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Functional properties for a class of analytic functions associated with the four-leaf domain

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Abstract

By making use of subordination between two analytic functions, the authors introduce a novel subclass of analytic functions related to four-leaf domain defined in the domain of open unit disk. This subclass is related to those defined by Ma and Minda [14] in 1992. We determine the upper bounds on the initial four coefficients of the function belongs to this new class and of the Fekete-Szegö functional and Hankel determinants of various orders. We prove that the Zalcman conjecture holds true for the class if $n = 3$. Further, we investigate the upper bounds of Vandermonde and Toeplitz determinant of different orders. The estimate on modular difference of two successive coefficient of the function for the said class is determined. The method we used consisting of two recent inequalities that are related to the first three coefficients and the inequality between arbitrary coefficients for carathéodory function due to Ravichandran and Verma.

Keywords: Analytic function, subordination, Hankel determinants, Toeplitz determinants, Vandermonde determinants, module difference, Krushkal inequality, Zalcman functional

2020 MSC: 30C45, 30C50, 30C80

1. Introduction

Let $\mathcal{A}$ denote the class of functions f that are analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the normalization $f(0) = f'(0) - 1 = 0$. Every function $f \in \mathcal{A}$ can be represented by a Taylor series expansion of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

For functions $f, g \in \mathcal{A}$, we say that f is *subordinate* to g, denoted by $f \prec g$, if there exists an analytic function γ in $\mathbb{D}$ with $\gamma(0) = 0$ and $|\gamma(z)| < 1$ for $z \in \mathbb{D}$ such that $f(z) = g(\gamma(z))$. According to the subordination principle, if g is univalent in $\mathbb{D}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. The study of coefficient functionals, including the estimation of bounds for the Fekete-Szegö functional and Hankel determinants, gained popularity due to the unsolved Bieberbach conjecture in the 1960s (*cf.* [8, 17]). Estimating the bounds of Hankel matrices is more than an exercise in complex analysis (*cf.* [29]); it is a gateway to understanding the coefficients of univalent functions.

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Given their widespread utility in areas like signal processing and operator theory, refining these estimates is a high-priority task in modern mathematics. In 1976, Noonan and Thomas [16] stated the q^{th} Hankel determinant for $q \geq 1$ and $n \geq 1$ of function f as:

$$H_q(n)(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \tag{1.2}$$

for $q, n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. It may be noted that for $q = 2,\ n = 1$

$$H_2(1)(f) = a_3 - a_2^2 \tag{1.3}$$

and $q = 2,\ n = 2$, we have

$$H_2(2)(f) = a_2 a_4 - a_3^2, \tag{1.4}$$

where $H_2(1)(f)$ is popularly known as Fekete-Szegö functional and $H_2(2)(f)$ is known as second Hankel determinant. While both Hankel and Toeplitz determinants are central to the study of analytic functions, they possess distinct structural properties. Specifically, Toeplitz matrices are characterized by constant entries along their main diagonals, whereas Hankel matrices exhibit constant values along their skew (or reverse) diagonals. Building upon these properties, Thomas and Halim [25] formulated the symmetric Toeplitz determinant $T_q(n)$ for functions within the class $\mathcal{A}$. For $n, q \in \mathbb{N}$ and assuming $a_1 = 1$, the determinant is defined by

$$T_q(n)(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix}. \tag{1.5}$$

Also, Allu et al. [1] studied Hankel, Toeplitz and Hermitian-Toeplitz determinant for certain class of close-to-convex functions. Sunthrayuth et al. [24] established sharp bounds for the first four initial coefficients a_n (n=2,3,4,5), alongside the Fekete-Szegö functional, the Zalcman and Krushkal inequalities, and the estimation of the second-order Hankel determinant $H_2(2)$ for a subclass of functions with bounded turning related to a four-leaf-type domain. Similarly, the work of Shi et al. [22] provided an investigation into the sharp coefficient bounds and the second and third-order Hankel determinants, $H_2(2)$ and $H_3(1)$, for a specific subclass of starlike functions. Furthermore, Kamali and Riskulova [7] analyzed the upper bounds of Toeplitz determinants for a subclass of analytic functions. Motivated by above mentioned researchers, we introduce the concept of Vandermonde matrix for our study.

Vandermonde matrix is a matrix with the terms of a geometric progression in each row. It is made popular by Alexandre–Theophile Vandermonde (*cf.* [15]). The q-th Vandermonde determinant $q \geq 1$ and $n \geq 1$ for a function $f \in \mathcal{A}$ is $q \times q$ matrix defined as follows:

$$V_{q,n}(f) = \begin{vmatrix} 1 & a_n & a_n^2 & \cdots & a_n^{q-1} \\ 1 & a_{n+1} & a_{n+1}^2 & \cdots & a_{n+1}^{q-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_{n+q-1} & a_{n+q-1}^2 & \cdots & a_{n+q-1}^{q-1} \end{vmatrix} = \prod_{1 \leq i < j \leq q} (a_{n+j-1} - a_{n+i-1}). \tag{1.6}$$

For $q = 2$ and $n = 1$:

$$V_{2,1}(f) = (a_2 - a_1). \tag{1.7}$$

For $q = 3$ and $n = 1$:

$$V_{3,1}(f) = (a_2 - a_1)(a_3 - a_1)(a_3 - a_2). \tag{1.8}$$

For $q = 4$ and $n = 1$:

$$V_{4,1}(f) = (a_2 - a_1)(a_3 - a_1)(a_4 - a_1)(a_3 - a_2)(a_4 - a_2)(a_4 - a_3). \tag{1.9}$$

For $q = 5$ and $n = 1$:

$$V_{5,1}(f) = (a_2 - a_1)(a_3 - a_1)(a_4 - a_1)(a_5 - a_1)(a_3 - a_2)(a_4 - a_2)(a_5 - a_2)(a_4 - a_3)(a_5 - a_3)(a_5 - a_4). \tag{1.10}$$

There are various applications of Vandermonde determinant in different fields. The study of extreme points of the Vandermonde determinant are seen to play a significant role in both physical and biological sciences based on the zeros of the classical orthogonal polynomials, the Gaussian ensembles and the Wishart ensembles in symmetric cones of Jordan algebras. Vijayalakshmi et al. [26] determined the coefficient bounds for the second and third order Vandermonde determinant for a Sakaguchi type function associated with Limacon domain. For more details about Vandermonde determinant see [20, 21, 27, 28].

In 2022, Alshehry et al. [2] (also see [6]) explored and examined various subclasses of analytic functions defined by their subordination to the four-leaf function $\phi_{4L}(z)$, as shown in Figure 1, generated using MAPLE™ 2023 software.

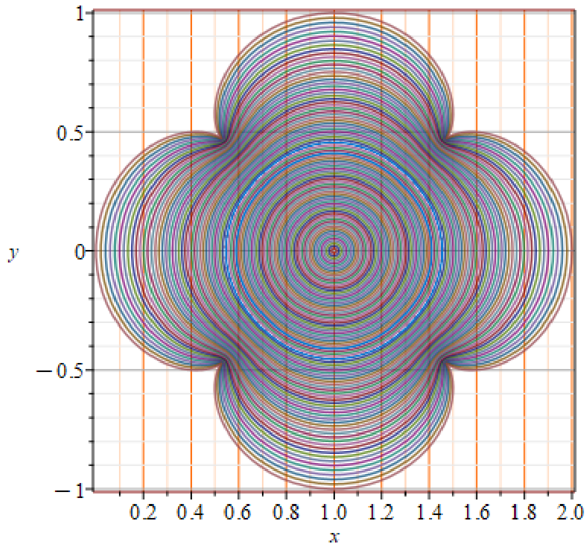


Figure 1. The image of $\phi_{4L}(z)$

Code for the image of $\phi_{4L}(z)$ as follows.

```
G := 1 + 5/6*z + z^5/6;
x:=cos(t);y:=sin(t);
expr_Re_G:=Re(eval(expr_G,z=(k/nr)*(x+I*y))):expr_Im_G:=Im(eval(expr_G,z=(k/nr)*(x+I*y))):
plot([seq([expr_Re_G,expr_Im_G,t=0..2*Pi],k=1..nr)],scaling=constrained,size = [400, 500]);
```

Using the four-leaf function, we define the following subclass of $\mathcal{A}$ through the concept of subordination.

