Montes Taurus J. Pure Appl. Math. 7 (3), 54–63, 2025

Montes Taurus Journal of Pure and Applied Mathematics
https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

Hermite-Hadamard type inequalities for certain classes of convex functions

Ghulam Farid [a], Sanja Kovač [b], Josip Pečarić [c], Mihaela Ribičić Penava [d]

[a] *Department of Mathematics, COMSATS University Islamabad, Attock Campus, Pakistan*
[b] *Faculty of Geotechnical Engineering, University of Zagreb, Hallerova aleja 7, 42 000 Varaždin, Croatia*
[c] *Croatian Academy of Sciences and Arts, Zagreb, Croatia*
[d] *School of Applied Mathematics and Informatics, Josip Juraj Strossmayer University of Osijek, Trg Ljudevita Gaja 6, 31000 Osijek, Croatia*

Abstract

The class of $(h, g; \alpha - m)$-convex functions not only includes a diverse selection of previously defined convexities but also generates new classes of convex functions. In this study, we present Hermite-Hadamard type inequalities for $(h, g; \alpha - m)$-convex functions, as well as some generalizations of Hermite-Hadamard type inequalities for $(h, g; m)$-convex functions. Finally, our results are compared with some previously known results from the literature.

Keywords: Integral inequalities, convex functions, Hermite-Hadamard inequalities

2020 MSC: 26A51, 26D10, 26D15

1. Introduction

The concept of convexity, alongside its diverse generalizations and extensions, plays a fundamental role in modern analysis, particularly in functional analysis, optimization theory, and the study of integral inequalities (*cf.* [1], [3]-[6], [11, 13, 16], and the references cited therein).

We begin by recalling the formal definition of a convex function.

Definition 1.1. A function $f : I \to \mathbb{R}$, $I \subset \mathbb{R}$, is said to be a convex function if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and all $\lambda \in [0, 1]$. If (1.1) holds in the opposite direction, then f is a concave function.

Several important inequalities concerning the integral mean of a convex function are the Hermite-Hadamard inequalities independently established by Hermite [9] and by Hadamard [8].

†*Article ID:* MTJPAM-D-25-00111

Email addresses: faridphdsms@outlook.com (Ghulam Farid), sanja.kovac@gfv.unizg.hr (Sanja Kovač), jopecaric@gmail.com (Josip Pečarić), mihaela@mathos.hr (Mihaela Ribičić Penava)

Received:30 April 2025, Accepted:17 September 2025, Published:1 February 2026

⋆*Corresponding Author:* Mihaela Ribičić Penava

Theorem 1.2 (The Hermite-Hadamard inequalities)**.** *Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

If f is a concave function, then the inequalities in (1.2) *are reversed.*

Over the past few decades Hermite-Hadamard inequalities have been extensively studied, and numerous versions have been published through further generalizations and extensions of convexity.

Definition 1.3 (*cf.* [1])**.** Let h be a non-negative function on $J \subset \mathbb{R}$, $(0, 1) \subset J$, $h \neq 0$, let g be a positive function on $I \subset \mathbb{R}$ and $m \in (0, 1]$. It is said that a function $f : I \to \mathbb{R}$ is $(h, g; m)$-convex if it is non-negative and satisfies the following inequality:

$$f(\lambda x + m(1-\lambda)y) \leq h(\lambda)f(x)g(x) + mh(1-\lambda)f(y)g(y), \tag{1.3}$$

for all $x, y \in I$ and all $\lambda \in (0, 1)$. If (1.3) in the opposite direction, then it is said that f is an $(h, g; m)$-concave function.

This class encompasses a certain range of convexity, allowing for generalizations of known results. For different choices of functions h, g, and parameter m, the class of $(h, g; m)$-convex functions can be reduced to a class of P-functions [7], h-convex functions [16], m-convex functions [15], $(h-m)$-convex functions [12], (s, m)-Godunova-Levin functions of the second kind [11], exponentially (s, m)-convex functions in the second sense [14], and so on.

The paper proceeds as follows. In Section 2, we consider new Hermite-Hadamard type inequalities for $(h, g; m)$-convex functions and improve the results presented in [2] and [1]. Further, we demonstrate that Hermite-Hadamard inequalities for $(h-m)$-convex functions, h-convex functions, m-convex functions, convex functions, and exponentially (s, m)-convex functions in the second sense can be derived as special cases of our main results. In Section 3, we introduce a further generalization of $(h, g; m)$-convexity by incorporating a parameter $\alpha \in (0, 1]$ in the definition of $(h, g; m)$-convexity. Finally, new Hermite-Hadamard type inequalities for the class of $(h, g; \alpha - m)$-convex functions are discussed. The main novelty of this work is a dual-parameter approach that unifies earlier convexity notions and yields sharper integral estimates than those in [1] and [2].

2. Main results

This section primarily aims to derive new Hermite-Hadamard type inequalities for $(h, g; m)$-convex functions, through the utilization of the properties of $(h, g; m)$-convexity and the properties of the definite integral.

