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Singular PEND overpartitions

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Abstract

In 2015, Andrews [1] defined combinatorial objects which he called as singular overpartition is an integer partition in which no part divisible by δ and parts $\equiv \pm i \pmod{\delta}$ may be overlined. In 2023, Ballantine and Welch [2] have considered various generalizations of POD and PED. In the process, they led to two new classes of POND and PEND partitions which are integer partitions with odd parts cannot be distinct (in case of POND) and even parts can not be distinct (in case of PEND) respectively. In this paper, we considered a new class of partition function called as singular PEND overpartition is a singular overpartition in which even parts not distinct, and obtain several Ramanujan-type infinite families of congruences.

Keywords: Partition congruences, partition identities, overpartitions, singular overpartitions, singular PEND overpartitions, theta-functions

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1. Introduction

Throughout this paper, we let $|q| < 1$. We use the standard notation

$$f_k := (q^k; q^k)_\infty.$$

Following Ramanujan, we define

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_\infty, \tag{1.1}$$

$$f^3(-q) = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}, \tag{1.2}$$

which are special cases of Ramanujan's general theta function (*cf.* [3]):

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \tag{1.3}$$

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In Ramanujan's notation, Jacobi's famous triple product identity becomes,

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \tag{1.4}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. Let $P(n)$ denote the number of partitions of n with $P(0) = 1$ and the generating function for $P(n)$ is

$$\sum_{n=0}^{\infty} P(n)q^n = \frac{1}{(q,q)_\infty}.$$

In 2004, Corteel and Lovejoy [4] introduced the notion of overpartitions, which are partition of a positive integer n in which first occurrence of each part may be overlined. The generating function is

$$\sum_{n=0}^{\infty} \overline{P}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}. \tag{1.5}$$

For example, there are 8 partitions for $\overline{P}(3)$, namely

$$3,\ \ \overline{3},\ \ 2{+}1,\ \ \overline{2}{+}1,\ \ 2{+}\overline{1},\ \ \overline{2}{+}\overline{1},\ \ 1{+}1{+}1,\ \ \overline{1}{+}1{+}1.$$

In 2015, Andrews [1] defined combinatorial objects which he called singular overpartitions and proved that these singular overpartitions which depends on two parameters δ and i can be enumerated by the function $\overline{C}_{\delta,i}(n)$ which gives the number of overpartitions of n in which no part divisible by δ and parts $\equiv \pm i \pmod{\delta}$ may be overlined. The generating function of $\overline{C}_{\delta,i}(n)$ is

$$\sum_{n=0}^{\infty} \overline{C}_{\delta,i}(n)q^n = \frac{(q^\delta;q^\delta)_\infty(-q^i;q^\delta)_\infty(-q^{\delta-i};q^\delta)_\infty}{(q;q)_\infty} = \frac{f(q^i,q^{\delta-i})}{(q;q)_\infty}. \tag{1.6}$$

For example, there are 10 partitions for $\overline{C}_{3,1}(4)$, namely

$$4,\ \ \overline{4},\ \ 2+2,\ \ \overline{2}+2,\ \ 2{+}1{+}1,\ \ \overline{2}{+}1{+}1,\ \ 2{+}\overline{1}+1,\ \ \overline{2}{+}\overline{1}+1,\ \ 1{+}1{+}1{+}1,\ \ \overline{1}{+}1{+}1{+}1.$$

He also proved that

$$\overline{C}_{3,1}(9n+3) = \overline{C}_{3,1}(9n+6) \equiv 0.$$

For more details one can see [8, 10] and [9].

In the following paper, Ballantine and Welch [2] considered various generalizations and refinements of POD and PED partitions. In the process they led to two new classes of POND and PEND partitions which are integer partitions with odd parts can not be distinct (in case of POND) and even parts can not be distinct (in case of PEND) respectively.

In 2025, Prasad et al. [11] have introduced a new class of partition function called as $[j,k]$-pond overpartitions or OPOND partitions which enumerates the number of partitions of a positive integer n in which first occurrence of each part congruent to j modulo k may be overlined and odd parts are not distinct (even parts are unrestricted) and obtained several infinite families of congruences.

In [12], Prasad et al. have defined and obtained infinite families of congruences for singular POND overpartitions which are singular overpartitions of a positive integer n in which odd parts are not distinct (even parts are unrestricted). Let $\overline{cpond}_{\delta,i}(n)$ denote the number of such partitions of n and the generating function be given by

$$\sum_{n=0}^{\infty} \overline{cpond}_{\delta,i}(n)q^n = \frac{f_4 f_6^2}{f_2^2 f_3 f_{12}} f(q^i,q^{\delta-i}). \tag{1.7}$$

A singular PEND overpartition is a singular overpartition of a positive integer n in which even parts are not distinct (odd parts are unrestricted). Let $\overline{cpend}_{\delta,i}(n)$ denote the number of such partitions of n and the generating function be given by

$$\sum_{n=0}^{\infty}\overline{cpend}_{\delta,i}(n)q^n = \frac{f_2f_{12}}{f_1f_4f_6}f(q^i,q^{\delta-i}). \tag{1.8}$$

Motivated by the above works, in this paper we obtain several Ramanujan-type infinite families of congruences for a new class of combinatorial objects called as singular PEND overpartitions by using elementary dissection formulas.

1.1. Preliminaries

In this section, we record several identities which are useful in proving our main results.

Lemma 1.1. *The following* 2*-dissection holds:*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5f_{16}^2} + 2q\frac{f_4^2f_{16}^2}{f_2^5f_8}, \tag{1.9}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}} \tag{1.10}$$

and

$$\frac{f_3^3}{f_1} = \frac{f_4^3f_6^2}{f_2^2f_{12}} + q\frac{f_{12}^3}{f_4}. \tag{1.11}$$

The equation (1.9) can be found in [5, p. 14, Eq. (1.9.4)]. The identity (1.10) can be found in [5, p. 15, Eq. (1.10.1)]. The equation (1.11) is same as equation (22.2.14) in [5] (after using equations (22.1.6) and (22.1.7) of [5]).

Lemma 1.2. *The following* 3*-dissections hold*

$$f_1^3 = \frac{f_6f_9^6}{f_3f_{18}^3} + 4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^2} - 3qf_9^3, \tag{1.12}$$

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3f_{18}^2}{f_6f_9} \tag{1.13}$$

and

$$\frac{f_2}{f_1f_4} = \frac{f_{18}^9}{f_3^2f_9^3f_{12}^2f_{36}^3} + q\frac{f_6^2f_{18}^3}{f_3^3f_{12}^3} + q^2\frac{f_6^4f_9^3f_{36}^3}{f_3^4f_{12}^4f_{18}^3}. \tag{1.14}$$

The equation (1.12) can be found as equation (14.8.5) in [5]. The equations (1.13) and (1.14) can be found in [7].

Lemma 1.3. *The following 5-dissection holds*

$$f_1 = f_{25}\left(a - q - q^2/a\right), \tag{1.15}$$

where

$$a := a(q^5) = \frac{f(-q^{10},-q^{15})}{f(-q^5,q^{20})}.$$

The identity (1.15) is essentially (8.1.1) in [5].

Lemma 1.4. *The following 7-dissection holds*

$$f_1 = f_{49}\left(\frac{B}{C} - q\frac{A}{B} - q^2 + q^5\frac{C}{A}\right), \tag{1.16}$$

where $A := A(q^7) = f(-q^{21},-q^{28}), B := B(q^7) = f(-q^{14},q^{35})$ *and* $C := C(q^7) = f(-q^7,-q^{42})$.

The equation (1.16) can be found in [5], one can also see [3, p. 303, Entry 17(v)].

2. Generating function for $\overline{cpend}_{\delta,i}(n)$

In this section, we provide proof for a generating function of a partition $\overline{cpend}_{\delta,i}(n)$.

Theorem 2.1. *For $1 \le i \le \delta$, we have*

$$\sum_{n=0}^{\infty} \overline{cpend}_{\delta,i}(n)q^n = \frac{f_2 f_{12}}{f_1 f_4 f_6} f(q^i, q^{\delta-i}). \tag{2.1}$$

Proof. By the definition,

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{cpend}_{\delta,i}(n)q^n = \frac{f(q^i, q^{\delta-i})}{(q;q^2)_\infty} \prod_{i=1}^{\infty} \left(\frac{1}{1-q^{2i}} - q^{2i} \right) &= \frac{f(q^i, q^{\delta-i})}{(q;q^2)_\infty} \prod_{i=1}^{\infty} \left(\frac{1-q^{2i}+(q^{2i})^2}{(1-q^{2i})} \right) \\ &= \frac{f(q^i, q^{\delta-i})}{(q;q^2)_\infty} \prod_{i=1}^{\infty} \left(\frac{(1+q^{6i})}{(1-q^{2i})(1+q^{2i})} \right) \\ &= \frac{f(q^i, q^{\delta-i})}{(q;q^2)_\infty} \frac{(-q^6;q^6)_\infty}{(q^4;q^4)_\infty} = \frac{f_2 f_{12}}{f_1 f_4 f_6} f(q^i, q^{\delta-i}).\end{aligned}$$

□

3. Congruences for $\overline{cpend}_{\delta,i}(n)$

In this section, we prove several infinite families of congruences for $\overline{cpend}_{\delta,i}(n)$.

Theorem 3.1. *We have*

$$\sum_{n=0}^{\infty} \overline{cpend}_{k,k}(n)q^n = 2\frac{f_2 f_{12} f_{2k}^2}{f_1 f_4 f_6 f_k}, \tag{3.1}$$

$$\sum_{n=0}^{\infty} \overline{cpend}_{4k,k}(n)q^n = \frac{f_2 f_{12} f_{2k}^2}{f_1 f_4 f_6 f_k} \tag{3.2}$$

and

$$\overline{cpend}_{k,k}(n) = 2\,\overline{cpend}_{4k,k}(n). \tag{3.3}$$

Proof. Putting $\delta = k$ and $i = k$ in (2.1), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{cpend}_{k,k}(n)q^n &= \frac{f_2 f_{12} f(q^k, 1)}{f_1 f_4 f_6} = 2\frac{f_2 f_{12} f(q^k, q^{3k})}{f_1 f_4 f_6} \\ &= 2\frac{f_2 f_{12} \psi(q^k)}{f_1 f_4 f_6},\end{aligned} \tag{3.4}$$

which implies (3.1).

Putting $\delta = 4k$ and $i = k$ in (2.1), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{cpend}_{4k,k}(n)q^n &= \frac{f_2 f_{12} f(q^k, q^{3k})}{f_1 f_4 f_6} = \frac{f_2 f_{12} \psi(q^k)}{f_1 f_4 f_6} \\ &= \frac{f_2 f_{12} f_{2k}^2}{f_1 f_4 f_6 f_k},\end{aligned} \tag{3.5}$$

which proves (3.2).

