
$\mathfrak{L}_{7,15} : A_7 \to A_5 \to A_2 \to A_1 \circlearrowleft$
$\mathfrak{L}_{8,3} : A_8 \to A_{16} \to A_{13} \to A_1 \circlearrowleft$
$\mathfrak{L}_{8,5} : A_8 \to A_3 \to A_{11} \to A_1 \circlearrowleft$
$\mathfrak{L}_{8,16} : A_8 \to A_5 \to A_2 \to A_1 \circlearrowleft$
$\mathfrak{L}_{10,7} : A_{10} \to A_{16} \to A_{16} \to A_1 \circlearrowleft$
$\mathfrak{L}_{10,12} : A_{10} \to A_3 \to A_3 \to A_1 \circlearrowleft$
$\mathfrak{L}_{10,14} : A_{10} \to A_5 \to A_5 \to A_1 \circlearrowleft$
$\mathfrak{L}_{12,8} : A_{12} \to A_5 \to A_6 \to A_1 \circlearrowleft$
$\mathfrak{L}_{12,10} : A_{12} \to A_3 \to A_3 \to A_1 \circlearrowleft$
$\mathfrak{L}_{12,15} : A_{12} \to A_{16} \to A_{13} \to A_1 \circlearrowleft$
$\mathfrak{L}_{14,8} : A_{14} \to A_{16} \to A_4 \to A_1 \circlearrowleft$
$\mathfrak{L}_{14,10} : A_{14} \to A_5 \to A_5 \to A_1 \circlearrowleft$
$\mathfrak{L}_{14,15} : A_{14} \to A_3 \to A_{11} \to A_1 \circlearrowleft$
$\mathfrak{L}_{15,3} : A_{15} \to A_5 \to A_6 \to A_1 \circlearrowleft$
$\mathfrak{L}_{15,5} : A_{15} \to A_{16} \to A_4 \to A_1 \circlearrowleft$
$\mathfrak{L}_{15,16} : A_{15} \to A_3 \to A_9 \to A_1 \circlearrowleft$