Definition 1.1. A function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^r_{\phi_{4L}}$ if

$$\frac{zf'(z)}{(1-r)f(z) + rz} \prec 1 + \frac{5}{6}z + \frac{1}{6}z^5 =: \phi_{4L}(z) \quad (0 \le r \le 1;\ z \in \mathbb{D}), \tag{1.11}$$

holds.
Note that, the function ϕ_{4L} maps the unit disk $\mathbb{D}$ onto a four-leaf shaped domain as shown in Figure 1.

Note that, for $r = 0$, the class $S^r_{\phi_{4L}}$ reduces to the class $\mathcal{S}^*_{\phi_{4\mathcal{L}}}$ as

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{5}{6}z + \frac{1}{6}z^5 \quad (z \in \mathbb{D})$$

(*cf.* [22]), and for $r = 1$, the class $S^r_{\phi_{4L}}$ reduces to the class $\mathcal{BT}_{\phi_{4\mathcal{L}}}$ as

$$f'(z) \prec 1 + \frac{5}{6}z + \frac{1}{6}z^5 \quad (z \in \mathbb{D})$$

(*cf.* [24]).

Remark 1.2. This subclass combines bounded turning for $r = 1$ and starlike for $r = 0$. For $r = 1$, the functions are bounded turning, meaning their images lie within a bounded region, while for $r = 0$, the functions are starlike, mapping the unit disk to a region star-shaped with respect to the origin.

Remark 1.3. Let f be analytic in $\mathbb{D}$. The function f is starlike with respect to $w_0 = f(0)$ if f is univalent in $\mathbb{D}$ and $f(\mathbb{D})$ is a starlike domain with respect to w_0, i.e., $[w_0, f(z)] \subset f(\mathbb{D})$ for all $z \in \mathbb{D}$. This holds if and only if $f'(0) \neq 0$ and

$$\Re\frac{zf'(z)}{f(z) - w_0} > 0, \quad z \in \mathbb{D}.$$

For ϕ_{4L}, since $\phi_{4L}(0) = 1$ and $\phi'_{4L}(0) = \frac{5}{6} \neq 0$,

$$\Re\frac{z\phi'_{4L}(z)}{\phi_{4L}(z) - \phi_{4L}(0)} = 5\Re\frac{1+z^4}{5+z^4} > 0, \quad z \in \mathbb{D}.$$

Thus, ϕ_{4L} is starlike in $\mathbb{D}$ with respect to $\phi_{4L}(0) = 1$. The symmetry $(\phi_{4L}(1) + \phi_{4L}(-1))/2 = 1$ and $\phi_{4L}(z) = \phi_{4L}(\bar{z})$ ensures $\phi_{4L}(\mathbb{D})$ is symmetric about both $w_0 = 1$ and the real axis.

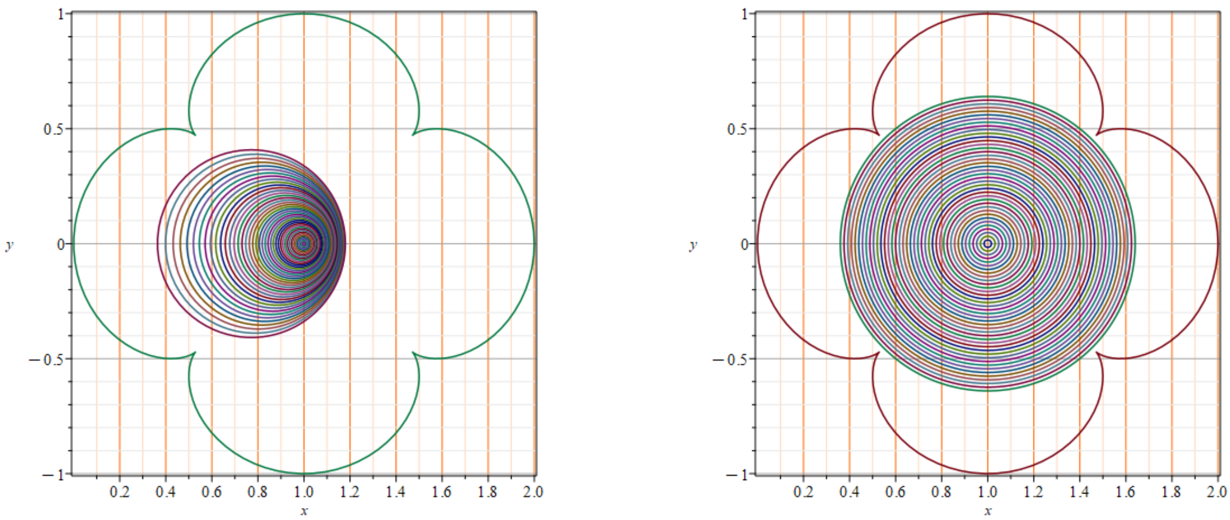
Finally, $\Re\{\phi_{4L}(z)\} > 0$ for $z \in \mathbb{D}$ since

$$\Re\{\phi_{4L}(z)\} = 1 + \Re\left(\frac{5}{6}z + \frac{1}{6}z^5\right) \geq 1 - \frac{5}{6}|z| - \frac{1}{6}|z|^5 \geq 1 - \frac{5}{6} - \frac{1}{6} = 0.$$

Remark 1.4. The class is well-defined if we can establish that it is non-empty. First, from Remark 1.3, we know that the function $\phi_{4L}(z) = 1 + \frac{5}{6}z + \frac{1}{6}z^5$ is univalent in $\mathbb{D}$. This demonstrates that there exists a function in the class. To further establish that the class $S^r_{\phi_{4L}} \neq \emptyset$ for $r = 0$, we choose an appropriate function $\mathfrak{m}$. Let us consider the function

$$\mathfrak{m}(z) := z + 0.28z^2 \in \mathcal{A}.$$

This function belongs to the class of analytic functions $\mathcal{A}$, and it is shown in Figure 2(a).



(a) Non empty for $r = 0$ (b) Non empty for $r = 1$

Figure 2. Figures for the Remark 1.4 of the class $S^r_{\phi_{4L}}$

Next, to prove that $S^r_{\phi_{4L}} \neq \emptyset$ for $r = 1$, we need to choose an appropriate function $\mathfrak{n}$ and demonstrate that it satisfies the conditions for bounded turning. Consider the function

$$\mathfrak{n}(z) := z + 0.32z^2 \in \mathcal{A}.$$

This function is also an element of the class $\mathcal{A}$, and it is shown in Figure 2(b).

