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Series and sums involving the floor function

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Abstract

Let $(a_n)_{n\geq 0}$ be an arbitrary sequence and $(a_{\lfloor n/k \rfloor})_{n\geq 0}$ its dual floor sequence. We study infinite series and finite generalized binomial sums involving $(a_{\lfloor n/k \rfloor})_{n\geq 0}$. As applications, we prove a range of new closed-form expressions for the Fibonacci (Lucas) series and binomial sum identities as particular cases.

Keywords: Floor function, power series, generating function, binomial transform, Fibonacci (Lucas) number, Gibonacci sequence

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1. Introduction

For $x \in \mathbb{R}$, Legendre introduced the concept of the integer part of x. The notation $[x]$ goes back to Gauss. Following Knuth [11], the floor and ceiling functions were introduced in the second half of the 20th century. The floor of x is defined as $\lfloor x \rfloor = \max\{k \in \mathbb{Z} \,|\, k \leq x\}$, while the ceiling of x is defined as $\lceil x \rceil = \min\{k \in \mathbb{Z} \,|\, k \geq x\}$. These functions are significant in number theory, combinatorics, and mathematical analysis. They frequently appear in asymptotic estimates, modular arithmetic, and summation formulas. Despite their simple definitions, expressions involving these functions often exhibit intricate and surprising properties, making them an intriguing subject of study.

In recent years, various summation techniques have been developed to evaluate series involving the floor and ceiling function. The study of such a series not only deepens our understanding of discrete mathematics and analytical techniques but also has applications in computational mathematics, algorithmic number theory, and special functions. For further reading, we refer to [8, 10], [15]-[17], [25]. Additionally, a detailed analysis of both finite and infinite series involving these functions is presented in [22, 23].

This paper aims to extend the existing body of work by systematically analyzing series and sums involving the floor function. We derive new identities, evaluate specific cases, and explore connections with classical number-theoretic functions. In doing so, we provide insights that may contribute to further theoretical developments and practical applications.

As usual, the Fibonacci numbers F_n and the Lucas numbers L_n are defined, for $n \in \mathbb{Z}$, through the recurrence relations $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, with initial values $F_0 = 0$, $F_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ with $L_0 = 2$, $L_1 = 1$. For

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negative subscripts we have $F_{-n} = (-1)^{n-1}F_n$ and $L_{-n} = (-1)^n L_n$. They possess the explicit formulas (Binet forms)

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \quad L_n = \alpha^n + \beta^n, \quad n \in \mathbb{Z}. \tag{1.1}$$

For more information we refer to Koshy [12] and Vajda [26] who have written high-quality books dealing with the Fibonacci and Lucas numbers.

If we were to specify a particular motivation, it would be the following series, which recently appeared as a problem proposal in [6]:

$$\sum_{n=0}^{\infty} \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\frac{F_n}{2^n} = \frac{32}{5} \qquad \text{and} \qquad \sum_{n=0}^{\infty} \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\frac{L_n}{2^n} = 16.$$

These series allow for generalizations in various directions. A Gibonacci version of the above series formulas is given by

$$\sum_{n=0}^{\infty} \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\frac{G_n}{2^n} = \frac{8(3a+4b)}{5},$$

where G_n denotes the Gibonacci number defined by the recurrence relation $G_n = G_{n-1} + G_{n-2}$, $n \geq 2$, with initial conditions $G_0 = a$ and $G_1 = b$, where a and b are arbitrary. The term *Gibonacci* was introduced by Benjamin and Quinn in [3]. Note that the Fibonacci sequence F_n corresponds to the case of G_n when $a = 0$ and $b = 1$, while the Lucas sequence L_n is obtained when $a = 2$ and $b = 1$. For $n \geq 0$, the Fibonacci and Gibonacci numbers satisfy the simple identity $G_n = aF_{n-1} + bF_n$ (see, e.g., [26]). The Binet-like formula for the Gibonacci sequence is given by

$$G_n = \frac{1}{\sqrt{5}}\left((a + (a-b)\beta)\alpha^{n+1} - (a + (a-b)\alpha)\beta^{n+1}\right).$$

In [3], it is shown that G_n counts the total weight of all n-tilings, where a tiling that ends in a domino has weight a and a tiling that ends in a square has weight b. Notable among the recent studies on the Gibonacci and the more general Horadam sequence are the papers [1, 2, 7, 9, 14, 24].

