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On the strong dual of $C_p(X)$ and distinguished $C_p(X)$ spaces

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Abstract

In this paper, we examine some topological properties of the strong dual $L_\beta(X)$ of the space $C_p(X)$ of real-valued continuous functions equipped with the pointwise topology, and provide a new characterization for $C_p(X)$ to be distinguished.

Keywords: Lindelöf space, μ-space, k-space, realcompact space, uniform space, positive cone, distinguished space

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1. Introduction

Unless otherwise stated, X will stand for an *infinite* Tychonoff space. We shall assume that all linear spaces are over the field $\mathbb{R}$ of real numbers and all topological spaces are Hausdorff. We denote by $C_p(X)$ the linear space $C(X)$ of real-valued continuous functions on X endowed with the pointwise topology τ_p. The topological dual of $C_p(X)$ is denoted by $L(X)$, or by $L_p(X)$ when equipped with the weak* topology $\sigma(L(X), C(X))$. We shall represent by $\delta(X)$ the canonical copy of X in $L(X)$ defined by the map $x \mapsto \delta_x$, which is a topological homeomorphism that embeds X into $L_p(X)$. When $L(X)$ is provided with the strong topology $\beta(L(X), C(X))$ we write $L_\beta(X)$ rather than $(L(X), \beta(L(X), C(X)))$, and we refer to $L_\beta(X)$ as the *strong dual* of $C_p(X)$. The topological dual $M(X)$ of $L_\beta(X)$ is called the *bidual* of $C_p(X)$. Algebraically one has $C(X) \subseteq M(X) \subseteq \mathbb{R}^X$. We shall assume $M(X)$ provided with the relative product topology of $\mathbb{R}^X$.

It is known that the locally convex space $C_p(X)$ is *distinguished* if and only if the strong topology $\beta(L(X), C(X))$ on $L(X)$ coincides with the strongest locally convex topology of $L(X)$ (*cf.* [6, Corollary 3.4]). Following [15] we denote by Δ the class of all those Tychonoff spaces X for which the space $C_p(X)$ is distinguished.

Since the pointwise topology τ_p on the linear space $C(X)$ is the same as the weak topology $\sigma(C(X), L(X))$, the τ_p-equicontinuous sets in $L(X)$ are the bounded finite-dimensional subsets of $L_p(X)$. Hence τ_p^{lf}, the strongest locally convex topology on $L(X)$ that coincides with $\sigma(L(X), C(X))$ on the τ_p-equicontinuous sets in $L(X)$ (*cf.* [21, Section 21.8]) is the strongest locally convex topology, i.e., $\tau_p^{lf} = \beta\left(L(X), \mathbb{R}^X\right)$. On the other hand, according to [21, Section 21.7, Theorem 3], the *polar topology* τ_p^0 to τ_p, i.e., the strongest locally convex topology on $L(X)$ of uniform convergence on a saturated family of bounded sets in $C_p(X)$ that coincides with $\sigma(L(X), C(X))$ on the τ_p-equicontinuous subsets of $L(X)$ (*cf.* [21, Section 21.7, Theorem 2]), is given by $\tau_p^0 = \beta(L(X), C(X))$. Indeed, if

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(E,τ) is a locally convex space τ^0 always coincides with the topology on E' of uniform convergence ρ_c on the τ-precompact subsets of E. So, since a set in $C(X)$ is τ_p-precompact if and only if it is pointwise bounded, one has that $\tau_p^0 = \rho_c = \beta(L(X), C(X))$ (*cf.* [21, Section 21.7, Theorem 3]).