Thus, by considering these specific examples of $m(z)$ and $n(z)$, we have demonstrated that the class $S^r_{\phi_{4L}}$ is not empty for both $r = 0$ and $r = 1$, for appropriate choices of the functions $\mathfrak{m}$ and $\mathfrak{n}$.

```
Code for the image of Non empty for the class $S^{r}_{\phi_{4L}}$ for $r=0$ as follows.
G := 1 + 5/6*z + z^5/6;
x:=cos(t);y:=sin(t);
Re_G:=Re(eval(expr_G,z=x+I*y)):Im_G:=Im(eval(expr_G,z=x+I*y)):
plot([Re_G,Im_G,t=0..2*Pi]);
F := 1 + a*w/(2*a*w + 1);U:=cos(u);V:=sin(u);
alpha := Re(eval(ex_F, w = U + V*I));beta := Im(eval(ex_F, w = U + V*I));
expr_Re_f:=Re(eval(ex_F,w=(k/nr)*(U+I*V))):expr_Im_f:=Im(eval(ex_F,w=(k/nr)*(U+I*V))):
plot([seq([expr_Re_f,expr_Im_f,u=0..2*Pi],k=1..nr)],scaling=constrained);
```

```
Code for the image of Non empty for the class $S^{r}_{\phi_{4L}}$ for $r=1$ as follows.
G := 1 + 5/6*z + z^5/6;
x:=cos(t);y:=sin(t);
Re_G:=Re(eval(expr_G,z=x+I*y)):Im_G:=Im(eval(expr_G,z=x+I*y)):
plot([Re_G,Im_G,t=0..2*Pi]);
F := 2*a*w + 1
U:=cos(u);V:=sin(u);
alpha := Re(eval(ex_F, w = U + V*I));beta := Im(eval(ex_F, w = U + V*I));
expr_Re_f:=Re(eval(ex_F,w=(k/nr)*(U+I*V))):expr_Im_f:=Im(eval(ex_F,w=(k/nr)*(U+I*V))):
plot([seq([expr_Re_f,expr_Im_f,u=0..2*Pi],k=1..nr)],scaling=constrained);
```

1.1. Preliminaries

In this study, we establish upper bounds for the coefficients of functions within the previously defined class. To derive our primary results, we utilize the following fundamental lemmas.

Lemma 1.5 (*cf.* [11])**.** *Let $\mathcal{P}$ denote the class of functions with a positive real part in $\mathbb{D}$. A function $p \in \mathcal{P}$ has the form*

$$p(z) = 1 + \sum_{n=1}^{\infty} d_n z^n, \quad z \in \mathbb{D}, \tag{1.12}$$

that is analytic in $\mathbb{D}$ and satisfies the condition $\Re(p(z)) > 0$. Then, for $n \geq 1$

$$|d_n| \leq 2 \tag{1.13}$$

and

$$2d_2 = d_1^2 + t\xi, \tag{1.14}$$

$$4d_3 = d_1^3 + 2d_1 t\xi - d_1 t\xi^2 + 2t\left(1 - |\xi|^2\right)\eta, \tag{1.15}$$

$$8d_4 = d_1^4 + 3d_1^2 t\xi + \left(4 - 3d_1^2\right)t\xi^2 + d_1^2 t\xi^3 + 4t\left(1 - |\xi|^2\right)\left(1 - |\eta|^2\right)a + 4t\left(1 - |\xi|^2\right)\left(d_1\eta - d_1\xi\eta - \bar{\xi}\eta^2\right), \tag{1.16}$$

for some $\xi, \eta, \gamma \in \bar{\mathbb{D}}$ and $t = \left(4 - d_1^2\right)$.

Lemma 1.6 (*cf.* [8, 14])**.** *Let $d \in \mathcal{P}$ be of the form* (1.12)*. If $\mu \geq 0$ then*

$$|d_{n+k} - \mu d_n d_k| \leq 2 \max\{1; |2\mu - 1|\} = \begin{cases} 2, & \text{if} \quad 0 \leq \mu \leq 1, \\ 2|2\mu - 1|, & \text{otherwise}. \end{cases} \tag{1.17}$$

If $0 < \mu < 1$, the inequality is sharp for the function $d(z) = \dfrac{1 + z^{n+k}}{1 - z^{n+k}}$. In the other cases, the inequality is sharp for the function $d(z) = \dfrac{1+z}{1-z}$.

Inequality (1.17) represents Lemma 2.3 of [18], that for $\mu = 1$ was proved in a more general form in [12, p. 546, Lemma 1]. We emphasize that the inequality (1.17) remains valid for all $\mu \in \mathbb{C}$ as it was proved in [5, Theorem 1].

Lemma 1.7 (*cf.* [3])**.** *If $p \in \mathcal{P}$ and has the series of the form* (1.12)*, then*

$$|Jc_1^3 - Kc_1c_2 + Lc_3| \leq 2|J| + 2|K - 2J| + 2|J - K + L|,$$

where J, K and L are real numbers.

2. Main results

This section is devoted to the derivation of coefficient bounds for the class $S^r_{\phi_{4L}}$. We focus specifically on the first four coefficients, the Fekete-Szegö functional and the second Hankel determinant. Furthermore, we extend our analysis to the Vandermonde determinant up to the fifth order to establish upper limits for functions within this class.

2.1. Initial coefficients estimates for the class $S^r_{\phi_{4L}}$

We initiate our study of the class $S^r_{\phi_{4L}}$ by determining the upper bounds for the initial coefficients a_n $(n = 2, 3, 4, 5)$. These preliminary estimates are essential for the evaluation of the second Hankel determinant and the Vandermonde determinant upto fifth order. Our main findings for these coefficients are summarized in the theorem below.

Theorem 2.1. *Let the function $f \in \mathcal{A}$ given by* (1.1) *be a member of the class $S^r_{\phi_{4L}}$. Then,*

$$|a_2| \leq \frac{5}{6(1+r)}, \quad |a_3| \leq \frac{5}{6(2+r)}, \quad |a_4| \leq \frac{5}{6(3+r)}, \quad |a_5| \leq \frac{5\,(7r+17)\,(2r+1)}{18\,(r+4)\,(r+3)\,(1+r)}. \tag{2.1}$$

Proof. If $f \in S^r_{\phi_{4L}}$, then by Definition 1.1 and the principle of subordination, there exists an analytic function γ satisfying $\gamma(0) = 0$ and $|\gamma(z)| < 1$ for $z \in \mathbb{D}$, such that:

$$\frac{zf'(z)}{(1-r)\,f(z) + rz} = 1 + \frac{5}{6}\gamma(z) + \frac{1}{6}\gamma(z)^5, \quad z \in \mathbb{D}. \tag{2.2}$$

Writing the analytic function γ in terms of $e \in \mathcal{P}$, that is

$$e(z) = \frac{1 + \gamma(z)}{1 - \gamma(z)} = 1 + e_1 z + e_2 z^2 + \cdots, \quad z \in \mathbb{D}.$$

By representing $\gamma(z)$ through the coefficients of $e(z)$, we arrive at the following expression

$$\gamma(z) = \frac{1}{2}e_1 z + \left(\frac{1}{2}e_2 - \frac{1}{4}e_1^2\right)z^2 + \left(\frac{1}{8}e_1^3 - \frac{1}{2}e_1e_2 + \frac{1}{2}e_3\right)z^3 + \left(\frac{1}{2}e_4 - \frac{1}{2}e_1e_3 - \frac{1}{4}e_2^2 - \frac{1}{16}e_1^4 + \frac{3}{8}e_1^2e_2\right)z^4 + \cdots, \quad z \in \mathbb{D}.$$

Upon substituting $\gamma(z)$ into the right-hand side of (2.2), we arrive at

$$\begin{aligned} 1 + \frac{5}{6}\gamma(z) + \frac{1}{6}\gamma(z)^5 \quad = \quad & 1 + \frac{5}{12}e_1 z + \left(\frac{5e_2}{12} - \frac{5e_1^2}{24}\right)z^2 + \left(\frac{5}{12}e_3 - \frac{5}{12}e_1e_2 + \frac{5}{48}e_1^3\right)z^3 \\ & + \left(\frac{5}{12}e_4 - \frac{5}{12}e_1e_3 - \frac{5}{24}e_2^2 + \frac{5}{16}e_2e_1^2 - \frac{5}{96}e_1^4\right)z^4 + \cdots. \end{aligned} \tag{2.3}$$