From (3.1) and (3.2), we obtain (3.3). □

Theorem 3.2. *For $\alpha \geq 0$ and $n \geq 0$, then we have for* (mod 32)

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(2 \cdot 3^{16\alpha+1}n + \frac{3^{16\alpha+2}-1}{2}\right)q^n \equiv 4f_1^9 f_3^3, \tag{3.6}$$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+9}n + \frac{13 \cdot 3^{16\alpha+8}-1}{2}\right) \equiv 0, \tag{3.7}$$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+9}n + \frac{3^{16\alpha+8}-1}{2}\right) \equiv \begin{cases} 16 & \text{if } n = \dfrac{k(3k-1)}{2} \\ 0 & \text{otherwise} \end{cases} \tag{3.8}$$

and

$$\overline{cpend}_{12,3}\left(2 \cdot 3^{16\alpha+17}n + \frac{3^{16\alpha+16}-1}{2}\right) \equiv 0, \tag{3.9}$$

where the dot symbol $\cdot$ denotes the classical multiplication.

Proof. Putting $k = 3$ in (3.2), we deduce that

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}(n)q^n = \frac{f_2 f_6 f_{12}}{f_1 f_4 f_3}. \tag{3.10}$$

Using (1.14) in (3.10), we get

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}(3n)q^n = \frac{f_2 f_6^9}{f_1^3 f_3^3 f_4 f_{12}^3}, \tag{3.11}$$

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}(3n+1)q^n = \frac{f_2^3 f_6^3}{f_1^4 f_4^2} \tag{3.12}$$

and

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}(3n+2)q^n = \frac{f_2^5 f_3^3 f_{12}^3}{f_1^5 f_4^3 f_6^3}. \tag{3.13}$$

Employing (1.10) in (3.12), we find that

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}(6n+1)q^n = \frac{f_2^{12} f_3^3}{f_1^{11} f_4^4} \tag{3.14}$$

and

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}(6n+4)q^n = \frac{4 f_3^3 f_4^4}{f_1^7}. \tag{3.15}$$

By using binomial theorem, we can easily prove

$$f_4^4 \equiv f_2^8 \equiv f_1^{16} \pmod 8. \tag{3.16}$$

Using (3.16) in (3.15), we get

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}(6n+4)q^n = 4f_1^9 f_3^3 \pmod{32}, \tag{3.17}$$

which is $\alpha = 0$ case of (3.6).

Suppose that the equation (3.6) is true for $\alpha \geq 0$.
Using (1.12) in (3.6), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+1}n+\frac{3^{16\alpha+2}-1}{2}\right)q^n &\equiv 4f_3^3\left(\frac{f_6f_9^6}{f_3f_{18}^3}+4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^3}-3qf_9^3\right)^3\\ &\equiv 4\frac{f_6^3f_9^2}{f_{18}}+4q^3f_3^3f_9^9+28\frac{f_3f_6^2f_{18}^2}{f_9}\\ &\quad +12q^2f_3^2f_6f_9^4f_{18}+16q^5\frac{f_6f_9^3f_{18}^6}{f_3} \pmod{32}. \qquad (3.18)\end{aligned}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} in (3.18), we find that for $\pmod{32}$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+2}n+\frac{3^{16\alpha+2}-1}{2}\right)q^n \equiv 4\frac{f_2^3f_3^2}{f_6}+4qf_1^3f_3^9, \qquad (3.19)$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+2}n+\frac{7\cdot 3^{16\alpha+1}-1}{2}\right)q^n \equiv 28\frac{f_1f_2^2f_6^2}{f_3} \qquad (3.20)$$

and

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+2}n+\frac{11\cdot 3^{16\alpha+1}-1}{2}\right)q^n \equiv 12f_1^2f_2f_3^4f_6+16\frac{qf_2f_3^3f_6^6}{f_1}. \qquad (3.21)$$

Using (1.12) in (3.19), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+2}n+\frac{3^{16\alpha+2}-1}{2}\right)q^n &\equiv 4\frac{f_3^2}{f_6}\left(\frac{f_{12}f_{18}^6}{f_6f_{36}^3}+4q^6\frac{f_6^2f_{36}^6}{f_{12}^2f_{18}^3}-3q^2f_{18}^3\right)\\ &\quad +4qf_3^9\left(\frac{f_6f_9^6}{f_3f_{18}^3}+4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^3}-3qf_9^3\right)\\ &\equiv 4\frac{f_3^2f_{12}f_{36}}{f_6^2f_{18}^2}+16q^6\frac{f_{18}^9}{f_{12}}+4q\frac{f_6^5f_{18}}{f_9^2}+16q^4\frac{f_{12}^2f_{18}^3f_9^3}{f_3}\\ &\quad +20q^2\frac{f_3^2f_{18}^3}{f_6}+20q^2f_3^9f_9^3 \pmod{32}. \qquad (3.22)\end{aligned}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} in (3.22), we get

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+3}n+\frac{3^{16\alpha+2}-1}{2}\right)q^n \equiv 4\frac{f_1^2f_4f_{12}}{f_2^2f_6^2}+16q^2\frac{f_6^9}{f_4}, \qquad (3.23)$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+3}n+\frac{5\cdot 3^{16\alpha+2}-1}{2}\right)q^n \equiv 4\frac{f_2^5f_6}{f_3^2}+16q\frac{f_4^2f_6^3f_3^3}{f_1} \qquad (3.24)$$

and

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+3}n+\frac{3^{16\alpha+4}-1}{2}\right)q^n \equiv 20\frac{f_1^2f_6^3}{f_2}+20f_1^9f_3^3. \qquad (3.25)$$

Using (1.12) and (1.13) in (3.25), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+3}n+\frac{3^{16\alpha+4}-1}{2}\right)q^n &\equiv 20f_6^3\left(\frac{f_9^2}{f_{18}}-2q\frac{f_3f_{18}^2}{f_6f_9}\right)+20f_3^3\left(\frac{f_6f_9^6}{f_3f_{18}^3}+4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^3}-3qf_9^3\right)^3\\ &\equiv 8\frac{f_6^3f_9^2}{f_{18}}+20q^3f_3^3f_9^9+4q\frac{f_3f_6^2f_{18}^2}{f_9}\\ &\quad+28q^2\frac{f_6^5f_9^4f_{18}}{f_3^6}+16q^5\frac{f_6f_9^3f_{36}^3}{f_3}. \end{aligned}\tag{3.26}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} in the above equation, we deduce that

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+4}n+\frac{3^{16\alpha+4}-1}{2}\right)q^n\equiv 8\frac{f_2^3f_3^2}{f_6}+20qf_1^3f_3^9,\tag{3.27}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+4}n+\frac{7\cdot 3^{16\alpha+3}-1}{2}\right)q^n\equiv 4\frac{f_1f_2^2f_6^2}{f_3}\tag{3.28}$$

and

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+4}n+\frac{11\cdot 3^{16\alpha+3}-1}{2}\right)q^n\equiv 28\frac{f_2^5f_3^4f_6}{f_1^6}+16\frac{qf_2f_3^3f_{12}^3}{f_1}.\tag{3.29}$$

Employing (1.12) in (3.27), we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+4}n+\frac{3^{16\alpha+4}-1}{2}\right)q^n &\equiv 8\frac{f_3^2}{f_6}\left(\frac{f_{12}f_{18}^6}{f_6f_{36}^3}+4q^6\frac{f_6^2f_{36}^6}{f_{12}^2f_{18}^3}-3q^2f_{18}^3\right)\\ &\quad+20qf_3^9\left(\frac{f_6f_9^6}{f_3f_{18}^3}+4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^3}-3qf_9^3\right)\\ &\equiv 8\frac{f_{12}f_{18}^2}{f_3^2f_{36}}+20q\frac{f_6^5f_9^2}{f_{18}}+16q^4\frac{f_6^4f_{18}^3f_9^3}{f_3}\\ &\quad+8q^2\frac{f_3^2f_{18}^3}{f_6}+4q^2f_3^9f_9^3. \end{aligned}\tag{3.30}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} in the above equation, we find that

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+5}n+\frac{3^{16\alpha+4}-1}{2}\right)q^n\equiv 8\frac{f_4f_6^2}{f_1^2f_{12}},\tag{3.31}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+5}n+\frac{5\cdot 3^{16\alpha+4}-1}{2}\right)q^n\equiv 20\frac{f_2^5f_3^2}{f_6}+16\frac{qf_2^4f_6^3f_3^3}{f_1}\tag{3.32}$$

and

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+5}n+\frac{3^{16\alpha+6}-1}{2}\right)q^n\equiv 8\frac{f_1^2f_6^3}{f_2}+4f_1^9f_3^3.\tag{3.33}$$

Using (1.12) and (1.13) in (3.33), we see that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+5}n+\frac{3^{16\alpha+6}-1}{2}\right)q^n &\equiv 8f_6^3\left(\frac{f_9^2}{f_{18}}-2q\frac{f_3f_{18}^2}{f_6f_9}\right)+4f_3^3\left(\frac{f_6f_9^6}{f_3f_{18}^3}+4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^3}-3qf_9^3\right)^3\\ &\equiv 12\frac{f_6^3f_9^2}{f_{18}}+4q^3f_3^3f_9^9+12q\frac{f_3f_6^2f_{18}^2}{f_9}\\ &\quad+12q^2\frac{f_6^5f_9^4f_{18}}{f_3^6}+16q^5\frac{f_6f_{18}^6f_9^3}{f_3}. \end{aligned}\tag{3.34}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} in the above equation, we get

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+6}n+\frac{3^{16\alpha+6}-1}{2}\right)q^n \equiv 12\frac{f_2^3f_3^2}{f_6}+4qf_1^3f_3^9, \tag{3.35}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+6}n+\frac{7\cdot 3^{16\alpha+5}-1}{2}\right)q^n \equiv 12\frac{f_1f_2^2f_6^2}{f_3} \tag{3.36}$$

and

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+6}n+\frac{11\cdot 3^{16\alpha+5}-1}{2}\right)q^n \equiv 12\frac{f_2^5f_3^4f_6}{f_1^6}+16\frac{qf_2f_6^6f_3^3}{f_1}. \tag{3.37}$$

Using (1.12) in (3.37), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+6}n+\frac{3^{16\alpha+6}-1}{2}\right)q^n &\equiv 12\frac{f_3^2}{f_6}\left(\frac{f_{12}f_{18}^6}{f_6f_{36}^3}+4q^6\frac{f_6^2f_{36}^6}{f_{12}^2f_{18}^3}-3q^2f_{18}^3\right)\\ &\quad +4qf_3^9\left(\frac{f_6f_9^6}{f_3f_{18}^3}+4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^3}-3qf_9^3\right)\\ &\equiv 12\frac{f_3^2f_{12}f_{36}}{f_6^2f_{18}^2}+16q^6\frac{f_{18}^9}{f_{12}}+4q\frac{f_6^5f_{18}}{f_9^2}\\ &\quad +16q^4\frac{f_3f_6^3f_{18}^5}{f_9}28q^2\frac{f_3^2f_{18}^3}{f_6}+20q^2f_3^9f_9^3. \end{aligned} \tag{3.38}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} in the above equation, we get