Recall that if $d := |X|$ and $\varphi_d := \varphi_d(\mathbb{R})$ denotes the locally convex direct sum of d real lines, the locally convex sum topology on φ_d coincides with the strong locally convex topology $\beta\left(\varphi_d, \mathbb{R}^X\right)$. If $d = \aleph_0$ we write as usual φ instead of φ_d. So, if we denote by τ^f the Kōmura's finest topology on φ_d that coincides with $\sigma\left(\varphi_d, \mathbb{R}^X\right)$ on the τ-equicontinuous subsets of φ_d, where τ designates here the product topology $\sigma\left(\mathbb{R}^X, \varphi_d\right)$ of $\mathbb{R}^X$, it turns out that τ^f equals the finest set topology on φ_d that coincides with $\sigma\left(\varphi_d, \mathbb{R}^X\right)$ on the $\beta\left(\varphi_d, \mathbb{R}^X\right)$-compact sets in φ_d, all them finite-dimensional. Thus, τ^f is exactly the *k-extension* (*cf.* [20, Chapter 7]) of the strongest locally convex topology $\beta\left(\varphi_d, \mathbb{R}^X\right)$ on φ_d, that is, of the original locally convex topology of φ_d. Consequently $\left(\varphi_d, \tau^f\right)$ is always a k-space. Let us point out that in general the k-extension τ_k of the topology τ of a Tychonoff space (X,τ) need not be completely regular (*cf.* [24]). But, as shown in [23], Kōmura's τ^f topology is always regular.

Research on distinguished $C_p(X)$ spaces has recently attracted some attention from analysts and topologists (see for instance [6]-[10], [15]-[17], [19, 22]). As mentioned in the abstract, in this paper we study some topological properties of the strong dual $L_\beta(X)$ of the space $C_p(X)$ and provide a new characterization for the space $C_p(X)$ to be distinguished or, equivalently, for X to be a Δ-space. Our notation and terminology are standard (*cf.* [1, 2, 13, 14, 21, 26, 27]).

1.1. Preliminary facts

A linear functional μ on $\mathbb{R}^X$, i.e., an element $\mu \in L(X)$, is called *positive* if $\langle \mu, f\rangle \geq 0$ for every $f \in \mathbb{R}^X$ with $f \geq 0$. If μ is a positive linear functional on $\mathbb{R}^X$ we shall write $\mu \geq \mathbf{0}$. Observe that $\mu \geq \mathbf{0}$ if and only if $\langle \mu, f\rangle \geq 0$ for every $f \in C(X)$ with $f \geq 0$. Indeed, if $\mu \geq \mathbf{0}$ and $f \in C(X)$, clearly $\langle \mu, f\rangle \geq 0$. Conversely if $\mu = \sum_{i=1}^n a_i\,\delta_{x_i}$ verifies that $\langle \mu, f\rangle \geq 0$ for every $f \in C(X)$ with $f \geq 0$, taking $g_i \in C(X)$ such that $0 \leq g_i \leq 1$ and $g_i\left(x_j\right) = \delta_{ij}$ for $1 \leq i, j \leq n$, as $g_j \geq 0$ one has

$$a_j = \sum_{i=1}^n a_i\, g_j(x_i) = \left\langle \mu, g_j \right\rangle \geq 0 \tag{1.1}$$

for $1 \leq j \leq n$, which implies that $\langle \mu, f\rangle \geq 0$ for every $f \in \mathbb{R}^X$ with $f \geq 0$. Actually, it turns out that $\mu = \sum_{i=1}^n a_i\,\delta_{x_i} \geq \mathbf{0}$ if and only if $a_i \geq 0$ for $1 \leq i \leq n$. In fact, this condition is clearly sufficient and conversely if $\mu = \sum_{i=1}^n a_i\,\delta_{x_i} \geq \mathbf{0}$, taking $g_i \in C(X)$ as before, we see that $a_j = \left\langle \mu, g_j\right\rangle \geq 0$ for $1 \leq j \leq n$.

The convex set $K(X) := \{\mu \in L(X) : \mu \geq \mathbf{0}\}$ is called the *positive cone* of $L(X)$. When we consider $K(X)$ as a topological subspace of $L_p(X)$ or $L_\beta(X)$, we shall refer to $K(X)$ as the positive cone of $L_p(X)$ or $L_\beta(X)$ and we shall frequently denote these objects as $K_p(X)$ and $K_\beta(X)$, respectively.