In this paper, we explore a general framework involving sequences and their transformations. Given an arbitrary sequence $(a_n)_{n\geq 0}$, we examine its associated dual floor sequence $(a_{\lfloor n/k \rfloor})_{n\geq 0}$. Our focus is on the study of infinite series and finite generalized binomial sums incorporating $(a_{\lfloor n/k \rfloor})_{n\geq 0}$.

In the first part of this study (Sections 2 and 3), we explored the properties of infinite series involving sequences of the form $(a_{\lfloor n/k \rfloor})_{n\geq 0}$ and derived closed-form evaluations for these series at specific Fibonacci (Lucas) arguments. Additionally, we obtained what appears to be a novel expression for the Riemann zeta function $\zeta(3)$, which involves harmonic numbers and the floor function (Theorem 2.24). In the second part (Sections 4 and 5), we focused on finite binomial sums, specifically examining three distinct sequence classes: $a_{\lfloor n/k \rfloor} = \lfloor \frac{n}{k} \rfloor$, $a_{\lfloor n/k \rfloor} = \lfloor \frac{n}{k} \rfloor^2$, and $a_{\lfloor n/k \rfloor} = (-1)^{\lfloor \frac{n}{k} \rfloor}$.

2. Some infinite series

The next fundamental lemma will be crucial in the first part of the study. It will enable us to establish many results for infinite series involving the floor function.

Lemma 2.1. *Let $k \geq 1$ be an integer and $z \in \mathbb{C}$ with $|z| < 1$. Let further $(a_n)_{n\geq 0}$ be an arbitrary sequence with ordinary generating function $F(z)$. Then the ordinary generating function of the sequence $\left(a_{\lfloor n/k \rfloor}\right)_{n\geq 0}$ is*

$$F^+(z,k) = \sum_{n=0}^{\infty} a_{\lfloor n/k \rfloor} z^n = \frac{1-z^k}{1-z} F(z^k). \tag{2.1}$$

Also, the ordinary generating function of the sequence $\left((-1)^n a_{\lfloor n/k \rfloor}\right)_{n\geq 0}$ is

$$F^-(z,k) = \sum_{n=0}^{\infty} (-1)^n a_{\lfloor n/k \rfloor} z^n = \frac{1 + (-1)^{k+1} z^k}{1+z} F((-1)^k z^k).$$

Proof. We have

$$F^+(z;k) = \sum_{n=0}^{\infty} a_{\lfloor n/k \rfloor} z^n = \sum_{n=0}^{\infty} a_n \sum_{j=0}^{k-1} z^{kn+j} = \sum_{j=0}^{k-1} z^j \sum_{n=0}^{\infty} a_n z^{kn} = \frac{1-z^k}{1-z} F(z^k).$$

The second identity follows from the first by replacing z with $-z$. □

Lemma 2.2. *For integer $k \geq 1$ and $|z| < 1$ the following expressions are valid:*

$$\sum_{n=0}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor z^n = \frac{z^k}{(1-z)(1-z^k)} \tag{2.2}$$

and

$$\sum_{n=0}^{\infty} (-1)^n \left\lfloor \frac{n}{k} \right\rfloor z^n = \frac{(-1)^k z^k}{(1+z)(1+(-1)^{k+1} z^k)}. \tag{2.3}$$

Proof. Use Lemma 2.1 with $a_n = n$ in conjunction with generating function $F(z) = \sum\limits_{n=0}^{\infty} n z^n = \frac{z}{(1-z)^2}$. The second identity follows from the first by replacing z with $-z$. □

Remark 2.3. Since for $k < 0$, $\left\lfloor \frac{n}{k} \right\rfloor = -1 - \left\lfloor \frac{n-1}{-k} \right\rfloor$, we can extend formulas (2.2) and (2.3) to negative integer k as follows:

$$\sum_{n=0}^{\infty} \left\lfloor \frac{n+1}{k} \right\rfloor z^n = \frac{z^k}{(1-z)(1-z^k)}$$

and

$$\sum_{n=0}^{\infty} (-1)^n \left\lfloor \frac{n+1}{k} \right\rfloor z^n = \frac{(-1)^k z^k}{(1+z)(1+(-1)^{k+1} z^k)}.$$

These results can be compared with those in [21].