B.5. Left amicable

B.5.1. $Ord(\mathfrak{L}_{i,j}) = 1$
$\mathfrak{L}_{2,5} : A_2 \rightleftarrows A_6$
$\mathfrak{L}_{2,7} : A_2 \rightleftarrows A_6$
$\mathfrak{L}_{2,13} : A_2 \rightleftarrows A_6$
$\mathfrak{L}_{2,15} : A_2 \rightleftarrows A_6$
$\mathfrak{L}_{3,5} : A_3 \rightleftarrows A_9$
$\mathfrak{L}_{3,6} : A_3 \rightleftarrows A_{11}$
$\mathfrak{L}_{3,7} : A_3 \rightleftarrows A_9$
$\mathfrak{L}_{3,8} : A_3 \rightleftarrows A_{11}$
$\mathfrak{L}_{3,13} : A_3 \rightleftarrows A_9$
$\mathfrak{L}_{3,14} : A_3 \rightleftarrows A_{11}$
$\mathfrak{L}_{3,15} : A_3 \rightleftarrows A_9$
$\mathfrak{L}_{3,16} : A_3 \rightleftarrows A_{11}$
$\mathfrak{L}_{4,6} : A_4 \rightleftarrows A_{13}$
$\mathfrak{L}_{4,8} : A_4 \rightleftarrows A_{13}$
$\mathfrak{L}_{4,14} : A_4 \rightleftarrows A_{13}$
$\mathfrak{L}_{4,16} : A_4 \rightleftarrows A_{13}$
$\mathfrak{L}_{5,3} : A_5 \rightleftarrows A_2$
$\mathfrak{L}_{5,4} : A_5 \rightleftarrows A_2$
$\mathfrak{L}_{5,7} : A_5 \rightleftarrows A_2$
$\mathfrak{L}_{5,8} : A_5 \rightleftarrows A_2$
$\mathfrak{L}_{5,11} : A_5 \rightleftarrows A_6$
$\mathfrak{L}_{5,12} : A_5 \rightleftarrows A_6$
$\mathfrak{L}_{5,15} : A_5 \rightleftarrows A_6$
$\mathfrak{L}_{5,16} : A_5 \rightleftarrows A_6$
$\mathfrak{L}_{6,5} : A_6 \rightleftarrows A_2$
$\mathfrak{L}_{6,8} : A_6 \rightleftarrows A_2$
$\mathfrak{L}_{6,9} : A_6 \rightleftarrows A_2$
$\mathfrak{L}_{6,12} : A_6 \rightleftarrows A_2$
$\mathfrak{L}_{7,1} : A_7 \rightleftarrows A_{10}$
$\mathfrak{L}_{8,10} : A_8 \rightleftarrows A_{10}$
$\mathfrak{L}_{9,3} : A_9 \rightleftarrows A_{11}$
$\mathfrak{L}_{9,4} : A_9 \rightleftarrows A_{11}$
$\mathfrak{L}_{9,7} : A_9 \rightleftarrows A_{11}$
$\mathfrak{L}_{9,8} : A_9 \rightleftarrows A_{11}$
$\mathfrak{L}_{11,2} : A_{11} \rightleftarrows A_9$
$\mathfrak{L}_{11,3} : A_{11} \rightleftarrows A_9$
$\mathfrak{L}_{11,14} : A_{11} \rightleftarrows A_9$
$\mathfrak{L}_{11,15} : A_{11} \rightleftarrows A_9$
$\mathfrak{L}_{12,1} : A_{12} \rightleftarrows A_{10}$
$\mathfrak{L}_{13,11} : A_{13} \rightleftarrows A_4$
$\mathfrak{L}_{13,12} : A_{13} \rightleftarrows A_4$
$\mathfrak{L}_{13,15} : A_{13} \rightleftarrows A_4$
$\mathfrak{L}_{13,16} : A_{13} \rightleftarrows A_4$
$\mathfrak{L}_{14,1} : A_{14} \rightleftarrows A_{10}$
$\mathfrak{L}_{15,10} : A_{15} \rightleftarrows A_{10}$
$\mathfrak{L}_{16,2} : A_{16} \rightleftarrows A_4$
$\mathfrak{L}_{16,3} : A_{16} \rightleftarrows A_4$
$\mathfrak{L}_{16,5} : A_{16} \rightleftarrows A_{13}$
$\mathfrak{L}_{16,8} : A_{16} \rightleftarrows A_{13}$
$\mathfrak{L}_{16,9} : A_{16} \rightleftarrows A_{13}$
$\mathfrak{L}_{16,12} : A_{16} \rightleftarrows A_{13}$
$\mathfrak{L}_{16,14} : A_{16} \rightleftarrows A_4$
$\mathfrak{L}_{16,15} : A_{16} \rightleftarrows A_4$

B.5.2. $Ord(\mathfrak{L}_{i,j}) = 2$
$\mathfrak{L}_{7,3} : A_7 \to A_9 \rightleftarrows A_3$
$\mathfrak{L}_{7,5} : A_7 \to A_2 \rightleftarrows A_5$
$\mathfrak{L}_{7,6} : A_7 \to A_4 \rightleftarrows A_{13}$
$\mathfrak{L}_{7,11} : A_7 \to A_{13} \rightleftarrows A_4$
$\mathfrak{L}_{8,4} : A_8 \to A_{13} \rightleftarrows A_4$
$\mathfrak{L}_{8,6} : A_8 \to A_2 \rightleftarrows A_6$
$\mathfrak{L}_{8,7} : A_8 \to A_4 \rightleftarrows A_{16}$