Lemma 1.1. *The following properties hold.*

- **(i)** *Algebraically* $\operatorname{co}(\delta(X)) \subseteq K(X)$*, where* $\operatorname{co}(\delta(X))$ *is the convex cover of* $\delta(X)$*.*
- **(ii)** *Both* $K(X)$ *and* $\operatorname{co}(\delta(X))$ *are closed subspaces of* $L_p(X)$
- **(iii)** $\delta(X)$ *is a discrete subspace of* $K(X)$ *under the relative weak* $\sigma(L(X), M(X))$ *topology.*

Proof. If $\mu \in \operatorname{co}(\delta(X))$ then $\mu = \sum_{i=1}^n a_i\,\delta_{x_i}$ with $x_i \in X$ and $a_i \geq 0$ for $1 \leq i \leq n$, and $\sum_{i=1}^n a_i = 1$, which implies that $\mu \geq \mathbf{0}$, hence $\mu \in K(X)$. For the first part of the second statement note that if $\{\mu_d\}_{d\in D}$ is a net in $K(X)$ such that $\mu_d \to \mu$ under the topology $\sigma(L(X), C(X))$ then $\langle \mu_d, f\rangle \to \langle \mu, f\rangle$ for every $f \in C(X)$, in particular for each $f \geq 0$. Since $\langle \mu_d, f\rangle \geq 0$ whenever $f \geq 0$, it follows that $\langle \mu, f\rangle \geq 0$ for every $f \in C(X)$ with $f \geq 0$. Therefore $\mu \in K(X)$. For the second part of the second statement it suffices to show that $\operatorname{co}(\delta(X))$ is closed in $K(X)$ as a subspace of $L_p(X)$, since we know that $K(X)$ is closed in $L_p(X)$. So, let $\mu = \sum_{i=1}^n a_i\,\delta_{x_i} \in K(X)$, with $a_i \geq 0$ and $x_i \in X$ for $1 \leq i \leq n$, be a point of the closure of $\operatorname{co}(\delta(X))$ in $K_p(X)$, and take $f \in C(X)$ such that $f(x) = 1$ for every $x \in X$. Given $\epsilon > 0$ choose $\lambda = \sum_{j=1}^m b_j\,\delta_{y_j} \in \operatorname{co}(\delta(X))$ with $|\langle f, \mu\rangle - \langle f, \lambda\rangle| \leq \epsilon$. Since $\langle f, \mu\rangle = \sum_{i=1}^n a_i$ and $\langle f, \lambda\rangle = \sum_{j=1}^m b_j = 1$, we have

$$-\epsilon + 1 = -\epsilon + \langle f, \lambda\rangle \leq \sum_{i=1}^n a_i \leq \epsilon + \langle f, \lambda\rangle = \epsilon + 1.$$

As this holds for all $\epsilon > 0$, we obtain the equality $\sum_{i=1}^{n} a_i = 1$. Thus $\mu \in \operatorname{co}(\delta(X))$. The third statement follows from [4, Lemma 18]. □

If $\mu = \sum_{i=1}^{n} a_i \delta_{x_i} \in L(X)$, we have seen that $\mu \geq \mathbf{0}$ if and only if $a_i \geq 0$ for $1 \leq i \leq n$. This ensures that any $\mu \in L(X)$ is of the form $\mu = \lambda - \nu$ with $\lambda, \nu \geq \mathbf{0}$, so that $L(X) = K(X) - K(X)$. Note that $K(X) \cap (-K(X)) = \{\mathbf{0}\}$, for if $\mu \in K(X) \cap (-K(X))$ with $\mu = \sum_{i=1}^{n} a_i \delta_{x_i}$ the fact that $\{\delta_x : x \in X\}$ is a Hamel basis of $L(X)$ guarantees that there is a unique way in which $\mu = \lambda - \nu$ with $\lambda, \nu \in K(X)$, namely $\lambda = \sum_{i \in P} a_i \delta_{x_i}$ and $\nu = \sum_{i \in N} a_i \delta_{x_i}$ with $P = \{i \in \{1, \ldots, n\} : a_i \geq 0\}$ and $N = \{i \in \{1, \ldots, n\} : a_i < 0\}$. So, on the one hand $\mu = \lambda$ due to $\mu \in K(X)$, and on the other hand $\mu = \nu$ since $\mu \in -K(X)$. This implies that $a_i = 0$ for $1 \leq i \leq n$, which means that $\mu = \mathbf{0}$. Since $L(X) = K(X) - K(X)$, the linear span $\operatorname{sp}(K(X))$ of $K(X)$ coincides with $L(X)$. Of course, this is also consequence of the first statement of Lemma 1.1.