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+7}n+\frac{3^{16\alpha+6}-1}{2}\right)q^n \equiv 12\frac{f_1^2f_4f_{12}}{f_2^2f_6^2}+16q^2\frac{f_6^9}{f_4}, \tag{3.39}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+7}n+\frac{5\cdot 3^{16\alpha+6}-1}{2}\right)q^n \equiv 4\frac{f_2^5f_6}{f_3^2}+16q\frac{f_1f_2^3f_6^5}{f_3} \tag{3.40}$$

and

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+7}n+\frac{3^{16\alpha+8}-1}{2}\right)q^n \equiv 28\frac{f_1^2f_6^3}{f_2}+20f_1^9f_3^3. \tag{3.41}$$

Using (1.12) and (1.13) in (3.41), we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+7}n+\frac{3^{16\alpha+8}-1}{2}\right)q^n &\equiv 28f_6^3\left(\frac{f_9^2}{f_{18}}-2q\frac{f_3f_{18}^2}{f_6f_9}\right)\\ &\quad +20f_3^3\left(\frac{f_6f_9^6}{f_3f_{18}^3}+4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^3}-3qf_9^3\right)^3\\ &\equiv 16f_3^6+20q^3f_3^3f_9^9+20q\frac{f_3f_6^2f_{18}^2}{f_9}\\ &\quad +28q^2f_3^2f_6f_9^4f_{18}+16q^5\frac{f_6f_{18}^6f_9^3}{f_3}. \end{aligned} \tag{3.42}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} in the above equation, we deduce that

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+8}n+\frac{3^{16\alpha+8}-1}{2}\right)q^n \equiv 16f_1^6+20qf_1^3f_3^9, \tag{3.43}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+8}n+\frac{7\cdot 3^{16\alpha+7}-1}{2}\right)q^n \equiv 20\frac{f_1f_2^2f_6^2}{f_3} \tag{3.44}$$

and

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+8}n+\frac{11\cdot 3^{16\alpha+7}-1}{2}\right)q^n \equiv 28f_1^2f_2f_3^4f_6+16q\frac{f_2f_6^6f_3^3}{f_1}. \tag{3.45}$$

Using (1.12) in (3.43), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+8}n+\frac{3^{16\alpha+8}-1}{2}\right)q^n &\equiv 16\left(\frac{f_6f_9^6}{f_3f_{18}^3}+4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^3}-3qf_9^3\right)^2+20qf_3^9\left(\frac{f_6f_9^6}{f_3f_{18}^3}+4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^3}-3qf_9^3\right)\\ &\equiv 16f_6+20q\frac{f_6^5f_{18}}{f_9^2}+16q^4\frac{f_6^4f_{18}^3f_9^3}{f_3}+4q^2f_3^9f_9^3+16q^2f_{18}^3. \end{aligned}\tag{3.46}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} in the above equation, we deduce that

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+9}n+\frac{3^{16\alpha+8}-1}{2}\right)q^n \equiv 16f_2, \tag{3.47}$$

which implies (3.7), (3.8) and

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+9}n+\frac{5\cdot 3^{16\alpha+8}-1}{2}\right)q^n \equiv 20\frac{f_2^5f_6}{f_3^2}+16q\frac{f_2^4f_6^3f_3^3}{f_1} \tag{3.48}$$

and

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+9}n+\frac{3^{16\alpha+10}-1}{2}\right)q^n \equiv 4f_1^9f_3^3+16f_6^3. \tag{3.49}$$

Employing (1.12) in (3.49), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+9}n+\frac{3^{16\alpha+10}-1}{2}\right)q^n &\equiv 4f_3^3\left(\frac{f_6f_9^6}{f_3f_{18}^3}+4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^3}-3qf_9^3\right)^3+16f_6^3\\ &\equiv 4\frac{f_6^3f_9^2}{f_{18}}+4q^3f_3^3f_9^9+16f_6^3+28q\frac{f_3f_6^2f_{18}^2}{f_9}\\ &\quad +12q^2f_3^2f_6f_9^4f_{18}+16q^5\frac{f_6f_9^3f_{18}^6}{f3}. \end{aligned}\tag{3.50}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} in the above equation, we deduce that

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+10}n+\frac{3^{16\alpha+10}-1}{2}\right)q^n \equiv 4\frac{f_2^3f_3^2}{f_6}+4qf_1^3f_3^9+16f_2^3, \tag{3.51}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+10}n+\frac{7\cdot 3^{16\alpha+9}-1}{2}\right)q^n \equiv 28\frac{f_1f_2^2f_6^2}{f_3} \tag{3.52}$$

and

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+10}n+\frac{11\cdot 3^{16\alpha+9}-1}{2}\right)q^n \equiv 12f_1^2f_2f_3^4f_6+16\frac{qf_2f_3^3f_6^6}{f_1}. \tag{3.53}$$

Employing (1.12) in (3.51), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+10}n+\frac{3^{16\alpha+10}-1}{2}\right)q^n &\equiv 4\frac{f_3^2}{f_6}\left(\frac{f_{12}f_{18}^6}{f_6f_{36}^3}+4q^6\frac{f_6^2f_{36}^6}{f_{12}^2f_{18}^3}-3q^2f_{18}^3\right)\\ &+4qf_3^9\left(\frac{f_6f_9^6}{f_3f_{18}^3}+4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^3}-3qf_9^3\right)\\ &+16\left(\frac{f_{12}f_{18}^6}{f_6f_{36}^3}+4q^6\frac{f_6^2f_{36}^6}{f_{12}^2f_{18}^3}-3q^2f_{18}^3\right)\\ &\equiv 4\frac{f_3^2f_{12}f_{36}}{f_6^2f_{18}^2}+16q^6\frac{f_{18}^9}{f_{12}}+4q\frac{f_6^5f_{18}}{f_9^2}+16q^4\frac{f_6^4f_9^3f_{18}^3}{f_3}\\ &+20q^2\frac{f_3^2f_{18}^3}{f_6}+20q^2f_3^9f_9^3+16q^2f_{18}^3. \end{aligned} \tag{3.54}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} in the above equation, we deduce that

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+11}n+\frac{3^{16\alpha+10}-1}{2}\right)q^n \equiv 4\frac{f_1^2f_4f_{12}}{f_2^2f_6^2}+16q^2\frac{f_6^9}{f_4}, \tag{3.55}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+11}n+\frac{5\cdot 3^{16\alpha+10}-1}{2}\right)q^n \equiv 4\frac{f_2^5f_6}{f_3^2}+16q\frac{f_2^4f_6^3f_3^3}{f_1} \tag{3.56}$$

and

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+11}n+\frac{3^{16\alpha+12}-1}{2}\right)q^n \equiv 20\frac{f_1^2f_6^3}{f_2}+20f_1^9f_3^3+16f_6^3. \tag{3.57}$$

Using (1.12) and (1.13) in (3.57), we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+11}n+\frac{3^{16\alpha+12}-1}{2}\right)q^n &\equiv 20f_6^3\left(\frac{f_9^2}{f_{18}}-2q\frac{f_3f_{18}^2}{f_6f_9}\right)\\ &+20f_3^3\left(\frac{f_6f_9^6}{f_3f_{18}^3}+4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^3}-3qf_9^3\right)^3+16f_6^3\\ &\equiv 8\frac{f_6^3f_9^2}{f_{18}}+20q^3f_3^3f_9^9+16f_6^3+4q\frac{f_3f_6^2f_{18}^2}{f_9}\\ &+28q^2f_3^2f_6f_9^4f_{18}+16q^5\frac{f_6f_9^3f_{18}^6}{f_3}. \end{aligned} \tag{3.58}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} in the above equation, we deduce that

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+12}n+\frac{3^{16\alpha+12}-1}{2}\right)q^n \equiv 8\frac{f_2^3f_3^2}{f_6}+20qf_1^3f_3^9+16f_2^3, \tag{3.59}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+12}n+\frac{7\cdot 3^{16\alpha+11}-1}{2}\right)q^n \equiv 4\frac{f_1f_2^2f_6^2}{f_3} \tag{3.60}$$

and

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+12}n+\frac{11\cdot 3^{16\alpha+11}-1}{2}\right)q^n \equiv 28f_1^2f_2f_3^4f_6+16q\frac{f_2f_3^3f_6^6}{f_1}. \tag{3.61}$$

Using (1.12) in (3.59), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+12}n+\frac{3^{16\alpha+12}-1}{2}\right)q^n &\equiv 8\frac{f_3^2}{f_6}\left(\frac{f_{12}f_{18}^6}{f_6f_{36}^3}+4q^6\frac{f_6^2f_{36}^6}{f_{12}^2f_{18}^3}-3q^2f_{18}^3\right)\\ &\quad +20qf_3^9\left(\frac{f_6f_9^6}{f_3f_{18}^3}+4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^3}-3qf_9^3\right)\\ &\quad +16\left(\frac{f_{12}f_{18}^6}{f_6f_{36}^3}+4q^6\frac{f_6^2f_{36}^6}{f_{12}^2f_{18}^3}-3q^2f_{18}^3\right)\\ &\equiv 8\frac{f_{12}f_{18}^2}{f_3^2f_{36}}+16f_6+20q\frac{f_6^5f_{18}}{f_9^2}+16q^4\frac{f_6^4f_9^3f_{18}^3}{f_3}\\ &\quad +8q^2\frac{f_3^2f_{18}^3}{f_6}+4q^2f_3^9f_9^3+16q^2f_{18}^3. \end{aligned}\tag{3.62}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} in the above equation, we deduce that

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+13}n+\frac{3^{16\alpha+12}-1}{2}\right)q^n\equiv 8\frac{f_4f_6^2}{f_1^2f_{12}}+16f_2,\tag{3.63}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+13}n+\frac{5\cdot 3^{16\alpha+12}-1}{2}\right)q^n\equiv 20\frac{f_2^5f_6}{f_3^2}+16q\frac{f_2^4f_3^3f_6^3}{f_1}\tag{3.64}$$

and

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+13}n+\frac{3^{16\alpha+14}-1}{2}\right)q^n\equiv 8\frac{f_1^2f_6^3}{f_2}+4f_1^9f_3^3+16f_6^3.\tag{3.65}$$

Using (1.12) and (1.13) in (3.65), we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+13}n+\frac{3^{16\alpha+14}-1}{2}\right)q^n &\equiv 8f_6^3\left(\frac{f_9^2}{f_{18}}-2q\frac{f_3f_{18}^2}{f_6f_9}\right)\\ &\quad +4f_3^9\left(\frac{f_6f_9^6}{f_3f_{18}^3}+4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^3}-3qf_9^3\right)^3+16f_6^3\\ &\equiv 12\frac{f_6^3f_9^2}{f_{18}}+4q^3f_3^3f_9^9+16f_6^3+12q\frac{f_3f_6^2f_{18}^2}{f_9}\\ &\quad +12q^2f_3^2f_6f_9^4f_{18}+16q^5\frac{f_6f_9^3f_{18}^6}{f_3}. \end{aligned}\tag{3.66}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} in the above equation, we deduce that

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+14}n+\frac{3^{16\alpha+14}-1}{2}\right)q^n\equiv 12\frac{f_2^3f_3^2}{f_6}+4qf_1^3f_3^9+16f_2^3,\tag{3.67}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+14}n+\frac{7\cdot 3^{16\alpha+13}-1}{2}\right)q^n\equiv 12\frac{f_1f_2^2f_6^2}{f_3}\tag{3.68}$$

and

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+14}n+\frac{11\cdot 3^{16\alpha+13}-1}{2}\right)q^n\equiv 12f_1^2f_2f_3^4f_6+16\frac{qf_2f_3^3f_6^6}{f_1}.\tag{3.69}$$