Expanding the left-hand side of (2.2) in a Maclaurin series with respect to z, we obtain

$$\begin{aligned}\frac{zf'(z)}{(1-r)f(z)+rz} &= 1 + a_2(1+r)z + \Big((r^2-1)a_2^2 + a_3(r+2)\Big)z^2 + \Big((1+r)(-1+r)^2a_2^3 \\ &\quad + a_3(2r^2+r-3)a_2 + a_4(r+3)\Big)z^3 + \Big((1+r)(-1+r)^3a_2^4 + (-1+r)^2(3r+4)a_3a_2^2 \\ &\quad + 2a_4(r+2)(-1+r)a_2 + r^2a_3^2 + \Big(a_3^2+a_5\Big)r - 2a_3^2 + 4a_5\Big)z^4 + \cdots . \end{aligned} \tag{2.4}$$

Equating the coefficients of z, z^2, z^3 and z^4 from the relations (2.3) and (2.4) we obtain

$$a_2 = \frac{5e_1}{12(1+r)}, \tag{2.5}$$

$$a_3 = \frac{5e_2}{12(r+2)} - \frac{5(11r+1)e_1^2}{144(1+r)(r+2)}, \tag{2.6}$$

$$a_4 = \frac{5\Big[(121r^2+88r+7)e_1^3 - \Big(264r^2+492r+108\Big)e_1e_2 + (144r^2+432r+288)e_3\Big]}{1728(1+r)(r+2)(r+3)}, \tag{2.7}$$

$$\begin{aligned} a_5 &= -\frac{(7920r^3+36720r^2+43920r+15120)e_2^2}{20736(1+r)(r+2)(r+3)(r+4)} + \frac{(21780r^3+71280r^2+54180r+8280)e_2e_1^2}{20736(1+r)(r+2)(r+3)(r+4)} \\ &\quad - \frac{(15840r^3+73440r^2+95040r+23040)e_1e_3}{20736(1+r)(r+2)(r+3)(r+4)} + \frac{(8640r^3+51840r^2+95040r+51840)e_4}{20736(1+r)(r+2)(r+3)(r+4)} \\ &\quad - \frac{(6655r^3+12705r^2+6105r+455)e_1^4}{20736(1+r)(r+2)(r+3)(r+4)}. \end{aligned} \tag{2.8}$$

Using (1.13) of Lemma 1.5 in (2.5) we get the bounds of $|a_2|$. From the relation (2.6), we have

$$a_3 = \frac{5}{12(2+r)}\left(e_2 - \frac{11r+1}{12(1+r)}e_1^2\right). \tag{2.9}$$

Taking modulus on both sides of (2.9) and applying (1.17) of Lemma 1.6, we get

$$|a_3| = \frac{5}{12(2+r)}\left|e_2 - \frac{11r+1}{12(1+r)}e_1^2\right| \leq \frac{5}{12(2+r)}\ 2\ \max\left\{1; \left|\frac{5r-5}{6r-6}\right|\right\} = \frac{5}{6(2+r)}.$$

Now rearranging the terms of (2.7) and applying modulus we get

$$|a_4| = \left|\frac{5(121r^2+88r+7)e_1^3}{1728(1+r)(r+2)(r+3)} - \frac{5\Big(264r^2+492r+108\Big)e_1e_2}{1728(1+r)(r+2)(r+3)} + \frac{5(144r^2+432r+288)e_3}{1728(1+r)(r+2)(r+3)}\right|. \tag{2.10}$$

Now applying Lemma 1.7 in (2.10) we get our desired estimate for $|a_4|$. Now rearranging the terms of (2.8) and applying triangle inequality, we get

$$\begin{aligned} |a_5| &\leq \frac{1}{20736(1+r)(r+2)(r+3)(r+4)}\Big[\Big|(8640r^3+51840r^2+95040r+51840)e_4 \\ &\quad -(7920r^3+36720r^2+43920r+15120)e_2^2\Big| \\ &\quad +|e_1|\Big|(6655r^3+12705r^2+6105r+455)e_1^3 - (21780r^3+71280r^2+54180r+8280)e_1e_2 \\ &\quad +(15840r^3+73440r^2+95040r+23040)e_3\Big|\Big] \\ &= \frac{(8640r^3+51840r^2+95040r+51840)}{20736(1+r)(r+2)(r+3)(r+4)}\left|e_4 - \frac{(7920r^3+36720r^2+43920r+15120)}{(8640r^3+51840r^2+95040r+51840)}e_2^2\right| \\ &\quad + \frac{|e_1|}{20736(1+r)(r+2)(r+3)(r+4)}\Big|(6655r^3+12705r^2+6105r+455)e_1^3 \\ &\quad -(21780r^3+71280r^2+54180r+8280)e_1e_2 + (15840r^3+73440r^2+95040r+23040)e_3\Big|. \end{aligned} \tag{2.11}$$

Using (1.13) of Lemma 1.5 and (1.17) of Lemma 1.6 and Lemma 1.7 in (2.11), we have the following:

$$|a_5| \leq \frac{5}{6(r+4)} \max\left\{1; \frac{5}{6}\left|\frac{r-1}{r+2}\right|\right\} + \frac{5\,(11r+13)\,(11r+7)\,(11r+1)}{5184\,(1+r)\,(r+2)\,(r+3)\,(r+4)} + \frac{5\,(77r+67)\left(11r^2+50r+11\right)}{2592\,(1+r)\,(r+2)\,(r+3)\,(r+4)} + \frac{5\,(r+17)\left(143r^2+542r+179\right)}{5184\,(1+r)\,(r+2)\,(r+3)\,(r+4)}.$$

After simplifying, we get

$$|a_5| \leq \frac{5}{6(r+4)} + \frac{5(11r^2+29r+8)}{18\,(r+4)\,(r+3)\,(1+r)} = \frac{5\,(7r+17)\,(2r+1)}{18\,(r+4)\,(r+3)\,(1+r)}.$$

□

For $r=0$, we obtain the following result due to Shi et al. [22].

Corollary 2.2 (*cf.* [22])**.** *If $f \in S^*_{\phi_{4L}}$, then*

$$|a_2| \leq \frac{5}{6},\;\; |a_3| \leq \frac{5}{12},\;\; |a_4| \leq \frac{5}{18},\;\; |a_5| \leq \frac{85}{216}.$$

The first three inequalities are sharp.

For $r=1$, we obtain the result of Sunthrayuth et al. [24] as given below:

Corollary 2.3 (*cf.* [24])**.** *If $f \in \mathcal{BT}_{\phi_{4L}}$, then*

$$|a_2| \leq \frac{5}{12},\;\; |a_3| \leq \frac{5}{18},\;\; |a_4| \leq \frac{5}{24},\;\; |a_5| \leq \frac{1}{2}.$$

The first three bounds are best possible.

Theorem 2.4. *Let the function $f \in S^r_{\phi_{4L}}$. Then*

$$|a_3-\mu a_2^2| \leq \frac{5}{6(2+r)} \max\left\{1, \frac{5}{6}\left|\frac{r^2-1+\mu(2+r)}{(1+r)^2}\right|\right\}. \tag{2.12}$$

Proof. If $f \in S^r_{\phi_{4L}}$, making use of (2.5) and (2.6), we obtain

$$a_3-\mu a_2^2 = \frac{5}{12(2+r)}e_2 - \frac{(1+11r)(1+r)+5\mu(2+r)}{12(1+r)^2}e_1^2. \tag{2.13}$$

Now taking modulus on both sides of (2.13) and application of (1.17) of Lemma 1.6 in the resulting equation, we get

$$\begin{aligned}|a_3-\mu a_2^2| &= \frac{5}{12(2+r)}\left|e_2 - \frac{\left(5\mu r+11r^2+10\mu+12r+1\right)(2+r)}{5\,(1+r)^2}e_1^2\right| \\ &\leq \frac{5}{6(2+r)} \max\left\{1, \frac{5}{6}\left|\frac{r^2-1+\mu(2+r)}{(1+r)^2}\right|\right\}.\end{aligned}$$

□

Letting $r=0$ in (2.12) we obtain the result of Shi et al. [22] as follows:

Corollary 2.5. *If $f \in S^*_{\phi_{4L}}$, then*

$$|a_3-\mu a_2^2| \leq \frac{5}{12}\max\left\{1, \left|\frac{5(2\mu-1)}{6}\right|\right\}.$$

This inequality is sharp.