Lemma 1.2. *The mapping $\varphi : K_p(X) \times K_p(X) \to L_p(X)$ defined by $\varphi(\mu, \nu) \to \mu - \nu$ is a surjective condensation.*

Proof. As $\delta(X) = \{\delta_x : x \in X\}$ is a Hamel basis for $L(X)$ clearly φ is one-to-one. Besides φ is onto since $L(X) = K(X) - K(X)$. Moreover, if $\{(\mu_d, \nu_d)\}_{d \in D}$ is a net in $K(X) \times K(X)$ that converges to (μ, ν) in $K_p(X) \times K_p(X)$, then $\mu_d \to \mu$ and $\nu_d \to \nu$ in $K_p(X)$. So $\langle f, \mu_d \rangle \to \langle f, \mu \rangle$ and $\langle f, \nu_d \rangle \to \langle f, \nu \rangle$ for every $f \in C(X)$. Consequently

$$\langle f, \varphi(\mu_d, \nu_d) \rangle = \langle f, \mu_d \rangle - \langle f, \nu_d \rangle \to \langle f, \mu \rangle - \langle f, \nu \rangle = \langle f, \varphi(\mu, \nu) \rangle,$$

which shows that φ is a continuous map. □

Proposition 1.3. *The space $L_p(X)$ is K-analytic, Lindelöf Σ or realcompact if and only if $K_p(X)$ is respectively K-analytic, Lindelöf Σ or realcompact.*

Proof. The first two statements are consequence of Lemmas 1.1 and 1.2. On the other hand, if $L_p(X)$ is realcompact then $K_p(X)$, as a closed subspace of $L_p(X)$, is also realcompact. Conversely, if $K_p(X)$ is realcompact, then $\delta(X)$, as a closed subspace of $K_p(X)$, is realcompact. So, $C_p(X)$ is bornological by the Buchwalter-Schmets theorem. If we choose a linear functional u on $C(X)$ which is continuous on the closed and separable linear subspaces of $C_p(X)$ then u is sequentially continuous on $C_p(X)$, hence continuous on $C_p(X)$ by [2, Proposition 3.6.6]. Thus $L_p(X)$ is realcompact by Corson's weak* realcompactness criterion (*cf.* [29, Chapter 1, Section 8, Theorems 6 and 7]). □

If Q is a bounded set in $\mathbb{R}^X$, define

$$\varphi_Q(x) = \sup\{|g(x)| : g \in Q\}$$

and denote by $M(X)$ the bidual of $C_p(X)$. Since a function $f \in \mathbb{R}^X$ belongs to $M(X)$ if and only if there exists a bounded set A in $C_p(X)$ such that $|f| \leq \varphi_A$ (*cf.* [5, Theorem 4]), it follows that if Q is a bounded set in $C(X)$ then $\varphi_Q \in M(X)$. In other words, there is a bounded set P in $C_p(X)$ such that $\varphi_Q \in \overline{P} \subseteq P^{00}$, the bipolar and the closure being taken in $M(X)$. Here it is worth to remark that P need not be the same as Q.

Theorem 1.4. *The strong topology $\beta(L(X), C(X))$ of $L(X)$ coincides on the positive cone $K(X)$ with the weak topology $\sigma(L(X), M(X))$ of $L(X)$.*

Proof. Since $M(X)$ is the topological dual of $L_\beta(X)$, the strong topology $\beta(L(X), C(X))$ of $L(X)$ is a locally convex topology of the dual pair $\langle L(X), M(X) \rangle$. Therefore, one has that $\beta(L(X), C(X)) \geq \sigma(L(X), M(X))$ on the whole of $L(X)$, and in particular on the positive cone $K(X)$ of $L(X)$.