Using (1.12) in (3.67), we see that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+14}n+\frac{3^{16\alpha+14}-1}{2}\right)q^n &\equiv 12\frac{f_3^2}{f_6}\left(\frac{f_{12}f_{18}^6}{f_6f_{36}^3}+4q^6\frac{f_6^2f_{36}^6}{f_{12}^2f_{18}^3}-3q^2f_{18}^3\right)\\ &\quad +4qf_3^9\left(\frac{f_6f_9^6}{f_3f_{18}^3}+4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^3}-3qf_9^3\right)\\ &\quad +16\left(\frac{f_{12}f_{18}^6}{f_6f_{36}^3}+4q^6\frac{f_6^2f_{36}^6}{f_{12}^2f_{18}^3}-3q^2f_{18}^3\right)\\ &\equiv 12\frac{f_3^2f_{12}f_{36}}{f_6^2f_{18}^2}+16q^6\frac{f_{18}^9}{f_{12}}+4q\frac{f_6^5f_{18}}{f_9^2}+16q^4\frac{f_6^4f_9^3f_{18}^3}{f_3}\\ &\quad +28q^2\frac{f_3^2f_{18}^3}{f_6}+20q^2f_3^9f_9^3+16q^2f_{18}^3. \end{aligned} \tag{3.70}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} in the above equation, we deduce that

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+15}n+\frac{3^{16\alpha+14}-1}{2}\right)q^n \equiv 12\frac{f_1^2f_4f_{12}}{f_2^2f_6^2}+16q^2\frac{f_6^9}{f_4}, \tag{3.71}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+15}n+\frac{5\cdot 3^{16\alpha+14}-1}{2}\right)q^n \equiv 4\frac{f_2^5f_6}{f_3^2}+16\frac{qf_2^4f_3^3f_6^3}{f_1} \tag{3.72}$$

and

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+15}n+\frac{3^{16\alpha+16}-1}{2}\right)q^n \equiv 28\frac{f_1^2f_6^3}{f_2}+20f_1^9f_3^3+16f_6^3. \tag{3.73}$$

Using (1.12) and (1.13) in (3.73), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+15}n+\frac{3^{16\alpha+16}-1}{2}\right)q^n &\equiv 28f_6^3\left(\frac{f_9^2}{f_{18}}-2q\frac{f_3f_{18}^2}{f_6f_9}\right)\\ &\quad +20f_3^3\left(\frac{f_6f_9^6}{f_3f_{18}^3}+4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^3}-3qf_9^3\right)^3+16f_6^3\\ &\equiv 20q^3f_3^3f_9^9+20q\frac{f_3f_6^2f_{18}^2}{f_9}+28q^2f_3^2f_6f_9^4f_{18}\\ &\quad +16q^5\frac{f_6f_9^3f_{18}^6}{f_3}. \end{aligned} \tag{3.74}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} in the above equation, we deduce that

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+16}n+\frac{3^{16\alpha+16}-1}{2}\right)q^n \equiv 20qf_1^3f_3^9, \tag{3.75}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+16}n+\frac{7\cdot 3^{16\alpha+15}-1}{2}\right)q^n \equiv 20\frac{f_1f_2^2f_6^2}{f_3} \tag{3.76}$$

and

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+16}n+\frac{11\cdot 3^{16\alpha+15}-1}{2}\right)q^n \equiv 28f_1^2f_2f_3^4f_6+16\frac{qf_2f_3^3f_6^6}{f_1}. \tag{3.77}$$

Using (1.12) in (3.75), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+16}n+\frac{3^{16\alpha+16}-1}{2}\right)q^n &\equiv 20qf_3^9\left(\frac{f_6f_9^6}{f_3f_{18}^3}+4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^3}-3qf_9^3\right)\\ &\equiv 20q\frac{f_6^5f_{18}}{f_9^2}+16q^4\frac{f_6^4f_9^3f_{18}^3}{f_3}+4q^2f_3^9f_9^3. \qquad (3.78)\end{aligned}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} in the above equation, we obtain (3.9),

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+17}n+\frac{5\cdot 3^{16\alpha+16}-1}{2}\right)q^n \equiv 20\frac{f_2^5f_6}{f_3^2}+16\frac{qf_2^4f_3^3f_6^3}{f_1} \qquad (3.79)$$

and

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+17}n+\frac{3^{16\alpha+18}-1}{2}\right)q^n \equiv 4f_1^9f_3^3, \qquad (3.80)$$

which is $\alpha+1$ case of (3.6). By mathematical induction, the equation (3.6) is hold for all $\alpha \geq 0$. □

Theorem 3.3. *For $\alpha, \beta \geq 0$ and $n \geq 0$, then we have for* (mod 8)

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+2}\cdot 5^{2\beta}n+\frac{7\cdot 3^{16\alpha+1}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 4f_1f_2^3, \qquad (3.81)$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+2}5^{2\beta+1}n+\frac{11\cdot 3^{16\alpha+1}5^{2\beta+1}-1}{2}\right)q^n \equiv 4qf_5f_{10}^3, \qquad (3.82)$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+2}\cdot 5^{2\beta}n+\frac{19\cdot 3^{16\alpha+1}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 4f_1f_6^3, \qquad (3.83)$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+2}5^{2\beta+1}n+\frac{23\cdot 3^{16\alpha+1}5^{2\beta+1}-1}{2}\right)q^n \equiv 4q^3f_5f_{30}^3, \qquad (3.84)$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+4}\cdot 5^{2\beta}n+\frac{7\cdot 3^{16\alpha+3}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 4f_1f_2^3, \qquad (3.85)$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+4}\cdot 5^{2\beta+1}n+\frac{11\cdot 3^{16\alpha+3}5^{2\beta+1}-1}{2}\right)q^n \equiv 4qf_5f_{10}^3, \qquad (3.86)$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+4}\cdot 5^{2\beta}n+\frac{19\cdot 3^{16\alpha+3}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 4f_1f_6^3, \qquad (3.87)$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+4}5^{2\beta+1}n+\frac{23\cdot 3^{16\alpha+3}5^{2\beta+1}-1}{2}\right)q^n \equiv 4q^3f_5f_{30}^3, \qquad (3.88)$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+6}\cdot 5^{2\beta}n+\frac{7\cdot 3^{16\alpha+5}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 4f_1f_2^3, \qquad (3.89)$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+2}5^{2\beta+1}n+\frac{11\cdot 3^{16\alpha+5}5^{2\beta+1}-1}{2}\right)q^n \equiv 4qf_5f_{10}^3, \qquad (3.90)$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+6}\cdot 5^{2\beta}n+\frac{19\cdot 3^{16\alpha+5}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 4f_1f_6^3, \qquad (3.91)$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+6}5^{2\beta+1}n+\frac{23\cdot 3^{16\alpha+5}5^{2\beta+1}-1}{2}\right)q^n\equiv 4q^3f_5f_{30}^3, \tag{3.92}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+8}\cdot 5^{2\beta}n+\frac{7\cdot 3^{16\alpha+7}\cdot 5^{2\beta}-1}{2}\right)q^n\equiv 4f_1f_2^3, \tag{3.93}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+8}5^{2\beta+1}n+\frac{11\cdot 3^{16\alpha+7}5^{2\beta+1}-1}{2}\right)q^n\equiv 4qf_5f_{10}^3, \tag{3.94}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+8}\cdot 5^{2\beta}n+\frac{19\cdot 3^{16\alpha+7}\cdot 5^{2\beta}-1}{2}\right)q^n\equiv 4f_1f_6^3, \tag{3.95}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+8}5^{2\beta+1}n+\frac{23\cdot 3^{16\alpha+7}5^{2\beta+1}-1}{2}\right)q^n\equiv 4q^3f_5f_{30}^3, \tag{3.96}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+10}\cdot 5^{2\beta}n+\frac{7\cdot 3^{16\alpha+9}\cdot 5^{2\beta}-1}{2}\right)q^n\equiv 4f_1f_2^3, \tag{3.97}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+10}5^{2\beta+1}n+\frac{11\cdot 3^{16\alpha+9}5^{2\beta+1}-1}{2}\right)q^n\equiv 4qf_5f_{10}^3, \tag{3.98}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+10}\cdot 5^{2\beta}n+\frac{19\cdot 3^{16\alpha+9}\cdot 5^{2\beta}-1}{2}\right)q^n\equiv 4f_1f_6^3, \tag{3.99}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+10}5^{2\beta+1}n+\frac{23\cdot 3^{16\alpha+9}5^{2\beta+1}-1}{2}\right)q^n\equiv 4q^3f_5f_{30}^3, \tag{3.100}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+12}\cdot 5^{2\beta}n+\frac{7\cdot 3^{16\alpha+11}\cdot 5^{2\beta}-1}{2}\right)q^n\equiv 4f_1f_2^3, \tag{3.101}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+12}5^{2\beta+1}n+\frac{11\cdot 3^{16\alpha+11}5^{2\beta+1}-1}{2}\right)q^n\equiv 4qf_5f_{10}^3, \tag{3.102}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+12}\cdot 5^{2\beta}n+\frac{19\cdot 3^{16\alpha+11}\cdot 5^{2\beta}-1}{2}\right)q^n\equiv 4f_1f_6^3, \tag{3.103}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+12}5^{2\beta+1}n+\frac{23\cdot 3^{16\alpha+11}5^{2\beta+1}-1}{2}\right)q^n\equiv 4q^3f_5f_{30}^3, \tag{3.104}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+14}\cdot 5^{2\beta}n+\frac{7\cdot 3^{16\alpha+13}\cdot 5^{2\beta}-1}{2}\right)q^n\equiv 4f_1f_2^3, \tag{3.105}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+14}5^{2\beta+1}n+\frac{11\cdot 3^{16\alpha+13}5^{2\beta+1}-1}{2}\right)q^n\equiv 4qf_5f_{10}^3, \tag{3.106}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+14}\cdot 5^{2\beta}n+\frac{19\cdot 3^{16\alpha+13}\cdot 5^{2\beta}-1}{2}\right)q^n\equiv 4f_1f_6^3, \tag{3.107}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+14}5^{2\beta+1}n+\frac{23\cdot 3^{16\alpha+13}5^{2\beta+1}-1}{2}\right)q^n\equiv 4q^3f_5f_{30}^3, \tag{3.108}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+16}\cdot 5^{2\beta}n+\frac{7\cdot 3^{16\alpha+15}\cdot 5^{2\beta}-1}{2}\right)q^n\equiv 4f_1f_2^3, \tag{3.109}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+16}5^{2\beta+1}n+\frac{11\cdot 3^{16\alpha+15}5^{2\beta+1}-1}{2}\right)q^n \equiv 4qf_5f_{10}^3, \tag{3.110}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+16}\cdot 5^{2\beta}n+\frac{19\cdot 3^{16\alpha+15}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 4f_1f_6^3 \tag{3.111}$$

and

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+16}5^{2\beta+1}n+\frac{23\cdot 3^{16\alpha+15}5^{2\beta+1}-1}{2}\right)q^n \equiv 4q^3f_5f_{30}^3. \tag{3.112}$$

Proof. From the equation (3.20), we have for $\pmod 8$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+2}n+\frac{7\cdot 3^{16\alpha+1}}{2}\right)q^n \equiv 4\frac{f_2^3f_3^3}{f_1}. \tag{3.113}$$

Using (1.11) in (3.113) and then extracting the terms involving q^{2n} and q^{2n+1}, we deduce that

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+2}n+\frac{7\cdot 3^{16\alpha+1}-1}{2}\right)q^n \equiv 4f_1f_2^3 \tag{3.114}$$

and

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+2}n+\frac{19\cdot 3^{16\alpha+1}-1}{2}\right)q^n \equiv 4f_1f_6^3. \tag{3.115}$$

The equation (3.114) is $\beta = 0$ case of (3.81).