Putting $r = 1$, in (2.12), the following result is obtained (*cf.* [24]).

Corollary 2.6. *If $f \in \mathcal{BT}_{\phi_{4\mathcal{L}}}$, then*

$$|a_3 - \mu a_2^2| \leq \frac{5}{18} \max\left\{1, \frac{5|\mu|}{8}\right\}.$$

This inequality is sharp.

Corollary 2.7. *If $f \in S^r_{\phi_{4L}}$, then*

$$|a_4 - a_2 a_3| \leq \frac{5}{6(3+r)}. \tag{2.14}$$

This inequality is sharp.

Proof. From (2.5), (2.6) and (2.7), we have

$$\begin{aligned} a_4 - a_2 a_3 &= \frac{(605r^3 + 1320r^2 + 1325r + 110)e_1^3}{1728\,(1+r)^2\,(r+2)\,(r+3)} - \frac{\left(1320r^3 + 4080r^2 + 4200r + 1440\right)e_1 e_2}{1728\,(1+r)^2\,(r+2)\,(r+3)} \\ &\quad + \frac{\left(720r^3 + 2880r^2 + 3600r + 1440\right)e_3}{1728\,(1+r)^2\,(r+2)\,(r+3)}. \end{aligned} \tag{2.15}$$

Now applying modulus on both sides of (2.15) and followed by application of Lemma 1.7 we get our desired result. □

Putting $r = 0$, in (2.14) we obtain the result due to Shi et al. [22].

Corollary 2.8. *If $f \in S^*_{\phi_{4L}}$, then*

$$|a_4 - a_2 a_3| \leq \frac{5}{18}.$$

This inequality is sharp.

Putting $r = 1$ in (2.14) we have the following corollary:

Corollary 2.9 (*cf.* [24])**.** *If $f \in \mathcal{BT}_{\phi_{4\mathcal{L}}}$, then*

$$|a_4 - a_2 a_3| \leq \frac{5}{24}.$$

This inequality is best possible.

Theorem 2.10. *If $f \in S^r_{\phi_{4L}}$ then*

$$|V_{2,1}(f)| \leq \frac{(11+6r)}{6(1+r)}, \tag{2.16}$$

$$|V_{3,1}(f)| \leq \frac{5(3+2r)(17+6r)(11+6r)}{216(1+r)^2(2+r)^2)}, \tag{2.17}$$

$$|V_{4,1}(f)| \leq \frac{125(864r^5 + 10800r^4 + 52560r^3 + 124520r^2 + 143606r + 64515)}{23328(3+r)^3(2+r)^2(1+r)^3}, \tag{2.18}$$

$$\begin{aligned} |V_{5,1}(f)| \leq{}& \frac{15625}{15116544(4+r)^4(3+r)^3(2+r)^3(1+r)^4}(20736r^8 + 483840r^7 + 4852224r^6 + 27315840r^5 \\ &+ 94403104r^4 + 205049440r^3 + 273268736r^2 + 204210680r + 65482725). \end{aligned} \tag{2.19}$$

Proof. By substituting $a_1 = 1$ into (1.7) and using (2.5), along with applying equation (1.13) of Lemma 1.5, we obtain the desired estimation for (2.16). Now

$$|V_{3,1}(f)| = |a_3 - a_2||a_3 - a_1||a_2 - a_1| = |a_3 - a_2||a_3 - a_1||V_{2,1}(f)|. \tag{2.20}$$

From (2.5) and (2.6), we have

$$a_3 - a_2 = \frac{5\,(1+r)\,e_2}{12\,(1+r)\,(r+2)} - \frac{5\,(11r+1)\,e_1^2}{144\,(1+r)\,(r+2)} - \frac{5}{12+12r}e_1. \tag{2.21}$$

Now taking modulus and applying triangle inequality in (2.21), we get

$$|a_3 - a_2| \leq \frac{5}{12(r+2)}\left|e_2 - \frac{(11r+1)\,e_1^2}{12\,(1+r)}\right| + \frac{5}{12+12r}|e_1|. \tag{2.22}$$

Applying (1.12) of Lemma 1.5 and (1.17) of Lemma 1.6, we get

$$|a_3 - a_2| \leq \frac{5(3+2r)}{6(1+r)(2+r)}. \tag{2.23}$$

Using (2.6) and $a_1 = 1$, we get

$$a_3 - a_1 = -\frac{5(-12-12r)e_2}{144(1+r)(r+2)} - \frac{5(11re_1^2 + e_1^2)}{144(1+r)(r+2)} - 1. \tag{2.24}$$

Now taking modulus on both sides of (2.24) and applying triangle inequality, we get

$$|a_3 - a_1| \leq \frac{5}{12(2+r)}\left|e_2 - \frac{11r+1}{12(1+r)}e_1^2\right| + 1. \tag{2.25}$$

Using (1.17) of Lemma 1.6 in (2.25), we get

$$|a_3 - a_1| \leq \frac{17+6r}{6(2+r)}. \tag{2.26}$$

Making use of (2.23), (2.26) and (2.16) in (2.20) we get the desired bound of $|V_{3,1}(f)|$. Now taking modulus in (1.9), we get

$$|V_{4,1}(f)| = |a_4 - a_3||a_4 - a_2||a_4 - a_1||V_{3,1}(f)|. \tag{2.27}$$

Now using (2.6) and (2.7), we get

$$\begin{aligned} a_4 - a_3 \;=\; & \frac{1}{1728\,(1+r)\,(r+2)\,(r+3)}\Big[\Big(605r^2 + 440r + 35\Big)e_1^3 - \Big(1320r^2 + 2460r + 540\Big)e_2e_1 \\ & + \Big(720r^2 + 2160r + 1440\Big)e_3 + \Big(-720r^2 - 2880r - 2160\Big)e_2 + \Big(660r^2 + 2040r + 180\Big)e_1^2\Big]. \end{aligned} \tag{2.28}$$

Now taking modulus and applying triangle inequality in (2.28) we get

$$\begin{aligned} |a_4 - a_3| \;\leq\; & \frac{1}{1728\,(1+r)\,(r+2)\,(r+3)}\Big[\Big|\Big(605r^2 + 440r + 35\Big)e_1^3 - \Big(1320r^2 + 2460r + 540\Big)e_2e_1 \\ & + \Big(720r^2 + 2160r + 1440\Big)e_3\Big| + \Big|\Big(-720r^2 - 2880r - 2160\Big)e_2 + \Big(660r^2 + 2040r + 180\Big)e_1^2\Big|\Big]. \end{aligned} \tag{2.29}$$

Now applying (1.17) of Lemma 1.6 and Lemma 1.7 in (2.29), we get

$$|a_4 - a_3| \leq \frac{5(720 + 1008r + 288r^2)}{6(1+r)(2+r)}. \tag{2.30}$$

By utilizing (2.5) and (2.7), we obtain

$$\begin{aligned} a_4 - a_2 &= \frac{1}{1728\,(1+r)\,(r+2)\,(r+3)}\Big[\Big(605r^2+440r+35\Big)e_1^3 - \Big(1320r^2+2460r+540\Big)e_2e_1 \\ &\quad + \Big(720r^2+2160r+1440\Big)e_3 + \Big(-720r^2-3600r-4320\Big)e_1\Big]. \end{aligned} \tag{2.31}$$