First, we introduce the following notation:

$$\Omega_1(f, g, h, u) = h\left(\frac{b-u}{b-a}\right)f(a)g(a)g(u) + mh\left(\frac{u-a}{b-a}\right)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)g(u), \tag{2.1}$$

$$\Omega_2(f, g, h, u) = h\left(\frac{u-a}{b-a}\right)f(b)g(b)g(u) + mh\left(\frac{b-u}{b-a}\right)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)g(u), \tag{2.2}$$

$$\Omega_3(f, g, h, u) = h\left(\frac{b-u}{b-a}\right)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)g\left(\frac{u}{m}\right) + mh\left(\frac{u-a}{b-a}\right)f\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right)g\left(\frac{u}{m}\right) \tag{2.3}$$

and

$$\Omega_4(f, g, h, u) = h\left(\frac{u-a}{b-a}\right)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)g\left(\frac{u}{m}\right) + mh\left(\frac{b-u}{b-a}\right)f\left(\frac{a}{m^2}\right)g\left(\frac{a}{m^2}\right)g\left(\frac{u}{m}\right). \tag{2.4}$$

In the paper [1], the following result was proved:

Lemma 2.1. *Let f be a non-negative $(h, g; m)$-convex function on $[0, \infty)$ where h is a non-negative function on $J \subset \mathbb{R}$, $h \neq 0$, g is a positive function on $[0, \infty)$ and $m \in (0, 1]$. Let $0 \leq a < b < \infty$ and $f, g, h \in L_1[a, b]$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b \left[f(u)g(u) + mf\left(\frac{u}{m}\right)g\left(\frac{u}{m}\right)\right]du. \tag{2.5}$$

In the next theorem, we present refinements of the Hermite-Hadamard inequalities for $(h, g; m)$-convex functions, as originally introduced in [1], by employing minima of four auxiliary expressions.

Theorem 2.2. *Let f be a non-negative $(h, g; m)$-convex function on $[0, \infty)$ where h is a non-negative function on $J \subset \mathbb{R}$, $h \neq 0$, g is a positive function on $[0, \infty)$ and $m \in (0, 1]$. Let $0 \leq a < b < \infty$ and $f, g, h \in L_1[a, b]$. Then the following inequalities hold:*

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b \left[f(u)g(u) + mf\left(\frac{u}{m}\right)g\left(\frac{u}{m}\right)\right]du \\
&\leq \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b \min\{\Omega_1(f,g,h,u) + m\Omega_3(f,g,h,u), \Omega_1(f,g,h,u) + m\Omega_4(f,g,h,u), \\
&\quad \Omega_2(f,g,h,u) + m\Omega_3(f,g,h,u), \Omega_2(f,g,h,u) + m\Omega_4(f,g,h,u)\}\,du \\
&\leq \frac{h\left(\frac{1}{2}\right)}{b-a}\min\left\{\int_a^b \left(f(a)g(a)g(u) + mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)g\left(\frac{u}{m}\right)\right)h\left(\frac{b-u}{b-a}\right)du\right. \\
&\quad + \int_a^b \left(mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)g(u) + m^2 f\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right)g\left(\frac{u}{m}\right)\right)h\left(\frac{u-a}{b-a}\right)du, \\
&\quad \int_a^b \left(f(a)g(a)g(u) + m^2 f\left(\frac{a}{m^2}\right)g\left(\frac{a}{m^2}\right)g\left(\frac{u}{m}\right)\right)h\left(\frac{b-u}{b-a}\right)du \\
&\quad + \int_a^b mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\left(g(u) + g\left(\frac{u}{m}\right)\right)h\left(\frac{u-a}{b-a}\right)du, \\
&\quad \int_a^b mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\left(g(u) + g\left(\frac{u}{m}\right)\right)h\left(\frac{b-u}{b-a}\right)du \\
&\quad + \int_a^b \left(f(b)\,g(b)\,g(u) + m^2 f\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right)g\left(\frac{u}{m}\right)\right)h\left(\frac{u-a}{b-a}\right)du, \\
&\quad \int_a^b \left(mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)g(u) + m^2 f\left(\frac{a}{m^2}\right)g\left(\frac{a}{m^2}\right)g\left(\frac{u}{m}\right)\right)h\left(\frac{b-u}{b-a}\right)du \\
&\quad \left. + \int_a^b \left(f(b)\,g(b)\,g(u) + mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)g\left(\frac{u}{m}\right)\right)h\left(\frac{u-a}{b-a}\right)du\right\}. \qquad (2.6)
\end{aligned}$$

For a non-negative $(h, g; m)$-concave function f, the inequalities in (2.6) are reversed by using the maximum instead of the minimum.