Remark 2.4. Since $\left\lfloor \frac{n}{k} \right\rfloor = \left\lceil \frac{n+1}{k} \right\rceil - 1$ for $k \geq 1$ and $\left\lfloor \frac{n+1}{k} \right\rfloor = \left\lceil \frac{n}{k} \right\rceil - 1$ for $k \leq -1$, from (2.2) and (2.3) we have infinite series involving the ceiling function:

$$\sum_{n=0}^{\infty} \left\lceil \frac{n+1}{k} \right\rceil z^n = \frac{1}{(1-z)(1-z^k)}, \qquad k \geq 1,$$
$$\sum_{n=0}^{\infty} \left\lceil \frac{n}{k} \right\rceil z^n = \frac{1}{(1-z)(1-z^k)}, \qquad k \leq -1,$$

and

$$\sum_{n=0}^{\infty} (-1)^n \left\lceil \frac{n+1}{k} \right\rceil z^n = \frac{1}{(1+z)(1-(-1)^k z^k)}, \qquad k \geq 1,$$
$$\sum_{n=0}^{\infty} (-1)^n \left\lceil \frac{n}{k} \right\rceil z^n = \frac{1}{(1+z)(1-(-1)^k z^k)}, \qquad k \leq -1.$$

Lemma 2.2 leads immediately to a range of new Fibonacci (Lucas) series evaluations.

Theorem 2.5. *For any integers m and s, an integer $k \geq 1$ and any $p > \alpha^s$, we have*

$$\sum_{n=0}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor \frac{F_{sn+m}}{p^{n+1}} = \frac{p^k(pF_{sk+m} - (-1)^s F_{s(k-1)+m}) - (-1)^{sk}(pF_m - (-1)^s F_{m-s})}{(p^2 - pL_s + (-1)^s)(p^{2k} - p^k L_{sk} + (-1)^{sk})}$$

and

$$\sum_{n=0}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor \frac{L_{sn+m}}{p^{n+1}} = \frac{p^k(pL_{sk+m} - (-1)^s L_{s(k-1)+m}) - (-1)^{sk}(pL_m - (-1)^s L_{m-s})}{(p^2 - pL_s + (-1)^s)(p^{2k} - p^k L_{sk} + (-1)^{sk})}.$$

Proof. Inserting $z = \frac{\alpha^s}{p}$ in (2.2) gives

$$\sum_{n=0}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor \frac{\alpha^{sn}}{p^n} = \frac{p\alpha^{sk}}{(p-\alpha^s)(p^k - \alpha^{sk})} \quad \text{or} \quad \sum_{n=0}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor \frac{\alpha^{sn+m}}{p^n} = \frac{p\alpha^{sk+m}}{(p-\alpha^s)(p^k - \alpha^{sk})}.$$

Similarly,

$$\sum_{n=0}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor \frac{\beta^{sn+m}}{p^n} = \frac{p\beta^{sk+m}}{(p-\beta^s)(p^k - \beta^{sk})}.$$

Now, we combine the results according to the Binet forms (1.1) using $\alpha\beta = -1$ and $\alpha + \beta = 1$. □

Corollary 2.6. *We have the following series valid for $k \geq 1$:*

$$\sum_{n=0}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor \frac{F_{n+m}}{2^{n+1}} = \frac{2^k F_{k+m+2} - (-1)^k F_{m+2}}{4^k - 2^k L_k + (-1)^k}, \qquad \sum_{n=0}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor \frac{L_{n+m}}{2^{n+1}} = \frac{2^k L_{k+m+2} - (-1)^k L_{m+2}}{4^k - 2^k L_k + (-1)^k},$$

$$\sum_{n=0}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor \frac{F_{n+m}}{3^{n+1}} = \frac{3^k L_{k+m+1} - (-1)^k L_{m+1}}{5(9^k - 3^k L_k + (-1)^k)}, \qquad \sum_{n=0}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor \frac{L_{n+m}}{3^{n+1}} = \frac{3^k F_{k+m+1} - (-1)^k F_{m+1}}{9^k - 3^k L_k + (-1)^k},$$

$$\sum_{n=0}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor \frac{F_{2n+m}}{4^{n+1}} = \frac{4^k L_{2k+m+1} - L_{m+1}}{5(16^k - 4^k L_{2k} + 1)}, \qquad \sum_{n=0}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor \frac{L_{2n+m}}{4^{n+1}} = \frac{4^k F_{2k+m+1} - F_{m+1}}{16^k - 4^k L_{2k} + 1},$$

$$\sum_{n=0}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor \frac{F_{2sn+m}}{L_{2s}^{n+1}} = \frac{L_{2s}^k F_{2s(k+1)+m} - F_{m+2s}}{L_{2s}^{2k} - L_{2s}^k L_{2sk} + 1}, \qquad \sum_{n=0}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor \frac{L_{2sn+m}}{L_{2s}^{n+1}} = \frac{L_{2s}^k L_{2s(k+1)+m} - L_{m+2s}}{L_{2s}^{2k} - L_{2s}^k L_{2sk} + 1},$$

$$\sum_{n=0}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor \frac{F_{2sn+m}}{F_p^{n+1}} = \frac{F_p^k(F_p F_{2sk+m} - F_{2s(k-1)+m}) - F_p F_m + F_{m-2s}}{(F_p^2 - F_p L_{2s} + 1)(F_p^{2k} - F_p^k L_{2sk} + 1)}, \qquad F_p > \alpha^s.$$