Suppose that the equation (3.81) hold for $\alpha,\beta \geq 0$.

Using (1.15) in (3.81) and then extracting the term involving q^{5n+2}, we obtain (3.82).

From equation (3.82), we find that

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+2}\cdot 5^{2\beta+2}n+\frac{7\cdot 3^{16\alpha+1}\cdot 5^{2\beta+2}-1}{2}\right)q^n \equiv 4f_1f_2^3, \tag{3.116}$$

which is $\beta+1$ case of (3.81). Hence by induction the equation (3.81) is hold for all $\alpha,\beta \geq 0$.

The equation (3.115) is $\beta = 0$ case of (3.83).

Suppose that the equation (3.83) hold for $\alpha,\beta \geq 0$.

Using (1.15) in (3.83) and then extracting the term involving q^{5n+4}, we obtain (3.84).

From equation (3.84), we find that

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+2}\cdot 5^{2\beta+2}n+\frac{19\cdot 3^{16\alpha+1}\cdot 5^{2\beta+2}-1}{2}\right)q^n \equiv 4f_1f_6^3, \tag{3.117}$$

which is $\beta+1$ case of (3.83). Hence by induction the equation (3.83) is hold for all $\alpha,\beta \geq 0$.

The proofs of (3.85)-(3.112) are similar to the proofs of (3.81)-(3.84). So, we omit the details. □

Theorem 3.4. *For $\alpha,\beta \geq 0$, then we have for* $\pmod 8$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+2}\cdot 5^{2\beta}n+\frac{11\cdot 3^{16\alpha+1}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 4f_2f_3^3, \tag{3.118}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+2}5^{2\beta+1}n+\frac{11\cdot 3^{16\alpha+1}5^{2\beta+1}-1}{2}\right)q^n \equiv 4q^2f_{10}f_{15}^3, \tag{3.119}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+4}\cdot 5^{2\beta}n+\frac{11\cdot 3^{16\alpha+3}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 4f_2f_3^3, \tag{3.120}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+4}5^{2\beta+1}n+\frac{7\cdot 3^{16\alpha+3}5^{2\beta+1}-1}{2}\right)q^n \equiv 4q^2 f_{10}f_{15}^3, \tag{3.121}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+6}\cdot 5^{2\beta}n+\frac{11\cdot 3^{16\alpha+5}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 4f_2f_3^3, \tag{3.122}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+6}5^{2\beta+1}n+\frac{7\cdot 3^{16\alpha+5}5^{2\beta+1}-1}{2}\right)q^n \equiv 4q^2 f_{10}f_{15}^3, \tag{3.123}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+8}\cdot 5^{2\beta}n+\frac{11\cdot 3^{16\alpha+7}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 4f_2f_3^3, \tag{3.124}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+8}5^{2\beta+1}n+\frac{7\cdot 3^{16\alpha+7}5^{2\beta+1}-1}{2}\right)q^n \equiv 4q^2 f_{10}f_{15}^3, \tag{3.125}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+10}\cdot 5^{2\beta}n+\frac{11\cdot 3^{16\alpha+9}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 4f_2f_3^3, \tag{3.126}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+10}5^{2\beta+1}n+\frac{7\cdot 3^{16\alpha+9}5^{2\beta+1}-1}{2}\right)q^n \equiv 4q^2 f_{10}f_{15}^3, \tag{3.127}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+12}\cdot 5^{2\beta}n+\frac{11\cdot 3^{16\alpha+11}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 4f_2f_3^3, \tag{3.128}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+12}5^{2\beta+1}n+\frac{7\cdot 3^{16\alpha+11}5^{2\beta+1}-1}{2}\right)q^n \equiv 4q^2 f_{10}f_{15}^3, \tag{3.129}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+14}\cdot 5^{2\beta}n+\frac{11\cdot 3^{16\alpha+13}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 4f_2f_3^3, \tag{3.130}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+14}5^{2\beta+1}n+\frac{7\cdot 3^{16\alpha+13}5^{2\beta+1}-1}{2}\right)q^n \equiv 4q^2 f_{10}f_{15}^3, \tag{3.131}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+16}\cdot 5^{2\beta}n+\frac{11\cdot 3^{16\alpha+15}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 4f_2f_3^3 \tag{3.132}$$

and

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+16}5^{2\beta+1}n+\frac{7\cdot 3^{16\alpha+15}5^{2\beta+1}-1}{2}\right)q^n \equiv 4q^2 f_{10}f_{15}^3. \tag{3.133}$$

Proof. From the equation (3.21), we obtain

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+2}n+\frac{11\cdot 3^{16\alpha+1}-1}{2}\right)q^n \equiv 12\frac{f_2^3f_6^3}{f_1^2} \pmod{16}. \tag{3.134}$$

Using (1.9) in (3.134) and then extracting the term involving q^{2n}, we find that

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+2}n+\frac{11\cdot 3^{16\alpha+1}-1}{2}\right)q^n \equiv 4f_2f_3^3 \pmod{8}, \tag{3.135}$$

which is $\beta = 0$ case of (3.118).

Suppose that the equation (3.118) hold for $\alpha, \beta \geq 0$.

Using (1.15) in (3.118) and then extracting the term involving q^{5n+1}, we obtain (3.119).

From equation (3.119), we find that

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+2}\cdot 5^{2\beta+2}n+\frac{11\cdot 3^{16\alpha+1}\cdot 5^{2\beta+2}-1}{2}\right)q^n\equiv 4f_2f_3^3, \tag{3.136}$$

which is $\beta+1$ case of (3.118). Hence, by induction the equation (3.118) is hold for all $\alpha,\beta\geq 0$.

The proofs of (3.120)-(3.133) are similar to the proofs of (3.118)-(3.119). So, we omit the details. □

Theorem 3.5. *For $\alpha,\beta\geq 0$, then we have for* $\pmod{16}$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+2}n+\frac{23\cdot 3^{16\alpha+1}-1}{2}\right)q^n\equiv 8f_2f_3^3f_4^3, \tag{3.137}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+4}n+\frac{23\cdot 3^{16\alpha+3}-1}{2}\right)q^n\equiv 8f_2f_3^3f_4^3, \tag{3.138}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+6}n+\frac{23\cdot 3^{16\alpha+5}-1}{2}\right)q^n\equiv 8f_2f_3^3f_4^3, \tag{3.139}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+8}n+\frac{23\cdot 3^{16\alpha+7}-1}{2}\right)q^n\equiv 8f_2f_3^3f_4^3, \tag{3.140}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+10}n+\frac{23\cdot 3^{16\alpha+9}-1}{2}\right)q^n\equiv 8f_2f_3^3f_4^3, \tag{3.141}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+12}n+\frac{23\cdot 3^{16\alpha+11}-1}{2}\right)q^n\equiv 8f_2f_3^3f_4^3, \tag{3.142}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+14}n+\frac{23\cdot 3^{16\alpha+13}-1}{2}\right)q^n\equiv 8f_2f_3^3f_4^3 \tag{3.143}$$

and

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+16}n+\frac{23\cdot 3^{16\alpha+15}-1}{2}\right)q^n\equiv 8f_2f_3^3f_4^3. \tag{3.144}$$

Proof. Using (1.9) in (3.134) and then extracting the term involving q^{2n+1}, we obtain (3.137).

The proofs of (3.138)-(3.144) are similar to the proof of (3.137). So, we omit the details. □

Theorem 3.6. *For $\alpha\geq 0$, then we have for* $\pmod{8}$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+3}n+\frac{3^{16\alpha+2}-1}{2}\right)\equiv\begin{cases}4 & \text{if } n=\dfrac{k(3k-1)}{2}\\ 0 & \text{otherwise},\end{cases} \tag{3.145}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+7}n+\frac{3^{16\alpha+6}-1}{2}\right)\equiv\begin{cases}4 & \text{if } n=\dfrac{k(3k-1)}{2}\\ 0 & \text{otherwise},\end{cases} \tag{3.146}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+11}n+\frac{3^{16\alpha+10}-1}{2}\right)\equiv\begin{cases}4 & \text{if } n=\dfrac{k(3k-1)}{2}\\ 0 & \text{otherwise}\end{cases} \tag{3.147}$$

and

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+15}n+\frac{3^{16\alpha+14}-1}{2}\right)\equiv\begin{cases}4 & \text{if } n=\dfrac{k(3k-1)}{2}\\ 0 & \text{otherwise}.\end{cases} \tag{3.148}$$

Proof. From the equation (3.23), we deduce that

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(2\cdot 3^{16\alpha+3}n+\frac{3^{16\alpha+2}-1}{2}\right)q^n \equiv 4\frac{f_4f_{12}}{f_1^2f_6^2} \pmod{16}. \tag{3.149}$$

Using (1.9) in (3.149) and then extracting the term involving q^{2n}, we find that

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+3}n+\frac{3^{16\alpha+2}-1}{2}\right)q^n \equiv 4f_1 \pmod{8}, \tag{3.150}$$

which implies (3.145).

The proofs of (3.146)-(3.148) are similar to the proof of (3.145). So, we omit the details. □

Theorem 3.7. *For $\alpha,\beta \geq 0$, then we have for* $\pmod{16}$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+3}\cdot 5^{2\beta}n+\frac{13\cdot 3^{16\alpha+2}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 8f_1f_4^3, \tag{3.151}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+3}5^{2\beta+1}n+\frac{17\cdot 3^{16\alpha+2}5^{2\beta+1}-1}{2}\right)q^n \equiv 8q^2f_5f_{20}^3, \tag{3.152}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+7}\cdot 5^{2\beta}n+\frac{13\cdot 3^{16\alpha+6}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 8f_1f_4^3, \tag{3.153}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+7}5^{2\beta+1}n+\frac{17\cdot 3^{16\alpha+6}5^{2\beta+1}-1}{2}\right)q^n \equiv 8q^2f_5f_{20}^3, \tag{3.154}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+11}\cdot 5^{2\beta}n+\frac{13\cdot 3^{16\alpha+10}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 8f_1f_4^3, \tag{3.155}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+11}5^{2\beta+1}n+\frac{17\cdot 3^{16\alpha+10}5^{2\beta+1}-1}{2}\right)q^n \equiv 8q^2f_5f_{20}^3, \tag{3.156}$$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+15}\cdot 5^{2\beta}n+\frac{13\cdot 3^{16\alpha+14}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 8f_1f_4^3 \tag{3.157}$$

and

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+15}5^{2\beta+1}n+\frac{17\cdot 3^{16\alpha+14}5^{2\beta+1}-1}{2}\right)q^n \equiv 8q^2f_5f_{20}^3. \tag{3.158}$$

Proof. Using (1.9) in (3.149) and then extracting the term involving q^{2n+1}, we find that

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+3}n+\frac{13\cdot 3^{16\alpha+2}-1}{2}\right)q^n \equiv 8f_1f_4^3 \pmod{16}, \tag{3.159}$$

which is $\beta = 0$ case of (3.151).