Proof. The first inequality in (2.6) follows directly from Lemma 2.1. To prove the second inequality in (2.6), we consider $(h, g; m)$-convexity of f. So, we get

$$f(u) = f\left(\frac{b-u}{b-a}a + m\frac{u-a}{b-a}\frac{b}{m}\right) \leq h\left(\frac{b-u}{b-a}\right)f(a)g(a) + mh\left(\frac{u-a}{b-a}\right)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right) \qquad (2.7)$$

and

$$f(u) = f\left(m\frac{b-u}{b-a}\frac{a}{m} + \frac{u-a}{b-a}b\right) \leq h\left(\frac{u-a}{b-a}\right)f(b)g(b) + mh\left(\frac{b-u}{b-a}\right)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right). \qquad (2.8)$$

By $(h, g; m)$-convexity of f we again get

$$\begin{aligned}
f\left(\frac{u}{m}\right) &= f\left(\frac{b-u}{b-a}\frac{a}{m} + m\frac{u-a}{b-a}\frac{b}{m^2}\right) \\
&\leq h\left(\frac{b-u}{b-a}\right)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right) + mh\left(\frac{u-a}{b-a}\right)f\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right) \qquad (2.9)
\end{aligned}$$

and

$$f\left(\frac{u}{m}\right) = f\left(m\frac{b-u}{b-a}\frac{a}{m^2}+\frac{u-a}{b-a}\frac{b}{m}\right) \leq h\left(\frac{u-a}{b-a}\right)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)+mh\left(\frac{b-u}{b-a}\right)f\left(\frac{a}{m^2}\right)g\left(\frac{a}{m^2}\right). \tag{2.10}$$

From inequalities (2.7), (2.8), (2.9) and (2.10), by using notations (2.1)-(2.4) and integrating on $[a,b]$ we obtain the second inequality in (2.6). Further, the third inequality in (2.6) derives from the properties of the definite integral and the minimum. Finally, if f is a non-negative $(h,g;m)$-concave function the inequalities in (2.7), (2.8), (2.9) and (2.10) hold in the opposite direction. Consequently, the inequalities in (2.6) hold in reverse, with the maximum replacing the minimum. □

Remark 2.3. In [1], the authors presented the following Hermite-Hadamard type inequalities for an $(h,g;m)$-convex function:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b \left[f(u)g(u)+mf\left(\frac{u}{m}\right)g\left(\frac{u}{m}\right)\right]du \\ &\leq \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b \left(f(a)g(a)g(u)+mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)g\left(\frac{u}{m}\right)\right)h\left(\frac{b-u}{b-a}\right)du \\ &\quad + \int_a^b \left(mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)g(u)+m^2f\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right)g\left(\frac{u}{m}\right)\right)h\left(\frac{u-a}{b-a}\right)du. \end{aligned} \tag{2.11}$$

So, our results in (2.6) improve (2.11).

Further, by employing particular choices of the functions h and/or g together with the parameter m, we obtain special cases of Theorem 2.2 that recover or refine known inequalities for $(h-m)$-convex functions, h-convex functions, m-convex functions, convex functions, and exponentially (s,m)-convex functions in the second sense.

In the next corollary, for $g \equiv 1$, we obtain Hermite-Hadamard inequalities for $(h-m)$-convex functions.

Corollary 2.4. *Let f be a non-negative $(h-m)$-convex function on $[0,\infty)$, where h is a non-negative function on $J \subset \mathbb{R}$, $h \neq 0$ and $m \in (0,1]$. Let $0 \leq a < b < \infty$ and $f,h \in L_1[a,b]$. Then the following inequalities hold:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b \left[f(u)+mf\left(\frac{u}{m}\right)\right]du \\ &\leq \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b \min\{\Omega_1(f,1,h,u)+m\Omega_3(f,1,h,u),\Omega_1(f,1,h,u)+m\Omega_4(f,1,h,u), \\ &\quad \Omega_2(f,1,h,u)+m\Omega_3(f,1,h,u),\Omega_2(f,1,h,u)+m\Omega_4(f,1,h,u)\}\,du \\ &\leq h\left(\frac{1}{2}\right)\min\left\{f(a)+mf\left(\frac{b}{m}\right)+mf\left(\frac{a}{m}\right)+m^2f\left(\frac{b}{m^2}\right),\right. \\ &\quad f(a)+2mf\left(\frac{b}{m}\right)+m^2f\left(\frac{a}{m^2}\right), f(b)+2mf\left(\frac{a}{m}\right)+m^2f\left(\frac{b}{m^2}\right), \\ &\quad \left. f(b)+mf\left(\frac{a}{m}\right)+mf\left(\frac{b}{m}\right)+m^2f\left(\frac{a}{m^2}\right)\right\}\int_0^1 h(t)\,dt. \end{aligned} \tag{2.12}$$

For a non-negative $(h-m)$-concave function f, the inequalities in (2.12) are reversed by using the maximum instead of the minimum.

Proof. Applying Theorem 2.2 for $g \equiv 1$ and using

$$\int_a^b h\left(\frac{b-u}{b-a}\right)du = \int_a^b h\left(\frac{u-a}{b-a}\right)du = (b-a)\int_0^1 h(t)\,dt,$$

we get above inequalities. □

For $g \equiv 1$ and $m = 1$, we get Hermite-Hadamard inequalities for h-convex functions, first proved in paper [16].