Proof. All results follow by taking particular values for the parameters s and p in Theorem 2.5. For instance, for the first two series take $s = 1$ and $p = 2$, and simplify. To get the third and fourth series take $s = 1$ and $p = 3$, and so on. □

We now present the alternating version of Theorem 2.5.

Theorem 2.7. *For any integers m and s, integer $k \geq 1$ and any $p > \alpha^s$, we have*

$$\sum_{n=0}^{\infty} (-1)^{n-k} \left\lfloor \frac{n}{k} \right\rfloor \frac{F_{sn+m}}{p^{n+1}} = \frac{p^k(pF_{sk+m} + (-1)^s F_{s(k-1)+m}) - (-1)^{k(s-1)}(pF_m + (-1)^s F_{m-s})}{(p^2 + pL_s + (-1)^s)(p^{2k} - (-1)^k p^k L_{sk} + (-1)^{sk})}$$

and

$$\sum_{n=0}^{\infty} (-1)^{n-k} \left\lfloor \frac{n}{k} \right\rfloor \frac{L_{sn+m}}{p^{n+1}} = \frac{p^k(pL_{sk+m} + (-1)^s L_{s(k-1)+m}) - (-1)^{k(s-1)}(pL_m + (-1)^s L_{m-s})}{(p^2 + pL_s + (-1)^s)(p^{2k} - (-1)^k p^k L_{sk} + (-1)^{sk})}.$$

Proof. Insert $z = \frac{\alpha^s}{p}$ and $z = \frac{\beta^s}{p}$, in turn, into (2.3), then multiply through by α^m (or β^m) and combine the resulting expressions according to the Binet forms (1.1), making use of $\alpha\beta = -1$ and $\alpha + \beta = 1$. □

Corollary 2.8. *For $k \geq 1$, we have the following series:*

$$\sum_{n=0}^{\infty}(-1)^{n-k}\left\lfloor\frac{n}{k}\right\rfloor\frac{F_{n+m}}{2^{n+1}} = \frac{2^k L_{k+m-1} - L_{m-1}}{5(4^k - (-2)^k L_k + (-1)^k)},$$
$$\sum_{n=0}^{\infty}(-1)^{n-k}\left\lfloor\frac{n}{k}\right\rfloor\frac{L_{n+m}}{2^{n+1}} = \frac{2^k F_{k+m-1} - F_{m-1}}{4^k - (-2)^{k+1} L_k + (-1)^k},$$
$$\sum_{n=0}^{\infty}(-1)^{n}\left\lfloor\frac{n}{k}\right\rfloor\frac{F_{2n+m}}{3^{n+1}} = \frac{(-3)^k(3F_{2k+m} + F_{2(k-1)+m}) - 3F_m - F_{m-2}}{19(9^k - (-3)^k L_{2k} + 1)},$$
$$\sum_{n=0}^{\infty}(-1)^{n}\left\lfloor\frac{n}{k}\right\rfloor\frac{L_{2n+m}}{3^{n+1}} = \frac{(-3)^k(3L_{2k+m} + L_{2(k-1)+m}) - 3L_m - L_{m-2}}{19(9^k - (-3)^k L_{2k} + 1)}$$

and

$$\sum_{n=0}^{\infty}(-1)^{n-k}\left\lfloor\frac{n}{k}\right\rfloor\frac{F_{2sn+m}}{L_{2s}^{n+1}} = \frac{L_{2s}^k(F_{2sk+m}L_{2s} + F_{2s(k-1)+m}) - (-1)^k(L_{2s}F_m + F_{m-2s})}{(2L_{2s}^2 + 1)(L_{2s}^{2k} - (-1)^k L_{2s}^k L_{2sk} + 1)},$$
$$\sum_{n=0}^{\infty}(-1)^{n-k}\left\lfloor\frac{n}{k}\right\rfloor\frac{L_{2sn+m}}{L_{2s}^{n+1}} = \frac{L_{2s}^k(L_{2sk+m}L_{2s} + L_{2s(k-1)+m}) - (-1)^k(L_{2s}L_m + L_{m-2s})}{(2L_{2s}^2 + 1)(L_{2s}^{2k} - (-1)^k L_{2s}^k L_{2sk} + 1)}.$$