Now taking modulus on both sides of (2.31) and applying triangle inequality we get

$$\begin{aligned} |a_4 - a_2| &\leq \frac{1}{1728\,(1+r)\,(r+2)\,(r+3)}\Big[\Big|\Big(605r^2+440r+35\Big)e_1^3 - \Big(1320r^2+2460r+540\Big)e_2e_1 \\ &\quad + \Big(720r^2+2160r+1440\Big)e_3\Big| + \Big|\Big(-720r^2-3600r-4320\Big)e_1\Big|\Big]. \end{aligned} \tag{2.32}$$

By using (1.13) of Lemma 1.5 and Lemma 1.7 in (2.32), we get

$$|a_4 - a_2| \leq \frac{5(r^2+4r+4)}{3(1+r)(2+r)(3+r)}. \tag{2.33}$$

Similarly using (2.7) we get

$$\begin{aligned} a_4 - a_1 &= \frac{5}{1728(1+r)(r+2)(r+3)}((121r^2+88r+7)e_1^3 - (264r^2+492r+108)e_1e_2 \\ &\quad +(144r^2+432r+288)e_3 - 1). \end{aligned} \tag{2.34}$$

Now taking modulus and applying triangle inequality in (2.34), we get

$$\begin{aligned} |a_4 - a_1| &\leq \frac{5}{1728(1+r)(r+2)(r+3)}\Big|(121r^2+88r+7)e_1^3 - (264r^2+492r+108)e_1e_2 \\ &\quad + (144r^2+432r+288)e_3\Big| + 1. \end{aligned} \tag{2.35}$$

Using Lemma 1.7 in (2.35), we get

$$|a_4 - a_1| \leq \frac{23+6r}{6(3+r)}. \tag{2.36}$$

Now using (2.30), (2.33), (2.36) and (2.17) in (2.27) we get our desired result in (2.18).
Now

$$|V_{5,1}(f)| = |a_5 - a_4||a_5 - a_3||a_5 - a_2||a_5 - a_1||V_{4,1}(f)|. \tag{2.37}$$

Using (2.7) and (2.8), we get

$$\begin{aligned} a_5 - a_4 &= \frac{\Big(-6655r^3-12705r^2-6105r-455\Big)e_1^4}{20736\,(1+r)\,(r+2)\,(r+3)\,(r+4)} + \frac{\Big(-7260r^3-34320r^2-21540r-1680\Big)e_1^3}{20736\,(1+r)\,(r+2)\,(r+3)\,(r+4)} \\ &\quad + \frac{\Big(21780r^3+71280r^2+54180r+8280\Big)e_2e_1^2}{20736\,(1+r)\,(r+2)\,(r+3)\,(r+4)} + \frac{\Big(15840r^3+92880r^2+124560r+25920\Big)e_1e_2}{20736\,(1+r)\,(r+2)\,(r+3)\,(r+4)} \\ &\quad + \frac{\Big(-15840r^3-73440r^2-95040r-23040\Big)e_3e_1}{20736\,(1+r)\,(r+2)\,(r+3)\,(r+4)} + \frac{\Big(-7920r^3-36720r^2-43920r-15120\Big)e_2^2}{20736\,(1+r)\,(r+2)\,(r+3)\,(r+4)} \\ &\quad + \frac{\Big(-8640r^3-60480r^2-120960r-69120\Big)e_3}{20736\,(1+r)\,(r+2)\,(r+3)\,(r+4)} + \frac{\Big(8640r^3+51840r^2+95040r+51840\Big)e_4}{20736\,(1+r)\,(r+2)\,(r+3)\,(r+4)}. \end{aligned} \tag{2.38}$$

Taking modulus and applying triangle inequality in (2.38) we get

$$\begin{aligned}|a_5-a_4| \quad \le \quad & \left|\frac{(8640r^3+51840r^2+95040r+51840)e_4}{20736(1+r)(r+2)(r+3)(r+4)}-\frac{(7920r^3+36720r^2+43920r+15120)e_2^2}{20736(1+r)(r+2)(r+3)(r+4)}\right| \\ & +|e_1|\left|\frac{(6655r^3+12705r^2+6105r+455)e_1^3}{20736(1+r)(r+2)(r+3)(r+4)}-\frac{(21780r^3+71280r^2+54180r+8280)e_1e_2}{20736(1+r)(r+2)(r+3)(r+4)}\right. \\ & \left.+\frac{(15840r^3+73440r^2+95040r+23040)e_3}{20736(1+r)(r+2)(r+3)(r+4)}\right|+\left|\frac{5(121r^2+88r+7)e_1^3}{1728(1+r)(r+2)(r+3)}\right. \\ & \left.-\frac{5\left(264r^2+492r+108\right)e_1e_2}{1728(1+r)(r+2)(r+3)}+\frac{5(144r^2+432r+288)e_3}{1728(1+r)(r+2)(r+3)}\right|. \end{aligned} \tag{2.39}$$

Using (1.13) of Lemma 1.5, (1.17) of Lemma 1.6 and Lemma 1.7 in (2.39), we get

$$|a_5-a_4| \le \frac{5(7+2r)}{6(3+r)(4+r)}. \tag{2.40}$$

Similarly, using (2.6) and (2.8) and taking modulus and applying triangle inequality, we get

$$\begin{aligned}|a_5-a_3| \quad \le \quad & \left|\frac{(8640r^3+51840r^2+95040r+51840)e_4}{20736(1+r)(r+2)(r+3)(r+4)}-\frac{(7920r^3+36720r^2+43920r+15120)e_2^2}{20736(1+r)(r+2)(r+3)(r+4)}\right| \\ & +|e_1|\left|\frac{(6655r^3+12705r^2+6105r+455)e_1^3}{20736(1+r)(r+2)(r+3)(r+4)}-\frac{(21780r^3+71280r^2+54180r+8280e_1^2)e_2e_1}{20736(1+r)(r+2)(r+3)(r+4)}\right. \\ & \left.+\frac{(15840r^3+73440r^2+95040r+23040)e_3}{20736(1+r)(r+2)(r+3)(r+4)}\right|+\left|-\frac{5(-12-12r)e_2}{144(1+r)(r+2)}-\frac{5(11r+1)e_1^2}{144(1+r)(r+2)}\right|. \end{aligned} \tag{2.41}$$

By virtue of (1.13) of Lemma 1.5 and (1.17) of Lemma 1.6 and Lemma 1.7, we get

$$|a_5-a_3| \le \frac{5(6+2r)}{6(2+r)(4+r)}. \tag{2.42}$$

Using equations (2.5) and (2.8), we proceed by taking the modulus on both sides and applying the triangle inequality, we obtain

$$\begin{aligned}|a_5-a_2| \quad \le \quad & \frac{\left|(8640r^3+51840r^2+95040r+51840)e_4-(7920r^3+36720r^2+43920r+15120)e_2^2\right|}{20736(1+r)(r+2)(r+3)(r+4)} \\ & +\frac{1}{20736(1+r)(r+2)(r+3)(r+4)}|e_1|\Big|(6655r^3+12705r^2+6105r+455)e_1^3 \\ & -(21780r^3e_1^2+71280r^2+54180r+8280)e_2e_1+(15840r^3+73440r^2+95040r+23040)e_3\Big|. \end{aligned} \tag{2.43}$$

Utilizing (1.13) of Lemma 1.5, in conjunction with the results (1.17) of Lemma 1.6 and Lemma 1.7 in (2.43), we deduce that

$$|a_5-a_2| \le \frac{5(5+2r)}{6(1+r)(4+r)}. \tag{2.44}$$

By using $a_1 = 1$ and (2.8) and applying the triangle inequality, we arrive at

$$\begin{aligned}|a_5-a_1| \quad \le \quad & \frac{\left|(8640r^3+51840r^2+95040r+51840)e_4-(7920r^3+36720r^2+43920r+15120)e_2^2\right|}{20736(1+r)(r+2)(r+3)(r+4)} \\ & +\frac{1}{20736(1+r)(r+2)(r+3)(r+4)}|e_1|\Big|(6655r^3+12705r^2+6105r+455)e_1^3 \\ & -(21780r^3+71280r^2+54180r+8280)e_1e_2+(15840r^3+73440r^2+95040r+23040)e_3\Big|+1. \end{aligned} \tag{2.45}$$