Corollary 2.5. *Let f be a non-negative h-convex function on $[0,\infty)$ where h is a non-negative function on $J \subset \mathbb{R}$, $h \neq 0$. Let $0 \le a < b < \infty$ and $f, h \in L_1[a,b]$. Then the following inequalities hold:*

$$\begin{aligned}\frac{1}{2} f\left(\frac{a+b}{2}\right) &\le \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b f(u)du \\ &\le h\left(\frac{1}{2}\right)[f(a)+f(b)]\int_0^1 h(t)\,dt. \end{aligned} \tag{2.13}$$

For a non-negative h-concave function f, the inequalities in (2.13) are reversed.

For $h(x) = x$ and $g \equiv 1$, we obtain Hermite-Hadamard inequalities for m-convex functions.

Corollary 2.6. *Let f be a non-negative m-convex function on $[0,\infty)$, for $m \in (0,1]$ and let $f \in L_1[a,b]$, $0 \le a < b < \infty$. Then the following inequalities hold:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{1}{2(b-a)}\int_a^b \left[f(u) + mf\left(\frac{u}{m}\right)\right]du \\ &\le \frac{1}{4}\min\left\{f(a) + mf\left(\frac{b}{m}\right) + mf\left(\frac{a}{m}\right) + m^2 f\left(\frac{b}{m^2}\right), f(a) + 2mf\left(\frac{b}{m}\right) + m^2 f\left(\frac{a}{m^2}\right),\right. \\ &\quad \left. f(b) + 2mf\left(\frac{a}{m}\right) + m^2 f\left(\frac{b}{m^2}\right), f(b) + mf\left(\frac{a}{m}\right) + mf\left(\frac{b}{m}\right) + m^2 f\left(\frac{a}{m^2}\right)\right\}. \end{aligned} \tag{2.14}$$

For a non-negative m-concave function f, the inequalities in (2.14) are reversed by using the maximum instead of the minimum.

Remark 2.7. Corollary 2.6 improves the results given by Dragomir in paper [5]. Furthermore, for $m = 1$, inequalities (2.14) reduce to the Hermite-Hadamard inequalities for convex functions (1.2).

Corollary 2.8. *Suppose that assumptions stated in Theorem 2.2 hold for an identity function h. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{1}{2(b-a)}\int_a^b \left[f(u)g(u) + mf\left(\frac{u}{m}\right)g\left(\frac{u}{m}\right)\right]du \\ &\le \frac{1}{2(b-a)}\int_a^b \min\{\Omega_1(f,g,x,u) + m\Omega_3(f,g,x,u), \Omega_1(f,g,x,u) + m\Omega_4(f,g,x,u), \\ &\quad \Omega_2(f,g,x,u) + m\Omega_3(f,g,x,u), \Omega_2(f,g,x,u) + m\Omega_4(f,g,x,u)\}\,du \\ &\le \frac{1}{2(b-a)^2}\min\left\{\int_a^b \left(f(a)g(a)g(u) + mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)g\left(\frac{u}{m}\right)\right)(b-u)\,du\right. \\ &\quad + \int_a^b \left(mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)g(u) + m^2 f\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right)g\left(\frac{u}{m}\right)\right)(u-a)\,du, \\ &\quad \int_a^b \left(f(a)g(a)g(u) + m^2 f\left(\frac{a}{m^2}\right)g\left(\frac{a}{m^2}\right)g\left(\frac{u}{m}\right)\right)(b-u)\,du \\ &\quad + \int_a^b mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\left(g(u) + g\left(\frac{u}{m}\right)\right)(u-a)\,du, \\ &\quad \int_a^b mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\left(g(u) + g\left(\frac{u}{m}\right)\right)(b-u)\,du \\ &\quad + \int_a^b \left(f(b)\,g(b)\,g(u) + m^2 f\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right)g\left(\frac{u}{m}\right)\right)(u-a)\,du, \\ &\quad \int_a^b \left(mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)g(u) + m^2 f\left(\frac{a}{m^2}\right)g\left(\frac{a}{m^2}\right)g\left(\frac{u}{m}\right)\right)(b-u)\,du \\ &\quad \left. + \int_a^b \left(f(b)\,g(b)\,g(u) + mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)g\left(\frac{u}{m}\right)\right)(u-a)\,du\right\}. \end{aligned} \tag{2.15}$$

Proof. Applying Theorem 2.2 to $h(x) = x$, we get above inequalities. □

Further, if we setting $h(x) = x^s$, $s \in (0,1]$, and $g(x) = e^{-\gamma x}$, $\gamma \in \mathbb{R}$, in Theorem 2.2, we obtain the Hermite-Hadamard inequality for exponentially (s,m)-convex functions in the second sense.