Proof. Take specific values for the parameters s and p in Theorem 2.7. □

Let $f^+(z,k)$ and $f^-(z,k)$ denote the functions in (2.2) and (2.3), respectively, i.e.,

$$f^+(z,k) = \sum_{n=0}^{\infty}\left\lfloor\frac{n}{k}\right\rfloor z^n = \frac{z^k}{(1-z)(1-z^k)}, \qquad k \geq 1,\ |z| < 1 \tag{2.4}$$

and

$$f^-(z,k) = \sum_{n=0}^{\infty}(-1)^n\left\lfloor\frac{n}{k}\right\rfloor z^n = \frac{(-1)^k z^k}{(1+z)(1+(-1)^{k+1}z^k)}, \qquad k \geq 1,\ |z| < 1. \tag{2.5}$$

Lemma 2.9. *If $|z| < 1$, then*

$$\sum_{n=1}^{\infty}\left\lfloor\frac{n}{2}\right\rfloor\frac{z^n}{n} = \frac{1}{4}\log\left|\frac{1-z}{1+z}\right| + \frac{z}{2(1-z)} \tag{2.6}$$

and

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left\lfloor\frac{n}{2}\right\rfloor\frac{z^n}{n} = \frac{1}{4}\log\left|\frac{1-z}{1+z}\right| + \frac{z}{2(1+z)}. \tag{2.7}$$

Proof. To get the first identity integrate $\frac{f^+(z,2)}{z}$ and use $f^+(0,2) = 0$. The second follows from working with $f^-(z,2)$. □

Corollary 2.10. *The following series are valid:*

$$\sum_{n=1}^{\infty}\frac{1}{5^n}\frac{n}{2n+1} = \frac{5}{8} - \frac{\sqrt{5}}{2}\log\alpha, \qquad \sum_{n=1}^{\infty}\left(\frac{5}{9}\right)^n\frac{n}{2n+1} = \frac{9}{8} - \frac{3\sqrt{5}}{5}\log\alpha,$$
$$\sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n\frac{n}{2n+1} = \frac{5}{2} - \frac{3\sqrt{5}}{4}\log\alpha, \qquad \sum_{n=1}^{\infty}\left(\frac{45}{49}\right)^n\frac{n}{2n+1} = \frac{49}{8} - \frac{14\sqrt{5}}{15}\log\alpha.$$

Proof. Set $z = \frac{\sqrt{5}}{5}$, $z = \frac{\sqrt{5}}{3}$, $z = \frac{2\sqrt{5}}{5}$, and $z = \frac{3\sqrt{5}}{7}$, in turn, into (2.6), (2.7), and then add both resulting formulas. □

Remark 2.11. One can obtain all of the above formulas from [19, Formula 5.2.4.]:

$$\sum_{k=0}^{\infty} \frac{x^k}{2k+m} = -\frac{1}{2x^{m/2}} \left(\log\left(1-\sqrt{x}\right) + (-1)^m \log\left(1+\sqrt{x}\right)\right).$$

For instance, to get the first formula, insert $x = \frac{1}{5}$ and $m = 1$ to get

$$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{5}\right)^k}{2k+1} = \frac{\sqrt{5}}{2} \log\left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right) \quad \text{or} \quad \sum_{k=0}^{\infty} \frac{1}{(2k+1)5^k} = \sqrt{5}\log\alpha.$$

The formula follows from $\frac{2n}{2n+1} = 1 - \frac{1}{2n+1}$ and $\sum_{k=1}^{\infty} \frac{1}{5^k} = \frac{1}{4}$.