$\mathfrak{L}_{8,9}$: $A_8 \to A_{11} \rightleftarrows A_9$
$\mathfrak{L}_{8,12}$: $A_8 \to A_9 \rightleftarrows A_3$
$\mathfrak{L}_{8,14}$: $A_8 \to A_6 \rightleftarrows A_5$
$\mathfrak{L}_{12,2}$: $A_{12} \to A_{11} \rightleftarrows A_9$
$\mathfrak{L}_{12,4}$: $A_{12} \to A_9 \rightleftarrows A_{11}$
$\mathfrak{L}_{12,5}$: $A_{12} \to A_6 \rightleftarrows A_5$
$\mathfrak{L}_{12,16}$: $A_{12} \to A_{13} \rightleftarrows A_{16}$
$\mathfrak{L}_{14,3}$: $A_{14} \to A_{11} \rightleftarrows A_3$
$\mathfrak{L}_{14,9}$: $A_{14} \to A_6 \rightleftarrows A_2$
$\mathfrak{L}_{14,13}$: $A_{14} \to A_2 \rightleftarrows A_6$
$\mathfrak{L}_{14,16}$: $A_{14} \to A_4 \rightleftarrows A_{16}$
$\mathfrak{L}_{15,2}$: $A_{15} \to A_6 \rightleftarrows A_2$
$\mathfrak{L}_{15,7}$: $A_{15} \to A_{13} \rightleftarrows A_{16}$
$\mathfrak{L}_{15,11}$: $A_{15} \to A_9 \rightleftarrows A_{11}$
$\mathfrak{L}_{15,12}$: $A_{15} \to A_{11} \rightleftarrows A_3$
$\mathfrak{L}_{15,13}$: $A_{15} \to A_4 \rightleftarrows A_{13}$
$\mathfrak{L}_{15,14}$: $A_{15} \to A_2 \rightleftarrows A_5$

B.6. Left sociable

B.6.1. $Ord(\mathfrak{L}_{i,j}) = 1$ and $Per(\mathfrak{L}_{i,j}) = 3$
$\mathfrak{L}_{8,1}$: $A_8 \underleftarrow{\to A_{15} \to A_{10}}$
$\mathfrak{L}_{15,1}$: $A_{15} \underleftarrow{\to A_8 \to A_{10}}$

B.6.2. $Ord(\mathfrak{L}_{i,j}) = 1$ and $Per(\mathfrak{L}_{i,j}) = 4$
$\mathfrak{L}_{7,2}$: $A_7 \underleftarrow{\to A_{12} \to A_{15} \to A_{10}}$
$\mathfrak{L}_{7,9}$: $A_7 \underleftarrow{\to A_{14} \to A_8 \to A_{10}}$
$\mathfrak{L}_{7,16}$: $A_7 \underleftarrow{\to A_7 \to A_{10} \to A_{10}}$
$\mathfrak{L}_{8,2}$: $A_8 \underleftarrow{\to A_{14} \to A_7 \to A_{10}}$
$\mathfrak{L}_{8,11}$: $A_8 \underleftarrow{\to A_{12} \to A_{14} \to A_{10}}$
$\mathfrak{L}_{8,13}$: $A_8 \underleftarrow{\to A_7 \to A_{12} \to A_{10}}$
$\mathfrak{L}_{10,3}$: $A_{10} \underleftarrow{\to A_{12} \to A_{12} \to A_{10}}$
$\mathfrak{L}_{10,5}$: $A_{10} \underleftarrow{\to A_{14} \to A_{14} \to A_{10}}$
$\mathfrak{L}_{10,16}$: $A_{10} \underleftarrow{\to A_7 \to A_7 \to A_{10}}$
$\mathfrak{L}_{12,3}$: $A_{12} \underleftarrow{\to A_{12} \to A_{10} \to A_{10}}$
$\mathfrak{L}_{12,6}$: $A_{12} \underleftarrow{\to A_7 \to A_8 \to A_{10}}$
$\mathfrak{L}_{12,13}$: $A_{12} \underleftarrow{\to A_{14} \to A_{15} \to A_{10}}$
$\mathfrak{L}_{14,4}$: $A_{14} \underleftarrow{\to A_{12} \to A_8 \to A_{10}}$
$\mathfrak{L}_{14,5}$: $A_{14} \underleftarrow{\to A_{14} \to A_{10} \to A_{10}}$
$\mathfrak{L}_{14,11}$: $A_{14} \underleftarrow{\to A_7 \to A_{15} \to A_{10}}$
$\mathfrak{L}_{15,4}$: $A_{15} \underleftarrow{\to A_7 \to A_{14} \to A_{10}}$
$\mathfrak{L}_{15,6}$: $A_{15} \underleftarrow{\to A_{14} \to A_{12} \to A_{10}}$
$\mathfrak{L}_{15,9}$: $A_{15} \underleftarrow{\to A_{12} \to A_7 \to A_{10}}$