Corollary 2.9. *Let f be a non-negative exponentially (s,m)-convex function in the second sense on $[0,\infty)$, for $s,m \in (0,1]$ and let $f \in L_1[a,b]$, $0 \le a < b < \infty$, $\gamma \in \mathbb{R}$. Consequently, the following inequalities hold:*

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &\le \frac{1}{2^s(b-a)}\int_a^b \left[f(u)e^{-\gamma u} + mf\left(\frac{u}{m}\right)e^{-\frac{\gamma u}{m}}\right]du \\
&\le \frac{1}{2^s(b-a)}\int_a^b \min\{\Omega_1(f,e^{-\gamma x},x^s,u) + m\Omega_3(f,e^{-\gamma x},x^s,u), \\
&\quad \Omega_1(f,e^{-\gamma x},x^s,u) + m\Omega_4(f,e^{-\gamma x},x^s,u), \Omega_2(f,e^{-\gamma x},x^s,u) + m\Omega_3(f,e^{-\gamma x},x^s,u), \\
&\quad \Omega_2(f,e^{-\gamma x},x^s,u) + m\Omega_4(f,e^{-\gamma x},x^s,u)\}\,du \\
&\le \frac{1}{2^s(b-a)^{s+1}}\min\left\{\int_a^b \left(f(a)e^{-\gamma a}e^{-\gamma u} + mf\left(\frac{a}{m}\right)e^{-\frac{\gamma a}{m}}e^{-\frac{\gamma u}{m}}\right)(b-u)^s\,du\right. \\
&\quad + \int_a^b \left(mf\left(\frac{b}{m}\right)e^{-\frac{\gamma b}{m}}e^{-\gamma u} + m^2 f\left(\frac{b}{m^2}\right)e^{-\frac{\gamma b}{m^2}}e^{-\frac{\gamma u}{m}}\right)(u-a)^s\,du, \\
&\quad \int_a^b \left(f(a)e^{-\gamma a}e^{-\gamma u} + m^2 f\left(\frac{a}{m^2}\right)e^{-\frac{\gamma a}{m^2}}e^{-\frac{\gamma u}{m}}\right)(b-u)^s\,du \\
&\quad + \int_a^b mf\left(\frac{b}{m}\right)e^{-\frac{\gamma b}{m}}\left(e^{-\gamma u} + e^{-\frac{\gamma u}{m}}\right)(u-a)^s\,du, \\
&\quad \int_a^b mf\left(\frac{a}{m}\right)e^{-\frac{\gamma a}{m}}\left(e^{-\gamma u} + e^{-\frac{\gamma u}{m}}\right)(b-u)^s\,du \\
&\quad + \int_a^b \left(f(b)\,e^{-\gamma b}e^{-\gamma u} + m^2 f\left(\frac{b}{m^2}\right)e^{-\frac{\gamma b}{m^2}}e^{-\frac{\gamma u}{m}}\right)(u-a)^s\,du, \\
&\quad \int_a^b \left(mf\left(\frac{a}{m}\right)e^{-\frac{\gamma a}{m}}e^{-\gamma u} + m^2 f\left(\frac{a}{m^2}\right)e^{-\frac{\gamma a}{m^2}}e^{-\frac{\gamma u}{m}}\right)(b-u)^s\,du \\
&\quad \left. + \int_a^b \left(f(b)\,e^{-\gamma b}e^{-\gamma u} + mf\left(\frac{b}{m}\right)e^{-\frac{\gamma b}{m}}e^{-\frac{\gamma u}{m}}\right)(u-a)^s\,du\right\}. \qquad (2.16)
\end{aligned}$$

If f is a non-negative exponentially (s,m)-concave function in the second sense then the inequalities in (2.16) *are reversed by using the maximum instead of the minimum.*

Corollary 2.10. *Let f be a non-negative exponentially convex function, $\gamma \neq 0$. If $f \in L_1[a,b]$, $0 \le a < b < \infty$, then the following inequalities hold:*

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &\le \frac{1}{b-a}\int_a^b f(u)e^{-\gamma u}du \\
&\le \frac{e^{-2\gamma a}f(a) - e^{-2\gamma b}f(b)}{\gamma(b-a)} + \frac{(e^{-\gamma b} - e^{-\gamma a})(e^{-\gamma a}f(a) - e^{-\gamma b}f(b))}{\gamma^2(b-a)^2}. \qquad (2.17)
\end{aligned}$$

If f is a non-negative exponentially concave function, then the inequalities in (2.17) *are reversed.*

Proof. A special case of Corollary 2.9 for $s = 1$ and $m = 1$. □

Remark 2.11. Inequalities (2.17) were first proved in paper [1].

3. Further generalizations

The class of $(h, g; m)$-convex functions can be further generalized by introducing another parameter $\alpha \in (0, 1]$ in the definition of $(h, g; m)$-convexity. In this way, one has the class of $(h, g; \alpha - m)$-convex functions (*cf.* [10, Definition 2.1. for $c = 0$]).

Definition 3.1. Let h be a non-negative function on $J \subset \mathbb{R}$, $(0, 1) \subset J$, $h \neq 0$ and let g be a positive function on $I \subset \mathbb{R}$ and $\alpha, m \in (0, 1]$. It is said that a function $f : I \to \mathbb{R}$ is $(h, g; \alpha - m)$-convex if it is non-negative and satisfies the following inequality:

$$f(\lambda x + m(1-\lambda)y) \leq h(\lambda^\alpha) f(x) g(x) + m h(1-\lambda^\alpha) f(y) g(y), \tag{3.1}$$

for all $\lambda \in (0, 1)$ and all $x, y \in I$. If (3.1) holds in the opposite direction, then f is said to be an $(h, g; \alpha - m)$-concave function.