On the other hand, let Q be a bounded set in $C_p(X)$ and let $\mu = \sum_{i=1}^{n} a_i \delta_{x_i}$ be a positive linear functional, i. e., such that $\langle \mu, f \rangle \geq 0$ for each $f \in \mathbb{R}^X$ with $f \geq 0$. Pick $\mu \in \{\varphi_Q\}^0$, that is, such that $\left|\langle \varphi_Q, \mu \rangle\right| \leq 1$. Since $\varphi_Q \in \mathbb{R}^X$ and $\varphi_Q \geq 0$ we have

$$\sum_{i=1}^{n} a_i \varphi_Q(x_i) = \langle \varphi_Q, \mu \rangle = \left|\langle \varphi_Q, \mu \rangle\right| \leq 1. \tag{1.2}$$

On the other hand, as $\mu \geq \mathbf{0}$, if $g \in Q$ it follows from (1.1) and (1.2) that

$$|\langle \mu, g \rangle| = \left|\sum_{i=1}^{n} a_i g(x_i)\right| \leq \sum_{i=1}^{n} a_i \, |g(x_i)| \leq \sum_{i=1}^{n} a_i \varphi_Q(x_i) \leq 1,$$

so that $\mu \in Q^0$. Hence, if $\mu \geq \mathbf{0}$ and $\mu \in \{\varphi_Q\}^0$ we have proven that $\mu \in Q^0$, that is, $\{\varphi_Q\}^0 \cap K(X) \subseteq Q^0 \cap K(X)$. Using the fact that $\varphi_Q \in M(X)$, as mentioned above, we conclude that $\sigma(L(X), M(X)) \geq \beta(L(X), C(X))$ on $K(X)$.

So, we see that the strong topology $\beta(L(X), C(X))$ actually coincides with the weak* topology $\sigma(L(X), M(X))$ on the positive cone $K(X)$ of $L(X)$, as stated. □

Example 1.5. If X is infinite $\beta(L(X), C(X))$ does not coincide with $\sigma(L(X), M(X))$ on the whole of $L(X)$. *In fact, if $Q = C(X, [-1, 1])$ then Q is an infinite-dimensional bounded set in $C_p(X)$. If $\beta(L(X), C(X)) = \sigma(L(X), M(X))$, there is a finite set F in $M(X)$ such that $F^0 \subseteq Q^0$, so that $Q \subseteq Q^{00} \subseteq F^{00} = \overline{\mathrm{abx}(F)}$, the bipolar and the closure being taken in $M(X)$. Hence Q must be finite-dimensional, a contradiction.*

2. Some topological properties of $L_\beta(X)$

As mentioned earlier, we denote by $L_\beta(X)$ the strong dual of the space $C_p(X)$. Let us recall that a topological space X is called a *k-space* if each subset A of X which meets every compact set K in a closed set of X is itself a compact set. A topological space X is said to be a *μ-space* if each functionally bounded set in X is relatively compact. In particular, every realcompact space is a μ-space, but the converse statement is not true. A topological space X is called *angelic* if for each relatively countably compact set $Y \subseteq X$ the following holds: (*i*) Y is a relatively compact, and (*ii*) if $x \in \overline{Y}$ there exists a sequence $\{x_n\}_{n=1}^\infty$ in Y which converges to x in X.

Theorem 2.1. *The following properties hold.*

- **(i)** *The strong dual $L_\beta(X)$ of $C_p(X)$ is a k-space if and only if X is countable.*
- **(ii)** *$L_\beta(X)$ is always a μ-space.*
- **(iii)** *$L_\beta(X)$ is always an angelic space.*
- **(iv)** *If $C_p(X)$ is distinguished the strong positive cone $K_\beta(X)$ is realcompact if and only if $|X|$ is nonmeasurable.*
- **(v)** *$L_\beta(X)$ is Lindelöf if and only if X is countable.*

Proof. As in the introduction, let τ_p^{lf} and τ_p^f be, respectively, the strongest locally convex topology on $L(X)$ that coincides with $\sigma(L(X), C(X))$ on the τ_p-equicontinuous sets in $L(X)$, and the Kōmura topology on φ_d with $d = |X|$.