Suppose that the equation (3.151) is true for all $\alpha,\beta \geq 0$.

Using (1.15) in (3.118) and then extracting the term involving q^{5n+3}, we obtain (3.152).

From (3.152), we find that for $\pmod{16}$

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+3}\cdot 5^{2\beta+2}n+\frac{13\cdot 3^{16\alpha+2}\cdot 5^{2\beta+2}-1}{2}\right)q^n \equiv 8f_1f_4^3, \tag{3.160}$$

which is $\beta + 1$ case of (3.151). So by induction, the equation (3.151) is hold for all $\alpha,\beta \geq 0$.

The proofs of (3.153)-(3.158) are similar to the proof of (3.151). So, we omit the details. □

Theorem 3.8. *For $\alpha \geq 0$, then we have for* (mod 16)

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+5}n + \frac{3^{16\alpha+4}-1}{2}\right) \equiv \begin{cases} 8 & \text{if } n = \dfrac{k(3k-1)}{2} \\ 0 & \text{otherwise} \end{cases} \tag{3.161}$$

and

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+13}n + \frac{3^{16\alpha+12}-1}{2}\right) \equiv \begin{cases} 8 & \text{if } n = \dfrac{k(3k-1)}{2} \\ 0 & \text{otherwise.} \end{cases} \tag{3.162}$$

Proof. From the equation (3.31), we have

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(2 \cdot 3^{16\alpha+5}n + \frac{3^{16\alpha+4}-1}{2}\right) q^n \equiv 8\frac{f_4 f_6^2}{f_1^2 f_{12}} \pmod{32}. \tag{3.163}$$

Using (1.9) in (3.163) and then extracting the term involving q^{2n}, we find that

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+5}n + \frac{3^{16\alpha+4}-1}{2}\right) q^n \equiv 8 f_1 \pmod{16}, \tag{3.164}$$

which implies (3.161).

The proof of (3.162) is similar to the proof of (3.161). So, we omit the details. □

Theorem 3.9. *For $\alpha, \beta \geq 0$, then we have for* (mod 32)

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+5} \cdot 5^{2\beta}n + \frac{13 \cdot 3^{16\alpha+4} \cdot 5^{2\beta}-1}{2}\right) q^n \equiv 16 f_1 f_4^3, \tag{3.165}$$

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+5} 5^{2\beta+1}n + \frac{17 \cdot 3^{16\alpha+4} 5^{2\beta+1}-1}{2}\right) q^n \equiv 16 q^2 f_5 f_{20}^3, \tag{3.166}$$

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+13} \cdot 5^{2\beta}n + \frac{13 \cdot 3^{16\alpha+12} \cdot 5^{2\beta}-1}{2}\right) q^n \equiv 16 f_1 f_4^3 \tag{3.167}$$

and

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+13} 5^{2\beta+1}n + \frac{17 \cdot 3^{16\alpha+12} 5^{2\beta+1}-1}{2}\right) q^n \equiv 16 q^2 f_5 f_{20}^3, \tag{3.168}$$

Proof. Using (1.9) in (3.163) and then extracting the term involving q^{2n+1}, we find that

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+5}n + \frac{13 \cdot 3^{16\alpha+4}-1}{2}\right) q^n \equiv 16 f_1 f_4^3 \pmod{32}, \tag{3.169}$$

which is $\beta = 0$ case of (3.165).

Suppose that the equation (3.165) is true for all $\alpha, \beta \geq 0$.

Using (1.15) in (3.165) and then extracting the term involving q^{5n+3}, we obtain (3.166).

From (3.166), we find that for (mod 32)

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+5} 5^{2\beta+2}n + \frac{13 \cdot 3^{16\alpha+4} 5^{2\beta+2}-1}{2}\right) q^n \equiv 8 f_1 f_4^3, \tag{3.170}$$

which is $\beta + 1$ case of (3.165). So by induction, the equation (3.165) is hold for all $\alpha, \beta \geq 0$.

The proof of (3.167)-(3.168) is similar to the proof of (3.165). So, we omit the details. □

Theorem 3.10. *For $\alpha, \beta \geq 0$ and $n \geq 0$, then we have for* (mod 8)

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+3} \cdot 7^{2\beta} n + \frac{5 \cdot 3^{16\alpha+2} \cdot 7^{2\beta} - 1}{2}\right) q^n \equiv 4 f_1 f_4, \tag{3.171}$$

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+3} \cdot 7^{2\beta+1} n + \frac{11 \cdot 3^{16\alpha+2} \cdot 7^{2\beta+1} - 1}{2}\right) q^n \equiv 4q f_7 f_{28}, \tag{3.172}$$

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+5} \cdot 7^{2\beta} n + \frac{5 \cdot 3^{16\alpha+4} \cdot 7^{2\beta} - 1}{2}\right) q^n \equiv 4 f_1 f_4, \tag{3.173}$$

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+5} \cdot 7^{2\beta+1} n + \frac{11 \cdot 3^{16\alpha+4} \cdot 7^{2\beta+1} - 1}{2}\right) q^n \equiv 4q f_7 f_{28}, \tag{3.174}$$

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+7} \cdot 7^{2\beta} n + \frac{5 \cdot 3^{16\alpha+6} \cdot 7^{2\beta} - 1}{2}\right) q^n \equiv 4 f_1 f_4, \tag{3.175}$$

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+7} \cdot 7^{2\beta+1} n + \frac{11 \cdot 3^{16\alpha+6} \cdot 7^{2\beta+1} - 1}{2}\right) q^n \equiv 4q f_7 f_{28}, \tag{3.176}$$

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+9} \cdot 7^{2\beta} n + \frac{5 \cdot 3^{16\alpha+8} \cdot 7^{2\beta} - 1}{2}\right) q^n \equiv 4 f_1 f_4, \tag{3.177}$$

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+9} \cdot 7^{2\beta+1} n + \frac{11 \cdot 3^{16\alpha+8} \cdot 7^{2\beta+1} - 1}{2}\right) q^n \equiv 4q f_7 f_{28}, \tag{3.178}$$

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+11} \cdot 7^{2\beta} n + \frac{5 \cdot 3^{16\alpha+10} \cdot 7^{2\beta} - 1}{2}\right) q^n \equiv 4 f_1 f_4, \tag{3.179}$$

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+11} \cdot 7^{2\beta+1} n + \frac{11 \cdot 3^{16\alpha+10} \cdot 7^{2\beta+1} - 1}{2}\right) q^n \equiv 4q f_7 f_{28}, \tag{3.180}$$

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+15} \cdot 7^{2\beta} n + \frac{5 \cdot 3^{16\alpha+14} \cdot 7^{2\beta} - 1}{2}\right) q^n \equiv 4 f_1 f_4, \tag{3.181}$$

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+15} \cdot 7^{2\beta+1} n + \frac{11 \cdot 3^{16\alpha+14} \cdot 7^{2\beta+1} - 1}{2}\right) q^n \equiv 4q f_7 f_{28}, \tag{3.182}$$

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+17} \cdot 7^{2\beta} n + \frac{5 \cdot 3^{16\alpha+16} \cdot 7^{2\beta} - 1}{2}\right) q^n \equiv 4 f_1 f_4 \tag{3.183}$$

and

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+17} \cdot 7^{2\beta+1} n + \frac{11 \cdot 3^{16\alpha+16} \cdot 7^{2\beta+1} - 1}{2}\right) q^n \equiv 4q f_7 f_{28}. \tag{3.184}$$

Proof. From the equation (3.32), we have

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(2 \cdot 3^{16\alpha+5} n + \frac{5 \cdot 3^{16\alpha+4} - 1}{2}\right) q^n \equiv 4 \frac{f_2^5 f_6}{f_3^2} \pmod{16}. \tag{3.185}$$

Using (1.9) in (3.185) and then extracting the term involving q^{2n}, we find that

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+5} n + \frac{5 \cdot 3^{16\alpha+4} - 1}{2}\right) q^n \equiv 4 f_1^5 \pmod{8}, \tag{3.186}$$

which is $\beta = 0$ case of (3.171).

Suppose that the equation (3.151) is true for all $\alpha, \beta \geq 0$.

Using (1.16) in (3.171) and then extracting the term involving q^{7n+3}, we obtain (3.172).

From (3.172), we find that

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+5} \cdot 7^{2\beta+2} n + \frac{5 \cdot 3^{16\alpha+4} \cdot 7^{2\beta+2} - 1}{2}\right) q^n \equiv 8 f_1 f_4 \pmod{8}, \tag{3.187}$$

which is $\beta + 1$ case of (3.171). So by induction, the equation (3.171) is hold for all $\alpha, \beta \geq 0$.

The proofs of (3.172)-(3.184) are similar to the proof of (3.171). So, we omit the details. □

Theorem 3.11. *For $\alpha, \beta \geq 0$ and $n \geq 0$, then we have for* $\pmod{16}$

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+3} n + \frac{17 \cdot 3^{16\alpha+2} - 1}{2}\right) q^n \equiv 8q f_1 f_4 f_{12}^3, \tag{3.188}$$

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+5} n + \frac{17 \cdot 3^{16\alpha+4} - 1}{2}\right) q^n \equiv 8q f_1 f_4 f_{12}^3, \tag{3.189}$$

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+7} n + \frac{17 \cdot 3^{16\alpha+6} - 1}{2}\right) q^n \equiv 8q f_1 f_4 f_{12}^3, \tag{3.190}$$

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+9} n + \frac{17 \cdot 3^{16\alpha+8} - 1}{2}\right) q^n \equiv 8q f_1 f_4 f_{12}^3, \tag{3.191}$$

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+11} n + \frac{17 \cdot 3^{16\alpha+10} - 1}{2}\right) q^n \equiv 8q f_1^5 f_{12}^3, \tag{3.192}$$

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+15} n + \frac{17 \cdot 3^{16\alpha+14} - 1}{2}\right) q^n \equiv 8q f_1^5 f_{12}^3 \tag{3.193}$$

and

$$\sum_{n=0}^{\infty} \overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+17} n + \frac{17 \cdot 3^{16\alpha+16} - 1}{2}\right) q^n \equiv 8q f_1^5 f_{12}^3. \tag{3.194}$$

Proof. Using (1.9) in (3.185) and then extracting the term involving q^{2n+1}, we obtain (3.188).

The proofs of (3.189)-(3.194) are similar to the proof of (3.188). So, we omit the details. □

4. Ramanujan-type congruences for $\overline{cpend}_{12,3}(n)$

In this section, we prove Ramanujan-type infinite families of congruences for $\overline{cpend}_{12,3}(n)$.