We will use the definition of an $(h, g; \alpha - m)$-convex function in the following form by setting $z = \lambda x + m(1-\lambda)y$, $z \in [x, y]$, so inequality (3.1) becomes

$$f(z) \leq h\left(\left(\frac{my - z}{my - x}\right)^\alpha\right) f(x)g(x) + mh\left(1 - \left(\frac{my - z}{my - x}\right)^\alpha\right) f(y)g(y). \tag{3.2}$$

Similarly, we get

$$f(z) \leq h\left(\left(\frac{z - mx}{y - mx}\right)^\alpha\right) f(y)g(y) + mh\left(1 - \left(\frac{z - mx}{y - mx}\right)^\alpha\right) f(x)g(x), \tag{3.3}$$

for $z \in [x, y]$.

Finally, we obtain

$$\begin{aligned} f(z) \leq \min \Bigg\{ & h\left(\left(\frac{my - z}{my - x}\right)^\alpha\right) f(x)g(x) + mh\left(1 - \left(\frac{my - z}{my - x}\right)^\alpha\right) f(y)g(y), \\ & h\left(\left(\frac{z - mx}{y - mx}\right)^\alpha\right) f(y)g(y) + mh\left(1 - \left(\frac{z - mx}{y - mx}\right)^\alpha\right) f(x)g(x) \Bigg\}, \end{aligned} \tag{3.4}$$

for $z \in [x, y]$.

By applying the following substitutions

$$\Omega(f, m, \alpha, x, y) = h\left(\left(\frac{my - z}{my - x}\right)^\alpha\right) f(x)g(x) + mh\left(1 - \left(\frac{my - z}{my - x}\right)^\alpha\right) f(y)g(y) \tag{3.5}$$

and

$$\Omega(f, m, \alpha, y, x) = h\left(\left(\frac{z - mx}{y - mx}\right)^\alpha\right) f(y)g(y) + mh\left(1 - \left(\frac{z - mx}{y - mx}\right)^\alpha\right) f(x)g(x), \tag{3.6}$$

inequality (3.4) can be rewritten in the subsequent form:

$$f(z) \leq \min \left\{ \Omega(f, m, \alpha, x, y), \Omega(f, m, \alpha, y, x) \right\}. \tag{3.7}$$

We are now equipped to formulate new Hermite-Hadamard inequalities for the class of $(h, g; \alpha - m)$-convex functions.

Theorem 3.2. *Let f be a non-negative $(h, g; \alpha - m)$-convex function on $[0, \infty)$ where h is a non-negative function on $J \subset \mathbb{R}$, $h \neq 0$, g is a positive function on $[0, \infty)$ and $\alpha, m \in (0, 1]$. Let $w : [a, b] \to \mathbb{R}$ be a non-negative function, $0 \leq a < b < \infty$ and $f, w, g, h \in L_1[a, b]$. Then the following inequality holds:*

$$\begin{aligned} \int_a^b f(u)w(u)du \quad \leq \quad \min \Bigg\{ & f(a)g(a) \int_a^b h\left(\left(\frac{mb - u}{mb - a}\right)^\alpha\right) w(u)du + mf(b)g(b) \int_a^b h\left(1 - \left(\frac{mb - u}{mb - a}\right)^\alpha\right) w(u)du, \\ & f(b)g(b) \int_a^b h\left(\left(\frac{u - ma}{b - ma}\right)^\alpha\right) w(u)du + mf(a)g(a) \int_a^b h\left(1 - \left(\frac{u - ma}{b - ma}\right)^\alpha\right) w(u)du \Bigg\}. \end{aligned} \tag{3.8}$$

If f is a non-negative $(h, g; \alpha - m)$-concave function, then inequalities in (3.8) are reversed with the maximum instead of the minimum.

Proof. Applying inequality (3.4) to a non-negative $(h, g; \alpha - m)$-convex function f and u in $[a, b]$, and multiplying above inequality with $w(u)$ on both sides and integrating it over $[a, b]$, we obtain

$$\begin{aligned}
\int_a^b f(u)w(u)du \quad & \leq \int_a^b \min\left\{h\left(\left(\frac{mb-u}{mb-a}\right)^{\alpha}\right)f(a)g(a) + mh\left(1-\left(\frac{mb-u}{mb-a}\right)^{\alpha}\right)f(b)g(b),\right. \\
& \left. h\left(\left(\frac{u-ma}{b-ma}\right)^{\alpha}\right)f(b)g(b) + mh\left(1-\left(\frac{u-ma}{b-ma}\right)^{\alpha}\right)f(a)g(a)\right\} w(u)du \\
& \leq \min\left\{f(a)g(a)\int_a^b h\left(\left(\frac{mb-u}{mb-a}\right)^{\alpha}\right)w(u)du + mf(b)g(b)\int_a^b h\left(1-\left(\frac{mb-u}{mb-a}\right)^{\alpha}\right)w(u)du,\right. \\
& \left. f(b)g(b)\int_a^b h\left(\left(\frac{u-ma}{b-ma}\right)^{\alpha}\right)w(u)du + mf(a)g(a)\int_a^b h\left(1-\left(\frac{u-ma}{b-ma}\right)^{\alpha}\right)w(u)du\right\}. \qquad (3.9)
\end{aligned}$$