(i) If $L_\beta(X)$ is a k-space, there is on $L(X)$ no stronger topology than $\beta(L(X), C(X))$ that coincides with $\beta(L(X), C(X))$ on the compact sets in $L_\beta(X)$. But since $C_p(X)$ is quasibarrelled [14, Corollary 11.7.3], every bounded set in $L_\beta(X)$ is finite-dimensional. So, as the closure of a bounded set in a finite-dimensional space is compact, there is on $L(X)$ no finer topology than $\beta(L(X), C(X))$ that coincides with $\beta(L(X), C(X))$ on the bounded finite-dimensional sets in $L_\beta(X)$, hence that coincides with $\sigma(L(X), C(X))$ on the bounded finite-dimensional subsets of $L_p(X)$; in other words, on the τ_p-equicontinuous subsets of $L(X)$. This ensures in particular that $\beta(L(X), C(X)) = \tau_p^{lf}$. Hence, $\beta(L(X), C(X))$ is the strongest locally convex topology on $L(X)$, that is, $\beta(L(X), C(X)) = \beta\left(L(X), \mathbb{R}^X\right)$. Therefore, our assumption ensures that

$$L_\beta(X) = \left(L(X), \beta\left(L(X), \mathbb{R}^X\right)\right) = \varphi_d$$

with $d = |X|$, is a k-space. But φ_d is a k-space if and only if $\tau^{lf} = \tau^f$, where here τ stands for the product topology of $\mathbb{R}^X$, and according to [21, Section 21.8, Theorem 2] this happens if and only if $d \leq \aleph_0$, i. e., if and only if X is countable. Conversely, if X is countable then $C_p(X)$ is distinguished (*cf.* [6, Theorem 3.3]). Therefore $L_\beta(X)$ carries the strongest locally convex topology, so that $L_\beta(X) = \varphi$, which is a k-space (*cf.* [25]).

(ii) Since $C_p(X)$ is always quasibarrelled [14, Corollary 11.7.3], every bounded set in $L_\beta(X)$ is finite-dimensional. So, in particular, every closed and functionally bounded set in $L_\beta(X)$ is a closed bounded subset of a finite-dimensional linear subspace, hence a compact set in $L_\beta(X)$.

(iii) Since $L_\beta(X)$ is a μ-space, each relatively countably compact set is relatively compact. As in addition each compact set in $L_\beta(X)$ is finite-dimensional, hence a Fréchet-Urysohn compact, it follows that $L_\beta(X)$ is an angelic space (*cf.* [12, Definition 3.3]).

(iv) If $K_\beta(X)$ is realcompact, since $\delta(X)$ is discrete (by the third statement of Lemma 1.1) and realcompact (due to $\delta(X)$ is closed in $K_\beta(X)$) under the relative topology, it follows that $|X|$ is nonmeasurable (*cf.* [13, Theorem 12.2]). Conversely, if $|X|$ is nonmeasurable then $\mathbb{R}^X$ is bornological by the Mackey-Ulam theorem (*cf.* [21, Section 28.8, Theorem 6]). As $C_p(X)$ is distinguished, we have that $M(X) = \mathbb{R}^X$ by [6, Theorem 3.3], so $M(X)$ is bornological. Consequently each linear form u on $M(X)$ which is continuous on each closed and separable linear subspace of $M(X)$ is continuous (*cf.* [2, Proposition 3.6.6]). Hence $(L(X), \sigma(L(X), M(X)))$ is realcompact by Corson's criterion (*cf.* [3, Theorem 28]). Thus $K(X)$, as a closed subspace of $(L(X), \sigma(L(X), M(X)))$, is also a realcompact space. So, Theorem 1.4 guarantees that $K_\beta(X)$ is realcompact.