B.6.3. $Ord(\mathfrak{L}_{i,j}) = 1$ and $Per(\mathfrak{L}_{i,j}) = 6$
$\mathfrak{L}_{7,12}$: $A_7 \underleftarrow{\to A_{15} \to A_{12} \to A_8 \to A_{14} \to A_{10}}$
$\mathfrak{L}_{7,14}$: $A_7 \underleftarrow{\to A_8 \to A_{14} \to A_{15} \to A_{12} \to A_{10}}$
$\mathfrak{L}_{8,15}$: $A_8 \underleftarrow{\to A_8 \to A_{15} \to A_{15} \to A_{10} \to A_{10}}$
$\mathfrak{L}_{10,8}$: $A_{10} \underleftarrow{\to A_{15} \to A_{15} \to A_8 \to A_8 \to A_{10}}$
$\mathfrak{L}_{10,15}$: $A_{10} \underleftarrow{\to A_8 \to A_8 \to A_{15} \to A_{15} \to A_{10}}$
$\mathfrak{L}_{12,7}$: $A_{12} \underleftarrow{\to A_8 \to A_7 \to A_{15} \to A_{14} \to A_{10}}$
$\mathfrak{L}_{12,14}$: $A_{12} \underleftarrow{\to A_{15} \to A_{14} \to A_8 \to A_7 \to A_{10}}$
$\mathfrak{L}_{14,7}$: $A_{14} \underleftarrow{\to A_{15} \to A_7 \to A_8 \to A_{12} \to A_{10}}$
$\mathfrak{L}_{14,12}$: $A_{14} \underleftarrow{\to A_8 \to A_{12} \to A_{15} \to A_7 \to A_{10}}$
$\mathfrak{L}_{15,8}$: $A_{15} \underleftarrow{\to A_{15} \to A_8 \to A_8 \to A_{10} \to A_{10}}$

C. Sequences of $\mathfrak{K}_i$

$\mathfrak{K}_1$: $K \to K_{1,1} \circlearrowleft$
$\mathfrak{K}_2$: $K \to K_{2,1} \circlearrowleft$
$\mathfrak{K}_3$: $K \to K_{2,1} \to K_{1,1} \circlearrowleft$
$\mathfrak{K}_4$: $K \to K_{2,1} \circlearrowleft$
$\mathfrak{K}_5$: $K \to K_{2,2} \to K_{1,1} \circlearrowleft$
$\mathfrak{K}_6$: $K \to K_{2,3} \circlearrowleft$
$\mathfrak{K}_7$: $K \rightleftarrows K$
$\mathfrak{K}_8$: $K \underleftarrow{\to K \to K}$
$\mathfrak{K}_9$: $K \to K_{2,2} \circlearrowleft$
$\mathfrak{K}_{10}$: $K \circlearrowleft$
$\mathfrak{K}_{11}$: $K \to K_{2,3} \circlearrowleft$
$\mathfrak{K}_{12}$: $K \rightleftarrows K$
$\mathfrak{K}_{13}$: $K \to K_{2,2} \circlearrowleft$
$\mathfrak{K}_{14}$: $K \rightleftarrows K$
$\mathfrak{K}_{15}$: $K \underleftarrow{\to K \to K}$
$\mathfrak{K}_{16}$: $K \to K_{2,3} \to K_{1,1} \circlearrowleft$