Since, for a non-negative $(h, g; \alpha - m)$-concave function f, the inequalities in (3.2) and (3.3) hold in the opposite direction, it easily follows that, in this case, the inequality in (3.8) holds in reverse, with the maximum replacing the minimum. □

Theorem 3.3. *Let f be a non-negative $(h, g; \alpha - m)$-convex function on $[0, \infty)$ where h is a non-negative function on $J \subset \mathbb{R}$, $h \neq 0$, g is a positive function on $[0, \infty)$ and $\alpha, m \in (0, 1]$. Let $0 \leq a < b < \infty$ and $f, w, g, h \in L_1[a, b]$. Then the following inequality holds:*

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) \quad & \leq \quad \frac{h\left(\frac{1}{2^{\alpha}}\right)}{b-a}\int_a^b f(u)g(u)du + \frac{h\left(1-\frac{1}{2^{\alpha}}\right)}{b-a}\int_a^b mf\left(\frac{u}{m}\right)g\left(\frac{u}{m}\right)du \\
& \leq \quad \frac{h\left(\frac{1}{2^{\alpha}}\right)}{b-a}\int_a^b \min\left\{\Omega(f, m, \alpha, a, b), \Omega(f, m, \alpha, b, a)\right\} g(u)du \\
& + \quad \frac{h\left(1-\frac{1}{2^{\alpha}}\right)m}{b-a}\int_a^b \min\left\{\Omega(f, m, \alpha, a/m, b/m), \Omega(f, m, \alpha, b/m, a/m)\right\} g\left(\frac{u}{m}\right)du \\
& \leq \quad \frac{h\left(\frac{1}{2^{\alpha}}\right)}{b-a}\min\left\{f(a)g(a)\int_a^b h\left(\left(\frac{mb-u}{mb-a}\right)^{\alpha}\right)g(u)du + mf(b)g(b)\int_a^b h\left(1-\left(\frac{mb-u}{mb-a}\right)^{\alpha}\right)g(u)du,\right. \\
& \left. f(b)g(b)\int_a^b h\left(\left(\frac{u-ma}{b-ma}\right)^{\alpha}\right)g(u)du + mf(a)g(a)\int_a^b h\left(1-\left(\frac{u-ma}{b-ma}\right)^{\alpha}\right)g(u)du\right\} \\
& + \quad \frac{h\left(1-\frac{1}{2^{\alpha}}\right)m}{b-a}\min\left\{f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\int_a^b h\left(\left(\frac{mb-u}{mb-a}\right)^{\alpha}\right)g\left(\frac{u}{m}\right)du\right. \\
& +mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\int_a^b h\left(1-\left(\frac{mb-u}{mb-a}\right)^{\alpha}\right)g\left(\frac{u}{m}\right)du, f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\int_a^b h\left(\left(\frac{u-ma}{b-ma}\right)^{\alpha}\right)g\left(\frac{u}{m}\right)du \\
& \left. +mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\int_a^b h\left(1-\left(\frac{u-ma}{b-ma}\right)^{\alpha}\right)g\left(\frac{u}{m}\right)du\right\}. \qquad (3.10)
\end{aligned}$$

If f is a non-negative $(h, g; \alpha - m)$-concave function, then inequalities in (3.10) are reversed with the maximum instead of the minimum.

Proof. Let f be a non-negative $(h, g; \alpha - m)$-convex function. For $\lambda = \frac{1}{2}$, we have

$$f\left(\frac{x+my}{2}\right) \leq h\left(\frac{1}{2^{\alpha}}\right)f(x)g(x) + mh\left(1-\frac{1}{2^{\alpha}}\right)f(y)g(y).$$

Substituting $y = y/m$ we get

$$f\left(\frac{x+y}{2}\right) \leq h\left(\frac{1}{2^{\alpha}}\right)f(x)g(x) + mh\left(1-\frac{1}{2^{\alpha}}\right)f(y/m)g(y/m).$$

Now, substituting $x = \lambda a + (1-\lambda)b$ and $y = (1-\lambda)a + \lambda b$ and integrating the above inequality we obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{h\left(\frac{1}{2^\alpha}\right)}{b-a}\int_a^b f(u)g(u)du + \frac{h\left(1-\frac{1}{2^\alpha}\right)}{b-a}\int_a^b mf\left(\frac{u}{m}\right)g\left(\frac{u}{m}\right)du. \tag{3.11}$$

Further, by using $(h, g; \alpha - m)$-convexity of f and inequality (3.7) with appropriate substitution, inequality (3.11) given above yields the second inequality in (3.10). The third inequality in (3.10) follows from the properties of the definite integral and the minimum. Finally, if f is a non-negative $(h, g; \alpha - m)$-concave function the inequalities in (3.2) and (3.3) hold in the opposite direction. Thus, we easily conclude that, in this case, the inequalities in (3.10) hold in reverse, with the maximum replacing the minimum. □

Remark 3.4. For $\alpha = 1$, $(h, g; \alpha - m)$-convexity reduces to $(h, g; m)$-convexity. Therefore, Theorem 3.3 generalizes Theorem 2.2.