Theorem 2.12. *If m is any integer, then*

$$\sum_{n=1}^{\infty} \left\lfloor \frac{n}{2} \right\rfloor \frac{L_{n+m}}{2^n n} = -\frac{3\sqrt{5}}{4} F_m \log\alpha - \frac{L_m}{8}\log 5 + \frac{L_{m+3}}{2} \tag{2.8}$$

and

$$\sum_{n=1}^{\infty} \left\lfloor \frac{n}{2} \right\rfloor \frac{F_{n+m}}{2^n n} = -\frac{3\sqrt{5}}{20} L_m \log\alpha - \frac{F_m}{8}\log 5 + \frac{F_{m+3}}{2}. \tag{2.9}$$

Proof. By substituting $z = \frac{\alpha}{2}$ and $z = \frac{\beta}{2}$ successively into (2.6), we obtain

$$\sum_{n=1}^{\infty} \left\lfloor \frac{n}{2} \right\rfloor \frac{\alpha^{n+m}}{2^n n} = \alpha^m \left(-\frac{3}{4}\log\alpha - \frac{1}{8}\log 5 + \alpha + \frac{1}{2}\right) \tag{2.10}$$

and

$$\sum_{n=1}^{\infty} \left\lfloor \frac{n}{2} \right\rfloor \frac{\beta^{n+m}}{2^n n} = \beta^m \left(\frac{3}{4}\log\alpha - \frac{1}{8}\log 5 + \beta + \frac{1}{2}\right). \tag{2.11}$$

Addition of (2.10) and (2.11) yields (2.8) while their difference gives (2.9). □

The alternating versions of (2.8) and (2.9) via (2.7) are stated next.

Theorem 2.13. *If m is any integer, then*

$$\sum_{n=1}^{\infty} (-1)^n \left\lfloor \frac{n}{2} \right\rfloor \frac{L_{n+m}}{2^n n} = \frac{L_m}{8}\log 5 + \frac{3\sqrt{5}}{4} F_m \log\alpha - \frac{F_m}{2} \tag{2.12}$$

and

$$\sum_{n=1}^{\infty} (-1)^n \left\lfloor \frac{n}{2} \right\rfloor \frac{F_{n+m}}{2^n n} = \frac{F_m}{8}\log 5 + \frac{3\sqrt{5}}{4} L_m \log\alpha - \frac{L_m}{10}. \tag{2.13}$$

Proof. Use of $z = \frac{\alpha}{2}$ and $z = \frac{\beta}{2}$, in turn, in (2.7) produces

$$\sum_{n=1}^{\infty} (-1)^n \left\lfloor \frac{n}{2} \right\rfloor \frac{\alpha^{n+m}}{2^n n} = \alpha^m \left(\frac{3}{4}\log\alpha + \frac{1}{8}\log 5 - \frac{\sqrt{5}}{10}\right)$$

and

$$\sum_{n=1}^{\infty} (-1)^n \left\lfloor \frac{n}{2} \right\rfloor \frac{\beta^{n+m}}{2^n n} = \beta^m \left(-\frac{3}{4}\log\alpha + \frac{1}{8}\log 5 + \frac{\sqrt{5}}{10}\right).$$

Adding the last two formulas yields (2.12), while their difference results in (2.13). □

Lemma 2.14. *If m is a non-negative integer and $|z| < 1$, then*

$$\sum_{n=1}^{\infty} \left\lfloor \frac{n}{2} \right\rfloor \binom{n}{m} z^{n-m} = -\frac{3}{4(1-z)^{m+1}} + \frac{(-1)^m}{4(1+z)^{m+1}} + \frac{m+1}{2(1-z)^{m+2}} \tag{2.14}$$

and

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left\lfloor \frac{n}{2} \right\rfloor \binom{n}{m} z^{n-m} = \frac{(-1)^m 3}{4(1+z)^{m+1}} - \frac{1}{4(1-z)^{m+1}} - \frac{(-1)^m (m+1)}{2(1+z)^{m+2}}. \tag{2.15}$$

Proof. Differentiate

$$f^+(z,2) := \sum_{n=0}^{\infty} \left\lfloor \frac{n}{2} \right\rfloor z^n = \frac{z^2}{(1-z)(1-z^2)} = \frac{1}{4}\left(-\frac{3}{1-z} + \frac{1}{1+z} + \frac{2}{(1-z)^2} \right),$$

where $f^+(z,k)$ defined in (2.4), m times with respect to z, for $|z| < 1$ we obtain (2.14). Since the proof of (2.15) follows similarly using $f^-(z,2)$, we omit the details. □

Theorem 2.15. *If m is a non-negative integer, then*

$$\sum_{n=1}^{\infty} \frac{n}{5^{n+1}} \binom{2n+1}{m} = \frac{(5m+1)F_{m+1} + (5m-3)F_m}{2^{m+4}}$$

and

$$\sum_{n=1}^{\infty} \frac{n}{5^{n+1}} \binom{2n}{m} = \frac{(3m+1)F_{m+1} + (m+1)F_m}{2^{m+4}}.$$