Now the use of (1.13) of Lemma 1.5 and (1.17) of Lemma 1.6 and Lemma 1.7, we get

$$|a_5 - a_1| \leq \frac{29+6r}{6(4+r)}. \tag{2.46}$$

Now substituting (2.40), (2.42), (2.44) and (2.46) in (2.37) we get our desired estimation in (2.19). □

2.2. Estimates for the second order Toeplitz determinant for the class $S^r_{\phi_{4L}}$

The study of coefficient functionals represents a foundational aspect of geometric function theory, with its origins linked to the historical investigation of the Bieberbach conjecture. Although individual coefficient bounds yield basic growth information, the symmetric Toeplitz determinant $T_q(n)$ is of particular interest because it captures the structural and geometric relationships among successive coefficients of functions in $\mathcal{A}$. This section establishes upper bounds for the second-order symmetric Toeplitz determinant.

Theorem 2.11. *Let* $0 \leq r \leq 1$, *and if the function* f *be of the form* (1.1) *belongs to the class* $S^r_{\phi_{4L}}$

$$|T_2(2)(f)| \leq \frac{25}{9(1+r)^2}. \tag{2.47}$$

Proof. By using (2.5) and (2.6), we get

$$T_2(2)(f) = \left(-\frac{5(-12-12r)e_2}{144(1+r)(r+2)} - \frac{5\left(11re_1^2+e_1^2\right)}{144(1+r)(r+2)}\right)^2 - \frac{25e_1^2}{(12+12r)^2}. \tag{2.48}$$

From equation (1.14) in Lemma 1.5, substituting e_2 in terms of e_1 and letting $e_1 = e \in [0,2]$, in (2.48) we get

$$T_2(2)(f) = \frac{625(-1+r)^2e^4}{20736(1+r)^2(r+2)^2} - \frac{25e^2}{144(1+r)^2} + \frac{125(1-r)e^2\left(-e^2+4\right)}{1728(1+r)(r+2)^2}\xi + \frac{25\left(-e^2+4\right)^2}{144(1+r)^2}\xi^2. \tag{2.49}$$

Taking modulus and substituting and $|\xi| = 1$ in (2.49), we have

$$\begin{aligned} |T_2(2)(f)| &\leq \frac{625(-1+r)^2e^4}{20736(1+r)^2(r+2)^2} + \frac{25e^2}{144(1+r)^2} + \frac{125(1-r)e^2\left(-e^2+4\right)}{1728(1+r)(r+2)^2} + \frac{25\left(-e^2+4\right)^2}{144(1+r)^2} \\ &= \phi(e)\ (say). \end{aligned}$$

Now differentiating ϕ with respect to e and solving $\phi'(e) = 0$, we have

$$\begin{aligned} \phi'(e) &= \frac{625(-1+r)^2e^3}{5184(1+r)^2(r+2)^2} + \frac{25e}{72(1+r)^2} + \frac{125(1-r)e\left(-e^2+4\right)}{864(1+r)(r+2)^2} \\ &\quad - \frac{125(1-r)e^3}{864(1+r)(r+2)^2} - \frac{25\left(-e^2+4\right)e}{36(1+r)^2}. \end{aligned}$$

It can be easily seen that solution of $\phi'(e) = 0$ are

$$e = 0, \quad e = \pm\frac{2\sqrt{6}\sqrt{(229r^2+526r+541)(26r^2+84r+79)}}{229r^2+526r+541}$$

and hence it is clearly observed that the maximum of $\phi(e)$ attains at $e = 0$.

$$|T_2(2)(f)| \leq \frac{25}{9(1+r)^2}.$$

This completes the prove of the Theorem 2.11. □

2.3. Successive coefficient difference

In this section, we will use the following lemma due to Sim and Thomas [23] for determining the upper bounds of some initial consecutive coefficients module difference for the class $S^r_{\phi_{4L}}$.

Lemma 2.12 (*cf.* [23])**.** *Let $e \in \mathcal{P}$ be given by* (1.1). *Let B_1, B_2 and B_3 be numbers such that $B_1 \geq 0$, $B_2 \in \mathbb{C}$ and $B_3 \in \mathbb{R}$. Define $\psi_+(e_1, e_2)$ and $\psi_-(e_1, e_2)$ by*

$$\psi_+(e_1, e_2) = |B_2 e_1^2 + B_3 e_2| - |B_1 e_1|$$

and

$$\psi_-(e_1, e_2) = -\psi_+(e_1, e_2).$$

Then

$$\psi_+(e_1, e_2) \leq \begin{cases} |4B_2 + 2B_3| - 2B_1, & when \quad |2B_2 + B_3| \geq |B_3| + B_1, \\ 2|B_3|, & otherwise \end{cases}$$

and

$$\psi_-(e_1, e_2) \leq \begin{cases} 2B_1 - B_4, & B_1 \geq B_4 + 2|B_3|, \\ 2B_1 \sqrt{\frac{2|B_3|}{B_4 + 2|B_3|}}, & when \quad B_1^2 \leq 2|B_3|(B_4 + 2|B_3|), \\ 2|B_3| + \frac{B_1^2}{B_4 + 2|B_3|}, & otherwise, \end{cases}$$

where $B_4 = |4B_2 + 2B_3|$.

Theorem 2.13. *Let $f \in \mathcal{A}$ given by* (1.1) *be a member of the function class $S^r_{\phi_{4L}}$. Then*

$$-\frac{5}{6(1+r)} \sqrt{\frac{6(1+r)}{(11+r)}} \leq |a_3| - |a_2| \leq \frac{5}{6(2+r)}. \tag{2.50}$$

Proof. Let $f \in \mathcal{A}$ given by (1.1) be in the class $S^r_{\phi_{4L}}$. From (2.5) and (2.6), we have

$$|a_3| - |a_2| = \left| \frac{5e_2}{12(r+2)} - \frac{5(11re_1^2 + e_1^2)}{144(1+r)(r+2)} \right| - \left| \frac{5e_1}{12(1+r)} \right| = \psi_+(e_1, e_2). \tag{2.51}$$

Taking $B_1 = \frac{5}{12(1+r)} \geq 0$, $B_2 = -\frac{5(11r+1)}{144(1+r)(r+2)}$ *and* $B_3 = \frac{5}{12(2+r)}$, which satisfies the condition in Lemma 2.12. We have

$$|2B_2 + B_3| = \left| \frac{25(1-r)}{36(1-r)(2+r)} \right|$$

and

$$|B_3| + B_1 = \frac{5(3+2r)}{12(1+r)(2+r)}.$$

Thus, $|2B_2 + B_3| \not\geq |B_3| + B_1$. Therefore, by application of Lemma 2.12 gives

$$\psi_+(e_1, e_2) \leq 2|B_3| = \frac{5}{6(2+r)}.$$

Hence relation (2.51) reduces to

$$|a_3| - |a_2| \leq \frac{5}{6(2+r)}. \tag{2.52}$$

Further, we have

$$|a_2| - |a_3| = -\psi_+(e_1, e_2) = \psi_-(e_1, e_2). \tag{2.53}$$

Letting $B_1 = \frac{5}{12(1+r)} \geq 0$, $B_2 = -\frac{5(11r+1)}{144(1+r)(r+2)}$ and $B_3 = \frac{5}{12(2+r)}$, we have

$$B_4 = |4B_2 + 2B_3| = \frac{25(1-r)}{36(1+r)(2+r)}$$

and

$$2|B_3|(B_4 + 2|B_3|) = \frac{25(11+r)}{216(1+r)(2+r)^2}, \quad B_1^2 = \frac{25}{144(1+r)^2}.$$