4. Conclusion

We derived refined Hermite-Hadamard type inequalities within the $(h, g; \alpha - m)$-convex framework, which unifies various convexity notions and improves earlier results. The paper presents a comprehensive unification of many Hermite-Hadamard inequalities under a versatile $(h, g; \alpha - m)$-convex framework and indeed yields sharper bounds. As a natural continuation of this research, future work will be devoted to investigating Jensen inequality and related inequalities for these classes of functions, which we expect will further broaden the applicability and depth of the theory.

Acknowledgments

This paper is dedicated to Professor Manuel López-Pellicer on the occasion of his 81st anniversary.

The authors thank the reviewers for their constructive comments and suggestions, which improved this work.

Author Contributions: All authors accept full responsibility for the content of this manuscript and have approved its submission to the journal.

Conflict of Interest: The authors state no conflict of interest.

Funding (Financial Disclosure): This work was supported by European union-NextGenerationEU, grant no. 581-UNIOS-55, OpHoMat.

References

[1] M. Andrić and J. Pečarić, *On $(h, g; m)$-convexity and the Hermite-Hadamard inequality*, J. Convex Anal. **29 (1)**, 257–268, 2022.
[2] M. Andrić, V. Čuljak, D. Pečarić, J. Pečarić and J. Perić, *New developments for Jensen and Lah-Ribarič inequalities*, Element, Zagreb, 2023.
[3] J. Barić, Lj. Kvesić, J. Pečarić and M. Ribičić Penava, *Estimations on some quadrature rules via weighted Hermite-Hadamard inequality*, Appl. Anal. Discrete Math. **16 (1)**, 232–245, 2022.
[4] J. Barić, Lj. Kvesić, J. Pečarić and M. Ribičić Penava, *Fejér type inequalities for higher order convex functions and quadrature formulae*, Aequationes Math. **96 (2)**, 417–430, 2022.
[5] S. S. Dragomir, *On some new inequalities of Hermite-Hadamard type for m-convex functions*, Tamkang J. Math. **33 (1)**, 45–56, 2002.
[6] S. S. Dragomir and C. Pearce, *Selected topics on Hermite-Hadamard inequalities and applications*, RGMIA Monographs, Victoria University, 2000.
[7] S. S. Dragomir, J. E. Pečarić and L. E. Persson, *Some inequalities of Hadamard type*, Soochow J. Math. **21 (3)**, 335–341, 1995.
[8] J. Hadamard, *Étude sur les propriétés des fonctions entiéres et en particulier d'une fonction considérée par Riemann*, J. Math. Pures Appl. **58**, 171–215, 1893.
[9] C. Hermite, *Sur deux limites d'une intégrale définée*, Mathesis **3 (1)**, 1–82, 1883.
[10] Y. Liu, G. Farid, J. Pečarić, J. Rod, M. Elamine and S. Abdel-Khalek, *Strongly $(g, h; \alpha - m)$-convex functions and the consequent Hermite-Hadamard-type inequalities*, Appl. Math. Sci. Eng. **33 (1)**, 2025; Article ID: 2471386, https://doi.org/10.1080/27690911.2025.2471386.
[11] M. A. Noor, K. I. Noor and M. U. Awan, *Fractional Ostrowski inequalities for (s,m)-Godunova-Levin functions*, Facta Univ. Ser. Math. Inform. **30 (4)**, 489–499, 2015.

[12] M. E. Özdemir, A. O. Akdemir and E. Set, *On $(h-m)$-convexity and Hadamard type inequalities*, Transylv. J. Math. Mech. **8** (**1**), 51–58, 2016.
[13] J. E. Pečarić, F. Proschan and Y. L. Tong, *Convex functions, partial orderings, and statistical applications*, Academic Press, San Diego, 1992.
[14] X. Qiang, G. Farid, J. Pečarić and S. B. Akbar, *Generalized fractional integral inequalities for exponentially (s,m)-convex functions*, J. Inequal. Appl. **2020**, 2020; Article ID: 70.
[15] G. Toader, *Some generalizations of the convexity*, In: Proceedings of the Colloquium on Approximation and Optimization, Univ. Cluj-Napoca (Romania), 1984, pp. 329–338.
[16] S. Varošanec, *On h-convexity*, J. Math. Anal. Appl. **326** (**1**), 303–311, 2007.

How to cite this article: G. Farid, S. Kovač, J. Pečarić and M. Ribičić Penava, *Hermite-Hadamard type inequalities for certain classes of convex functions*, Montes Taurus J. Pure Appl. Math. **7** (**3**), 54–63, 2025; Article ID: MTJPAM-D-25-00111.