(v) If $L_\beta(X)$ is a Lindelöf space, clearly the $\beta(L(X), C(X))$-closed set $\delta(X)$ is also a Lindelöf subspace of $L_\beta(X)$. As $\delta(X)$ is a discrete subspace of $L_\beta(X)$, the set X must be countable. Conversely, if X is countable then $C_p(X)$ is distinguished, so $M(X) = \mathbb{R}^X$ (*cf.* [6, Theorem 3.3]). Since for countable X the product $\mathbb{R}^X$ is a Fréchet space, so a locally convex space in the class $\mathfrak{G}$ of Cascales and Orihuela [18, Chapter 11], the space $(L(X), \sigma(L(X), M(X)))$ is quasi-Suslin (*cf.* [11, Theorem 4]). In addition $L_\beta(X)$, as the strong dual of a Fréchet space, is a complete locally convex space, which guarantees that $L(X)$ is a μ-space under the weak topology $\sigma(L(X), M(X))$ (*cf.* [28, Theorem 3]). So, the fact that $L(X)$ is a quasi-Suslin μ-space under $\sigma(L(X), M(X))$ ensures that $(L(X), \sigma(L(X), M(X)))$ is a K-analytic space (*cf.* [3, Theorem 57]). But then $\left(L(X), \tau^f\right)$ is also K-analytic whenever τ is the product topology of $\mathbb{R}^X = M(X)$, since the k-extension of the topology of a K-analytic space is also K-analytic. Hence $L_\beta(X)$ is K-analytic, as a consequence of the fact that $\beta\left(L(X), \mathbb{R}^X\right) = \tau_p^{lf} \leq \tau^f$ (see the Introduction). Thus $L_\beta(X)$ is a Lindelöf space. Alternatively, as $L(X)$ is K-analytic under the topology $\sigma(L(X), M(X))$, the closed subspace $K(X)$ is also K-analytic. Hence $K_\beta(X)$ is K-analytic by Theorem 1.4. This implies that $K_\beta(X) \times K_\beta(X)$ is K-analytic. So, as in Lemma 1.2, the condensation map φ from $K_\beta(X) \times K_\beta(X)$ onto $L_\beta(X)$ defined by $\varphi(\mu, \lambda) = \mu - \lambda$ ensures that $L_\beta(X)$ is K-analytic, hence Lindelöf. □

3. On distinguished $C_p(X)$ spaces

Let $\mathcal{M}$ denote the uniformity for X generated by the pseudometrics

$$d_A(x, y) = \sup_{g \in A} |g(x) - g(y)|$$

for each bounded set A in $C_p(X)$, and let $\mathcal{N}$ be the usual uniformity for $\mathbb{R}$ defined by the metric $d(s, t) = |t - s|$ for $s, t \in \mathbb{R}$.

Proposition 3.1. *If $C_p(X)$ is distinguished, then each $f \in \mathbb{R}^X$ is uniformly continuous when regarded as a map from $(X, \mathcal{M})$ into $(\mathbb{R}, \mathcal{N})$.*

Proof. Assume that $C_p(X)$ is distinguished. If $f \in \mathbb{R}^X$, according to [6] the fact that $C_p(X)$ is distinguished provides a bounded set Q_f in $C_p(X)$ such that $f \in \overline{Q_{f,\epsilon}}$, where the closure is taken in $\mathbb{R}^X$. Since the pointwise topology on $C(X)$ coincides with the weak locally convex topology $\sigma(C(X), L(X))$ of the dual pair $\langle C(X), L(X)\rangle$, given $\epsilon > 0$, if $(x, y) \in X \times X$ satisfies $d_{Q_f}(x, y) < \delta$ with $\delta = \epsilon$, regarding the functions of $\mathbb{R}^X$ as linear functionals on the whole of $L(X)$, it follows that

$$\begin{aligned} d(f(x), f(y)) &= |f(x) - f(y)| = \left|\left\langle f, \delta_x - \delta_y \right\rangle\right| \leq \sup_{g \in \overline{Q_f}} \left|\left\langle g, \delta_x - \delta_y \right\rangle\right| \\ &= \sup_{g \in Q_f} \left|\left\langle g, \delta_x - \delta_y \right\rangle\right| = \sup_{g \in Q_f} \left|\langle g, \delta_x \rangle - \left\langle g, \delta_y \right\rangle\right| = \sup_{g \in Q_f} |g(x) - g(y)| < \epsilon. \end{aligned}$$

Hence, f is uniformly continuous as a map from $(X, \mathcal{M})$ into $(\mathbb{R}, \mathcal{N})$. □

It is worth noting that the uniform topology $\tau_{\mathcal{M}}$ induced on X by the uniformity $\mathcal{M}$ coincides with the discrete topology (as follows from Lemma 1.2). In the next lemma $\mathcal{M}$ is regarded as the uniformity for $K(X)$ generated by the pseudometrics

$$d_A(\mu, \lambda) = \sup_{g \in A} |\langle g, \mu \rangle - \langle g, \lambda \rangle| = \sup_{g \in A} |\langle g, \mu - \lambda \rangle|$$

for each bounded set A in $C_p(X)$, where $\mu, \lambda \in K(X)$.