Proof. Setting $z = \frac{\sqrt{5}}{5}$ into (2.14) and (2.15), we obtain, respectively,

$$\sum_{n=1}^{\infty} \left\lfloor \frac{n}{2} \right\rfloor \frac{\binom{n}{m}}{\sqrt{5}^n} = \frac{1}{2^{m+3}} \left(5(m+1)\alpha^{m+2} - 3\sqrt{5}\alpha^{m+1} - \sqrt{5}\beta^{m+1} \right)$$

and

$$\sum_{n=1}^{\infty} (-1)^n \left\lfloor \frac{n}{2} \right\rfloor \frac{\binom{n}{m}}{\sqrt{5}^n} = \frac{1}{2^{m+3}} \left(5(m+1)\beta^{m+2} + 3\sqrt{5}\beta^{m+1} + \sqrt{5}\alpha^{m+1} \right).$$

Adding and subtracting these expressions, followed by applying the Binet forms (1.1) and simplifying, yields the desired results. □

Theorem 2.16. *If m is a non-negative integer, then*

$$\sum_{n=1}^{\infty} n \binom{2n+1}{2m} \left(\frac{5}{9}\right)^n = \frac{9}{16} \left(\frac{5}{4}\right)^m ((6m+3)F_{4m+4} - 4F_{4m+2}), \tag{2.16}$$

$$\sum_{n=1}^{\infty} n \binom{2n+1}{2m+1} \left(\frac{5}{9}\right)^n = \frac{9}{32} \left(\frac{5}{4}\right)^m ((6m+6)L_{4m+6} - 4L_{4m+4}), \tag{2.17}$$

$$\sum_{n=1}^{\infty} n \binom{2n}{2m} \left(\frac{5}{9}\right)^n = \frac{3}{16} \left(\frac{5}{4}\right)^m ((6m+3)L_{4m+4} - 2L_{4m+2}) \tag{2.18}$$

and

$$\sum_{n=1}^{\infty} n \binom{2n}{2m+1} \left(\frac{5}{9}\right)^n = \frac{15}{16} \left(\frac{5}{4}\right)^m ((3m+3)F_{4m+6} - F_{4m+4}). \tag{2.19}$$

Proof. Setting $z = \frac{\sqrt{5}}{3}$ in (2.14) and (2.15), we get

$$\sum_{n=1}^{\infty}\left\lfloor\frac{n}{2}\right\rfloor\binom{n}{m}\Big(\frac{\sqrt{5}}{3}\Big)^{n-m} = \frac{3}{8}\left(\frac{3}{2}\right)^{m}\left(3(m+1)\alpha^{2m+4} - 3\alpha^{2m+2} + (-1)^m\beta^{2m+2}\right)$$

and

$$\sum_{n=1}^{\infty}(-1)^n\left\lfloor\frac{n}{2}\right\rfloor\binom{n}{m}\Big(\frac{\sqrt{5}}{3}\Big)^{n-m} = \frac{3}{8}\left(\frac{3}{2}\right)^{m}\left(3(-1)^m(m+1)\beta^{2m+4} - 3(-1)^m\beta^{2m+2} + \alpha^{2m+2}\right).$$

Adding and subtracting these identities and simplifying, we obtain (2.16) and (2.18) for even m, and (2.17) and (2.19) for odd m. □

Theorem 2.17. *For p an integer and m a non-negative integer, we have the identities*

$$\sum_{n=1}^{\infty}\left\lfloor\frac{n}{2}\right\rfloor\binom{n}{m}\frac{F_{n+p}}{2^{n-1}} = 4(m+1)F_{3m+p+4} - 3F_{3m+p+2} + (-1)^m\begin{cases}5^{-(m+2)/2}L_{p-1}, & m \text{ even};\\ 5^{-(m+1)/2}F_{p-1}, & m \text{ odd}\end{cases}$$

and

$$\sum_{n=1}^{\infty}\left\lfloor\frac{n}{2}\right\rfloor\binom{n}{m}\frac{L_{n+p}}{2^{n-1}} = 4(m+1)L_{3m+p+4} - 3L_{3m+p+2} + (-1)^m\begin{cases}5^{-m/2}F_{p-1}, & m \text{ even};\\ 5^{-(m+1)/2}L_{p-1}, & m \text{ odd}.\end{cases}$$