Therefore, $B_1^2 < 2|B_3|(B_4 + 2|B_3|)$. Hence using Lemma 2.12, we get

$$\psi_-(e_1, e_2) \leq 2B_1 \sqrt{\frac{2|B_3|}{B_4 + 2|B_3|}} = \frac{5}{6(1+r)} \sqrt{\frac{6(1+r)}{(11+r)}}.$$

Hence, we obtain

$$|a_2| - |a_3| \leq \frac{5}{6(1+r)} \sqrt{\frac{6(1+r)}{(11+r)}},$$

which implies

$$|a_3| - |a_2| \geq -\frac{5}{6(1+r)} \sqrt{\frac{6(1+r)}{(11+r)}}. \tag{2.54}$$

From relations (2.52) and (2.54) we obtain the required estimate as stated in (2.50). Thus, the proof of Theorem 2.13 is complete. □

2.4. Krushkal inequality for the class $S^r_{\phi_{4L}}$

In this section, we provide a direct proof of the inequality

$$\left|a_n^p - a_2^{p(n-1)}\right| \leq 2^{p(n-1)-n^p}, \tag{2.55}$$

for the class $S^r_{\phi_{4L}}$, focusing on the specific cases where $n = 4$, $p = 1$, and $n = 5$, $p = 1$. This inequality, originally introduced by Krushkal [10], has been proven for the broader class of univalent functions. Our approach highlights its validity within this specialized subclass, emphasizing its significance in the context of coefficient bounds.

Theorem 2.14. *If $f \in S^r_{\phi_{4L}}$ and is of the form* (1.1), *then*

$$\left|a_4 - a_2^3\right| \leq \frac{5}{6r + 18}. \tag{2.56}$$

These estimations are sharp for $r = 0$ and $r = 1$. The sharpness at $r = 0$ reflects the initial boundary condition, while the sharpness at $r = 1$ demonstrates the extremal behavior of the function within the given domain.

Proof. By utilizing equations (2.5) and (2.7), we derive the following relationship

$$\begin{aligned} a_4 - a_2^3 \quad = \quad & \frac{\left(605r^4 + 1650r^3 + 1395r^2 - 115r - 715\right) e_1^3}{1728\,(1+r)^3\,(r+2)\,(r+3)} - \frac{\left(1320r^4 + 5100r^3 + 6780r^2 + 3540r + 540\right) e_2 e_1}{1728\,(1+r)^3\,(r+2)\,(r+3)} \\ & + \frac{\left(720r^4 + 3600r^3 + 6480r^2 + 5040r + 1440\right) e_3}{1728\,(1+r)^3\,(r+2)\,(r+3)}. \end{aligned} \tag{2.57}$$

Taking modulus in (2.57) and applying Lemma 1.7, we arrive at the required estimate. □

2.5. The Zalcman functional

Lawrence Zalcman posed the conjecture that if $f \in \mathcal{S}$ given by (1.1), then

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2, \quad (n \geq 2). \tag{2.58}$$

Equality in (2.58) holds for the Koebe function $k(z) = \frac{z}{(1-z)^2}$ $(z \in \mathbb{D})$ or its rotation. The area theorem shows that the conjecture is true for $n = 2$ (*cf.* [4]). Krushkal [9] proved that the conjecture is true for $n = 3$ and latter for $n = 4, 5, 6$. However, the Zalcaman conjecture remains an open problem for $n > 6$. For $f \in \mathcal{S}$, Ma [13] proposed the generalized Zalcman conjecture

$$J_{m,n}(f) = |a_n a_m - a_{n+m-1}| \leq (n-1)(m-1), \quad (n,\ m \geq 2) \tag{2.59}$$

and has proved this conjecture for the classes $\mathcal{S}^*$ and the class of all functions in $\mathcal{S}$ with real coefficients. In 2017, Ravichandran and Verma [19] proved it for the classes of starlike and convex functions of given order and for the class of functions with bounded turning. Now we prove that the inequality (2.59) holds for $n = m = 3$.

Theorem 2.15. *Let* $f \in S^r_{\phi_{4L}}$ *be of the form* (1.1). *Then*

$$J_{3,3}(f) = |a_5 - a_3^2| \leq \frac{70r^2 + 205r + 85}{18r^3 + 144r^2 + 342r + 216}. \tag{2.60}$$

Proof. Substituting the values of a_3 and a_5, from the relation (2.6) and (2.8) in non linear functional $a_5 - a_3^2$, we obtain following.

$$\begin{aligned} a_5 - a_3^2 = \; & \frac{1}{20736\,(1+r)^2\,(r+2)^2\,(r+3)\,(r+4)} \Big[\Big(6655r^5 + 35695r^4 + 79255r^3 + 84355r^2 + 20350r + 1210\Big) e_1^4 \\ & + \Big(-21780r^5 - 143220r^4 - 364980r^3 - 443580r^2 - 223800r - 23760\Big) e_2 e_1^2 \\ & + \Big(15840r^5 + 120960r^4 + 347040r^3 + 455040r^2 + 259200r + 46080\Big) e_3 e_1 \\ & + \Big(7920r^5 + 64080r^4 + 202320r^3 + 317520r^2 + 244800r + 73440\Big) e_2^2 \\ & + \Big(-8640r^5 - 77760r^4 - 267840r^3 - 440640r^2 - 345600r - 103680\Big) e_4 \Big]. \end{aligned} \tag{2.61}$$

Taking modulus on the both sides of (2.61) and followed by triangle inequality yield

$$\begin{aligned} |a_5 - a_3^2| \leq \; & \frac{|e_1|}{20736\,(1+r)^2\,(r+2)^2\,(r+3)\,(r+4)} \Big[\Big| \Big(6655r^5 + 35695r^4 + 79255r^3 + 84355r^2 + 20350r + 1210\Big) e_1^3 \\ & - \Big(21780r^5 + 143220r^4 + 364980r^3 + 443580r^2 + 223800r + 23760\Big) e_2 e_1 \\ & + \Big(15840r^5 + 120960r^4 + 347040r^3 + 455040r^2 + 259200r + 46080\Big) e_3 \Big| \\ & + \Big| \Big(7920r^5 + 64080r^4 + 202320r^3 + 317520r^2 + 244800r + 73440\Big) e_2^2 \\ & + \Big(-8640r^5 - 77760r^4 - 267840r^3 - 440640r^2 - 345600r - 103680\Big) e_4 \Big| \Big]. \end{aligned} \tag{2.62}$$

By using (1.13) of Lemma 1.5 and (1.17) of Lemmas 1.6 and 1.7 we get our desired result. This completes the proof of Theorem 2.15. □

3. Conclusion

In this study, we have conducted a rigorous investigation into a novel subclass of univalent functions, successfully characterizing its fundamental analytic properties. Our research established definitive initial coefficient bounds and provided a detailed analysis of the Fekete-Szegö inequality and the second Hankel determinant. By determining these core metrics, we have laid a robust mathematical foundation for this class, clarifying the relationship between its coefficients and its underlying geometric structure.

Beyond these primary results, our exploration addressed complex functional inequalities and determinant properties, including the second-order Toeplitz determinant and the modulus difference of the initial coefficients. We successfully verified Krushkal's inequality for the parameters $n = 4, p = 1$, and confirmed the generalized Zalcman functional for $n = m = 3$. While we extended our computations to the fifth-order Vandermonde determinant, the attainment of a sharp estimate remains an open challenge. These findings provide significant new insights into the field and highlight a clear path for future research to refine these estimates and apply them to broader classes of functions. We mention here that all the results in this paper are not sharp i.e. not best possible. Finding sharp upper bound remain an open challenges for researcher in the field of geometric function theory.

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