Lemma 3.2. *A function $f \in \mathbb{R}^X$ is uniformly continuous when regarded as a map from $(K(X), \mathcal{M})$ into $(\mathbb{R}, \mathcal{N})$ if and only if $f \in M(X)$.*

Proof. If $f \in \mathbb{R}^X$ is uniformly continuous when regarded as a map from $(K(X), \mathcal{M})$ into $(\mathbb{R}, \mathcal{N})$, given $\epsilon > 0$ there exist $\delta(\epsilon) > 0$ and a bounded set $Q_{f,\epsilon}$ in $C_p(X)$ such that if $(\mu, \lambda) \in K(X) \times K(X)$ satisfies $d_{Q_{f,\epsilon}}(\mu, \lambda) \leq \delta$ then $d(\langle f, \mu \rangle, \langle f, \lambda \rangle) < \epsilon$. Now, since $L(X) = K(X) - K(X)$, if $\nu \in L(X)$ is of the form $\nu = \mu - \lambda$ with $\nu \in \delta Q^0_{f,\epsilon}$ we have

$$d_{Q_{f,\epsilon}}(\mu, \lambda) = \sup_{g \in Q_{f,\epsilon}} |\langle g, \mu - \lambda \rangle| = \sup_{g \in Q_{f,\epsilon}} |\langle g, \nu \rangle| \leq \delta.$$

So, necessarily

$$|\langle f, \nu \rangle| = |\langle f, \mu \rangle - \langle f, \lambda \rangle| = d(\langle f, \mu \rangle, \langle f, \lambda \rangle) < \epsilon,$$

which implies that the linear functional $f : L(X) \to \mathbb{R}$ is continuous at the origin $\mathbf{0} \in L(X)$ under the strong topology $\beta(L(X), C(X))$. Consequently $f \in M(X)$.

Conversely, if $f \in M(X)$ there exists a bounded set Q_f in $C_p(X)$ such that $f \in \overline{Q_{f,\epsilon}}$, closure in $\mathbb{R}^X$. Since $\tau_p = \sigma(C(X), L(X))$, given $\epsilon > 0$, if $(\mu, \lambda) \in K(X) \times K(X)$ satisfies $d_{Q_f}(\mu, \lambda) < \delta$ with $\delta = \epsilon$, it follows that

$$d(\langle f, \mu \rangle, \langle f, \lambda \rangle) = |\langle f, \mu - \lambda \rangle| \leq \sup_{g \in \overline{Q_f}} |\langle g, \mu - \lambda \rangle| = \sup_{g \in Q_f} |\langle g, \mu - \lambda \rangle| = d_{Q_{f,\epsilon}}(\mu, \lambda) < \epsilon.$$

Hence, f is uniformly continuous as a map from $(K(X), \mathcal{M})$ into $(\mathbb{R}, \mathcal{N})$. □

Theorem 3.3. *A space $C_p(X)$ is distinguished if and only if every function $f \in \mathbb{R}^X$ is uniformly continuous when regarded as a map from $(K(X), \mathcal{M})$ into $(\mathbb{R}, \mathcal{N})$.*

Proof. If $C_p(X)$ is distinguished, [6, Theorem 3.3] tell us that every $f \in \mathbb{R}^X$ belongs to $M(X)$. So, by Lemma 3.2 each $f \in \mathbb{R}^X$ is uniformly continuous as a mapping from $(K(X), \mathcal{M})$ into $(\mathbb{R}, \mathcal{N})$. Conversely, if each $f \in \mathbb{R}^X$ is uniformly continuous as a map from $(K(X), \mathcal{M})$ into $(\mathbb{R}, \mathcal{N})$, Lemma 3.2 ensures that $M(X) = \mathbb{R}^X$. Hence $C_p(X)$ is distinguished by [10, Theorem 14]. □

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