Theorem 4.1. *Let p be a prime such that $p \equiv 13, 17, 19$ or $23 \pmod{24}$. Then for all $k, m, \alpha, \beta \geq 0$ such that $p \nmid m$, we have for* $\pmod{8}$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+2} 5^{2\beta} p^{2k+1} m + \frac{7 \cdot 3^{16\alpha+1} 5^{2\beta} p^{2k+2} - 1}{2}\right) \equiv 0, \tag{4.1}$$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+4} 5^{2\beta} p^{2k+1} m + \frac{7 \cdot 3^{16\alpha+3} 5^{2\beta} p^{2k+2} - 1}{2}\right) \equiv 0, \tag{4.2}$$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+6} 5^{2\beta} p^{2k+1} m + \frac{7 \cdot 3^{16\alpha+5} 5^{2\beta} p^{2k+2} - 1}{2}\right) \equiv 0, \tag{4.3}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+8}5^{2\beta}p^{2k+1}m+\frac{7\cdot 3^{16\alpha+7}5^{2\beta}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.4}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+10}5^{2\beta}p^{2k+1}m+\frac{7\cdot 3^{16\alpha+9}5^{2\beta}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.5}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+12}5^{2\beta}p^{2k+1}m+\frac{7\cdot 3^{16\alpha+11}5^{2\beta}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.6}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+14}5^{2\beta}p^{2k+1}m+\frac{7\cdot 3^{16\alpha+13}5^{2\beta}p^{2k+2}-1}{2}\right)\equiv 0 \tag{4.7}$$

and

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+16}5^{2\beta}p^{2k+1}m+\frac{7\cdot 3^{16\alpha+15}5^{2\beta}p^{2k+2}-1}{2}\right)\equiv 0. \tag{4.8}$$

Proof. Using (1.1) and (1.2) in (3.81), we find that

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+2}\cdot 5^{2\beta}n+\frac{7\cdot 3^{16\alpha+1}\cdot 5^{2\beta}-1}{2}\right)q^n\equiv 4\sum_{k=-\infty}^{\infty}\sum_{l=0}^{\infty}q^{\frac{k(3k-1)}{2}+l(l+1)},$$

which implies $n=\frac{k(3k-1)}{2}+l(l+1)$. This is equivalent to

$$24n+7=(6k-1)^2+6(2l+1)^2=x^2+6y^2.$$

Thus if $24n+7\neq x^2+6y^2$, then

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+2}\cdot 5^{2\beta}n+\frac{7\cdot 3^{16\alpha+1}\cdot 5^{2\beta}-1}{2}\right)\equiv 0 \pmod 8.$$

Taking $n=p^{2k+1}m+\frac{7(p^{2k+2}-1)}{24}\neq x^2+6y^2$, we obtain (4.1).

The proofs of (4.2)-(4.8) are similar to the proof of (4.1). So, we omit the details. □

Theorem 4.2. *Let p be a prime such that $p\equiv 13,17,19$ or $23 \pmod{24}$. Then for all $k,m,\alpha,\beta\geq 0$ such that $p\nmid m$, we have for* $\pmod 8$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+2}5^{2\beta+1}p^{2k+1}m+\frac{11\cdot 3^{16\alpha+1}5^{2\beta+1}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.9}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+4}5^{2\beta+1}p^{2k+1}m+\frac{11\cdot 3^{16\alpha+3}5^{2\beta+1}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.10}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+2}5^{2\beta+1}p^{2k+1}m+\frac{11\cdot 3^{16\alpha+5}5^{2\beta+1}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.11}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+8}5^{2\beta+1}p^{2k+1}m+\frac{11\cdot 3^{16\alpha+7}5^{2\beta+1}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.12}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+10}5^{2\beta+1}p^{2k+1}m+\frac{11\cdot 3^{16\alpha+9}5^{2\beta+1}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.13}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+12}5^{2\beta+1}p^{2k+1}m+\frac{11\cdot 3^{16\alpha+11}5^{2\beta+1}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.14}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+14}5^{2\beta+1}p^{2k+1}m+\frac{11\cdot 3^{16\alpha+13}5^{2\beta+1}p^{2k+2}-1}{2}\right)\equiv 0 \tag{4.15}$$

and

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+16}5^{2\beta+1}p^{2k+1}m+\frac{11\cdot 3^{16\alpha+15}5^{2\beta+1}p^{2k+2}-1}{2}\right)\equiv 0. \tag{4.16}$$

Proof. The proofs of (4.9)-(4.16) are similar to the proof of (4.1). So, we omit the details. □

Theorem 4.3. *Let p be a prime such that $p \equiv 5$ or $7 \pmod 8$. Then for all $k, m, \alpha, \beta \geq 0$ such that $p \nmid m$, we have for* (mod 8)

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+2}5^{2\beta}p^{2k+1}m+\frac{19\cdot 3^{16\alpha+1}5^{2\beta}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.17}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+4}5^{2\beta}p^{2k+1}m+\frac{19\cdot 3^{16\alpha+3}5^{2\beta}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.18}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+6}5^{2\beta}p^{2k+1}m+\frac{19\cdot 3^{16\alpha+5}5^{2\beta}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.19}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+8}5^{2\beta}p^{2k+1}m+\frac{19\cdot 3^{16\alpha+7}5^{2\beta}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.20}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+10}5^{2\beta}p^{2k+1}m+\frac{19\cdot 3^{16\alpha+9}5^{2\beta}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.21}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+12}5^{2\beta}p^{2k+1}m+\frac{19\cdot 3^{16\alpha+11}5^{2\beta}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.22}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+14}5^{2\beta}p^{2k+1}m+\frac{19\cdot 3^{16\alpha+13}5^{2\beta}p^{2k+2}-1}{2}\right)\equiv 0 \tag{4.23}$$

and

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+16}5^{2\beta}p^{2k+1}m+\frac{19\cdot 3^{16\alpha+15}5^{2\beta}p^{2k+2}-1}{2}\right)\equiv 0. \tag{4.24}$$

Proof. The proofs of (4.17)-(4.24) are similar to the proof of (4.1). So, we omit the details. □

Theorem 4.4. *Let p be a prime such that $p \equiv 5$ or $7 \pmod 8$. Then for all $k, m, \alpha, \beta \geq 0$ such that $p \nmid m$, we have for* (mod 8)

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+2}5^{2\beta+1}p^{2k+1}m+\frac{23\cdot 3^{16\alpha+1}5^{2\beta+1}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.25}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+4}5^{2\beta+1}p^{2k+1}m+\frac{23\cdot 3^{16\alpha+3}5^{2\beta+1}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.26}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+6}5^{2\beta+1}p^{2k+1}m+\frac{23\cdot 3^{16\alpha+5}5^{2\beta+1}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.27}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+8}5^{2\beta+1}p^{2k+1}m+\frac{23\cdot 3^{16\alpha+7}5^{2\beta+1}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.28}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+10}5^{2\beta+1}p^{2k+1}m+\frac{23\cdot 3^{16\alpha+9}5^{2\beta+1}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.29}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+12}5^{2\beta+1}p^{2k+1}m+\frac{23\cdot 3^{16\alpha+11}5^{2\beta+1}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.30}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+14}5^{2\beta+1}p^{2k+1}m+\frac{23\cdot 3^{16\alpha+13}5^{2\beta+1}p^{2k+2}-1}{2}\right)\equiv 0 \tag{4.31}$$

and

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+16}5^{2\beta+1}p^{2k+1}m+\frac{23\cdot 3^{16\alpha+15}5^{2\beta+1}p^{2k+2}-1}{2}\right)\equiv 0. \tag{4.32}$$

Proof. The proofs of (4.25)-(4.32) are similar to the proof of (4.1). So, we omit the details. □

Theorem 4.5. *Let p be a prime such that $p \equiv 5$ or $7 \pmod 8$. Then for all $k, m, \alpha, \beta \geq 0$, we have for $\pmod 8$*

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+2}5^{2\beta}p^{2k+1}m + \frac{11 \cdot 3^{16\alpha+1}5^{2\beta}p^{2k+2} - 1}{2}\right) \equiv 0, \tag{4.33}$$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+4}5^{2\beta}p^{2k+1}m + \frac{11 \cdot 3^{16\alpha+3}5^{2\beta}p^{2k+2} - 1}{2}\right) \equiv 0, \tag{4.34}$$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+6}5^{2\beta}p^{2k+1}m + \frac{11 \cdot 3^{16\alpha+5}5^{2\beta}p^{2k+2} - 1}{2}\right) \equiv 0, \tag{4.35}$$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+8}5^{2\beta}p^{2k+1}m + \frac{11 \cdot 3^{16\alpha+7}5^{2\beta}p^{2k+2} - 1}{2}\right) \equiv 0, \tag{4.36}$$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+10}5^{2\beta}p^{2k+1}m + \frac{11 \cdot 3^{16\alpha+9}5^{2\beta}p^{2k+2} - 1}{2}\right) \equiv 0, \tag{4.37}$$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+12}5^{2\beta}p^{2k+1}m + \frac{11 \cdot 3^{16\alpha+11}5^{2\beta}p^{2k+2} - 1}{2}\right) \equiv 0, \tag{4.38}$$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+14}5^{2\beta}p^{2k+1}m + \frac{11 \cdot 3^{16\alpha+13}5^{2\beta}p^{2k+2} - 1}{2}\right) \equiv 0 \tag{4.39}$$

and

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+16}5^{2\beta}p^{2k+1}m + \frac{11 \cdot 3^{16\alpha+15}5^{2\beta}p^{2k+2} - 1}{2}\right) \equiv 0. \tag{4.40}$$

Proof. Using (1.1) and (1.2) in (3.118) and then extracting the term involving q^n, we see that

$$n = k(k-1) + \frac{3l(l+1)}{2} \implies 24n + 11 = 2(6k-1)^2 + 9(2l+1)^2 = 2x^2 + y^2.$$

Thus if $24n + 11$ is not of the form $2x^2 + y^2$, then

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+2} \cdot 5^{2\beta}n + \frac{11 \cdot 3^{16\alpha+1} \cdot 5^{2\beta} - 1}{2}\right) \equiv 0.$$

Taking $n = p^{2k+1}m + \frac{11(p^{2k+2}-1)}{24}$, which canot be written in the form $2x^2 + y^2$. For such n, we obtain (4.33). The proofs of (4.34)-(4.40) are similar to the proof of (4.33). So, we omit the details. □

Theorem 4.6. *Let p be a prime such that $p \equiv 5$ or $7 \pmod 8$. Then for all $k, m, \alpha, \beta \geq 0$, we have for $\pmod 8$*

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+2} \cdot 5^{2\beta+1}p^{2k+1}m + \frac{11 \cdot 3^{16\alpha+1} \cdot 5^{2\beta+1}p^{2k+2} - 1}{2}\right) \equiv 0, \tag{4.41}$$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+4} \cdot 5^{2\beta+1}p^{2k+1}m + \frac{7 \cdot 3^{16\alpha+3} \cdot 5^{2\beta+1}p^{2k+2} - 1}{2}\right) \equiv 0, \tag{4.42}$$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+6} \cdot 5^{2\beta+1}p^{2k+1}m + \frac{7 \cdot 3^{16\alpha+5} \cdot 5^{2\beta+1}p^{2k+2} - 1}{2}\right) \equiv 0, \tag{4.43}$$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+8} \cdot 5^{2\beta+1}p^{2k+1}m + \frac{7 \cdot 3^{16\alpha+7} \cdot 5^{2\beta+1}p^{2k+2} - 1}{2}\right) \equiv 0, \tag{4.44}$$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+10} \cdot 5^{2\beta+1}p^{2k+1}m + \frac{7 \cdot 3^{16\alpha+9} \cdot 5^{2\beta+1}p^{2k+2} - 1}{2}\right) \equiv 0, \tag{4.45}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+12}\cdot 5^{2\beta+1}p^{2k+1}m+\frac{7\cdot 3^{16\alpha+11}\cdot 5^{2\beta+1}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.46}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+14}\cdot 5^{2\beta+1}p^{2k+1}m+\frac{7\cdot 3^{16\alpha+13}\cdot 5^{2\beta+1}p^{2k+2}-1}{2}\right)\equiv 0 \tag{4.47}$$

and

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+16}\cdot 5^{2\beta+1}p^{2k+1}m+\frac{7\cdot 3^{16\alpha+15}\cdot 5^{2\beta+1}p^{2k+2}-1}{2}\right)\equiv 0. \tag{4.48}$$

Proof. The proofs of (4.41)-(4.48) are similar to the proof of (4.1). So, we omit the details. □

Theorem 4.7. *Let p be a prime such that $p\equiv 5$ or $11 \pmod{12}$. Then for all $k,m,\alpha,\beta\geq 0$ such that $p\nmid m$, we have for* $\pmod{16}$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+3}5^{2\beta}p^{2k+1}m+\frac{13\cdot 3^{16\alpha+2}5^{2\beta}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.49}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+7}5^{2\beta}p^{2k+1}m+\frac{13\cdot 3^{16\alpha+6}5^{2\beta}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.50}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+11}5^{2\beta}p^{2k+1}m+\frac{13\cdot 3^{16\alpha+10}5^{2\beta}p^{2k+2}-1}{2}\right)\equiv 0 \tag{4.51}$$

and

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+15}\cdot 5^{2\beta}p^{2k+1}m+\frac{13\cdot 3^{16\alpha+14}\cdot 5^{2\beta}p^{2k+2}-1}{2}\right)\equiv 0. \tag{4.52}$$

Proof. The proofs of (4.49)-(4.52) are similar to the proof of (4.1). So, we omit the details. □

Theorem 4.8. *Let p be a prime such that $p\equiv 5$ or $11 \pmod{12}$. Then for all $k,m,\alpha,\beta\geq 0$ such that $p\nmid m$, we have for* $\pmod{16}$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+3}5^{2\beta+1}p^{2k+1}m+\frac{17\cdot 3^{16\alpha+2}5^{2\beta+1}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.53}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+7}5^{2\beta+1}p^{2k+1}m+\frac{17\cdot 3^{16\alpha+6}5^{2\beta+1}p^{2k+2}-1}{2}\right)\equiv 0, \tag{4.54}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+11}5^{2\beta+1}p^{2k+1}m+\frac{17\cdot 3^{16\alpha+10}5^{2\beta+1}p^{2k+2}-1}{2}\right)\equiv 0 \tag{4.55}$$

and

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+15}5^{2\beta+1}p^{2k+1}m+\frac{17\cdot 3^{16\alpha+14}5^{2\beta+1}p^{2k+2}-1}{2}\right)\equiv 0. \tag{4.56}$$

Proof. The proofs of (4.53)-(4.56) are similar to the proof of (4.1). So, we omit the details. □

Theorem 4.9. *Let p be a prime such that $p\equiv 5$ or $11 \pmod{12}$. Then for all $k,m,\alpha,\beta\geq 0$ such that $p\nmid m$, we have for* $\pmod{32}$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+5}5^{2\beta}p^{2k+1}m+\frac{13\cdot 3^{16\alpha+4}5^{2\beta}p^{2k+2}-1}{2}\right)\equiv 0 \tag{4.57}$$

and

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+13}5^{2\beta}p^{2k+1}m+\frac{13\cdot 3^{16\alpha+12}5^{2\beta}p^{2k+2}-1}{2}\right)\equiv 0. \tag{4.58}$$

Proof. The proofs of (4.57)-(4.58) are similar to the proof of (4.1). So, we omit the details. □

Theorem 4.10. *Let p be a prime such that $p \equiv 5$ or $11 \pmod{12}$. Then for all $k, m, \alpha, \beta \geq 0$ such that $p \nmid m$, we have for* $\pmod{32}$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+5}5^{2\beta+1}p^{2k+1}m + \frac{17 \cdot 3^{16\alpha+4}5^{2\beta+1}p^{2k+2} - 1}{2}\right) \equiv 0 \tag{4.59}$$

and

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+13}5^{2\beta+1}p^{2k+1}m + \frac{17 \cdot 3^{16\alpha+12}5^{2\beta+1}p^{2k+2} - 1}{2}\right) \equiv 0. \tag{4.60}$$

Proof. The proofs of (4.59)-(4.60) are similar to the proof of (4.1). So, we omit the details. □

Theorem 4.11. *Let p be a prime such that $p \equiv 3 \pmod 4$. Then for all $k, m, \alpha, \beta \geq 0$ with $p \nmid m$, we have for* $\pmod 8$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+3}7^{2\beta}p^{2k+1}m + \frac{5 \cdot 3^{16\alpha+2}7^{2\beta}p^{2k+2} - 1}{2}\right) \equiv 0, \tag{4.61}$$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+5}7^{2\beta}p^{2k+1}m + \frac{5 \cdot 3^{16\alpha+4}7^{2\beta}p^{2k+2} - 1}{2}\right) \equiv 0, \tag{4.62}$$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+7}7^{2\beta}p^{2k+1}m + \frac{5 \cdot 3^{16\alpha+6}7^{2\beta}p^{2k+2} - 1}{2}\right) \equiv 0, \tag{4.63}$$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+9}7^{2\beta}p^{2k+1}m + \frac{5 \cdot 3^{16\alpha+8}7^{2\beta}p^{2k+2} - 1}{2}\right) \equiv 0, \tag{4.64}$$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+11}7^{2\beta}p^{2k+1}m + \frac{5 \cdot 3^{16\alpha+10}7^{2\beta}p^{2k+2} - 1}{2}\right) \equiv 0, \tag{4.65}$$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+15}7^{2\beta}p^{2k+1}m + \frac{5 \cdot 3^{16\alpha+14}7^{2\beta}p^{2k+2} - 1}{2}\right) \equiv 0 \tag{4.66}$$

and

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+17}7^{2\beta}p^{2k+1}m + \frac{5 \cdot 3^{16\alpha+16}7^{2\beta}p^{2k+2} - 1}{2}\right) \equiv 0. \tag{4.67}$$

Proof. The proofs of (4.61)-(4.67) are similar to the proof of (4.1). So, we omit the details. □

Theorem 4.12. *Let p be a prime such that $p \equiv 3 \pmod 4$. Then for all $k, m, \alpha, \beta \geq 0$ with $p \nmid m$, we have for* $\pmod 8$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+3}7^{2\beta+1}p^{2k+1}m + \frac{11 \cdot 3^{16\alpha+2}7^{2\beta+1}p^{2k+2} - 1}{2}\right) \equiv 0, \tag{4.68}$$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+5}7^{2\beta+1}p^{2k+1}m + \frac{11 \cdot 3^{16\alpha+4}7^{2\beta+1}p^{2k+2} - 1}{2}\right) \equiv 0, \tag{4.69}$$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+7}7^{2\beta+1}p^{2k+1}m + \frac{11 \cdot 3^{16\alpha+6}7^{2\beta+1}p^{2k+2} - 1}{2}\right) \equiv 0, \tag{4.70}$$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+9}7^{2\beta+1}p^{2k+1}m + \frac{11 \cdot 3^{16\alpha+8}7^{2\beta+1}p^{2k+2} - 1}{2}\right) \equiv 0, \tag{4.71}$$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+11}7^{2\beta+1}p^{2k+1}m + \frac{11 \cdot 3^{16\alpha+10}7^{2\beta+1}p^{2k+2} - 1}{2}\right) \equiv 0, \tag{4.72}$$

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+15}7^{2\beta+1}p^{2k+1}m + \frac{11 \cdot 3^{16\alpha+14}7^{2\beta+1}p^{2k+2} - 1}{2}\right) \equiv 0 \tag{4.73}$$

and

$$\overline{cpend}_{12,3}\left(4 \cdot 3^{16\alpha+17}7^{2\beta+1}p^{2k+1}m + \frac{11 \cdot 3^{16\alpha+16}7^{2\beta+1}p^{2k+2} - 1}{2}\right) \equiv 0. \tag{4.74}$$

Proof. The proofs of (4.68)-(4.74) are similar to the proof of (4.1). So, we omit the details. □

Theorem 4.13. *If n is not a sum of one, a pentagonal number, four times a pentagonal number and twelve times triangular number, then we have for* (mod 16)

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+3}n+\frac{17\cdot 3^{16\alpha+2}-1}{2}\right)\equiv 0, \tag{4.75}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+5}n+\frac{17\cdot 3^{16\alpha+4}-1}{2}\right)\equiv 0, \tag{4.76}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+7}n+\frac{17\cdot 3^{16\alpha+6}-1}{2}\right)\equiv 0, \tag{4.77}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+9}n+\frac{17\cdot 3^{16\alpha+8}-1}{2}\right)\equiv 0, \tag{4.78}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+11}n+\frac{17\cdot 3^{16\alpha+10}-1}{2}\right)\equiv 0, \tag{4.79}$$

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+15}n+\frac{17\cdot 3^{16\alpha+14}-1}{2}\right)\equiv 0 \tag{4.80}$$

and

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+17}n+\frac{17\cdot 3^{16\alpha+16}-1}{2}\right)\equiv 0. \tag{4.81}$$

Proof. Using (1.1) and (1.2) in (3.188), we find that for (mod 16)

$$\sum_{n=0}^{\infty}\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+3}n+\frac{17\cdot 3^{16\alpha+2}-1}{2}\right)q^n\equiv 8\sum_{k,l=-\infty}^{\infty}\sum_{m=0}^{\infty}q^{1+\frac{k(3k-1)}{2}+2l(3l-1)+6m(m+1)}, \tag{4.82}$$

which implies $n=1+\frac{k(3k-1)}{2}+2l(3l-1)+6m(m+1)$. This is equivalent to

$$24n+17=(6k-1)^2+(12l-2)^2+(12m+6)^2=x^2+y^2+z^2.$$

Thus if $24n+17\neq(6k-1)^2+(12l-2)^2+(12m+6)^2=x^2+y^2+z^2$, then we have

$$\overline{cpend}_{12,3}\left(4\cdot 3^{16\alpha+3}n+\frac{17\cdot 3^{16\alpha+2}-1}{2}\right)\equiv 0.$$

This proves (4.75).

The proofs of (4.76)-(4.81) are similar to the proof of (4.75). So, we omit the details. □

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