Proof. Insert $z = \frac{\alpha}{2}$ and $z = \frac{\beta}{2}$, in turn, into (2.14), and then multiply through by α^{m+p} and β^{m+p}, to obtain

$$\sum_{n=1}^{\infty}\left\lfloor\frac{n}{2}\right\rfloor\binom{n}{m}\frac{\alpha^{n+p}}{2^n} = \frac{3\alpha^{m+p+1}}{2(2-\alpha)^{m+2}} + \frac{(2m-1)\alpha^{m+p}}{(2-\alpha)^{m+2}} + \frac{(-1)^m\alpha^{m+p}}{2(2+\alpha)^{m+1}}$$

and

$$\sum_{n=1}^{\infty}\left\lfloor\frac{n}{2}\right\rfloor\binom{n}{m}\frac{\alpha^{n+p}}{2^n} = \frac{3\alpha^{m+p+1}}{2(2-\alpha)^{m+2}} + \frac{(2m-1)\alpha^{m+p}}{(2-\alpha)^{m+2}} + \frac{(-1)^m\alpha^{m+p}}{2(2+\alpha)^{m+1}}.$$

Then, combine the resulting expressions above according to the Binet forms (1.1), making use of the relations $2-\alpha = \alpha^{-2}$, $2+\alpha = \sqrt{5}\alpha$, $2-\beta = \alpha^2$, and $2+\beta = -\sqrt{5}\beta$. □

Corollary 2.18. *For a positive integer k and $|z| < 1$,*

$$\sum_{n=1}^{\infty}\left\lfloor\frac{n}{k}\right\rfloor\left(\frac{z^n}{n} - \frac{z^{n+1}}{n+1}\right) = -\frac{1}{k}\log\left(1-z^k\right) \tag{2.20}$$

and

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left\lfloor\frac{n}{k}\right\rfloor\left(\frac{z^n}{n} + \frac{z^{n+1}}{n+1}\right) = \frac{1}{k}\log\left(1-(-1)^k z^k\right). \tag{2.21}$$

Proof. Write (2.2) as

$$\sum_{n=1}^{\infty}\left\lfloor\frac{n}{k}\right\rfloor(1-z)z^{n-1} = \frac{z^{k-1}}{1-z^k}$$

and integrate both sides with respect to z to obtain (2.20). The second identity follows from (2.20) by replacing z with $-z$. □

Theorem 2.19. *If $k \geq 1$ is an integer, r is any integer and m is an even integer, then*

$$\sum_{n=1}^{\infty} \frac{\left\lfloor \frac{n}{k} \right\rfloor}{L_m^{n+1}} \left(\frac{L_{mn+r} L_m}{n} - \frac{L_{m(n+1)+r}}{n+1} \right) = -\frac{\beta^r}{k} \log \left(L_m^{2k} - L_m^k L_{mk} + 1 \right) - \frac{\sqrt{5}}{k} F_r \log \left(L_m^k - \alpha^{mk} \right) + L_r \log \left(L_m \right) \tag{2.22}$$

and

$$\sum_{n=1}^{\infty} \frac{\left\lfloor \frac{n}{k} \right\rfloor}{L_m^{n+1}} \left(\frac{F_{mn+r} L_m}{n} - \frac{F_{m(n+1)+r}}{n+1} \right) = \frac{\beta^r}{\sqrt{5}\, k} \log \left(L_m^{2k} - L_m^k L_{mk} + 1 \right) - \frac{L_r}{\sqrt{5}\, k} \log \left(L_m^k - \alpha^{mk} \right) + F_r \log \left(L_m \right). \tag{2.23}$$

Proof. Choosing $z = \frac{\alpha^m}{L_m}$ and $z = \frac{\beta^m}{L_m}$ in (2.20) produces

$$\sum_{n=1}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor \left(\frac{\alpha^{mn+r}}{n L_m^n} - \frac{\alpha^{m(n+1)+r}}{(n+1) L_m^{n+1}} \right) = -\frac{\alpha^r}{k} \log \left(\frac{L_m^k - \alpha^{mk}}{L_m^k} \right)$$

and

$$\sum_{n=1}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor \left(\frac{\beta^{mn+r}}{n L_m^n} - \frac{\beta^{m(n+1)+r}}{(n+1) L_m^{n+1}} \right) = -\frac{\beta^r}{k} \log \left(\frac{L_m^k - \beta^{mk}}{L_m^k} \right),$$

from which (2.22) and (2.23) follow. □

Theorem 2.20. *If $k \geq 1$ is an integer, r is any integer and m is an odd integer, then*

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\left\lfloor \frac{n}{k} \right\rfloor}{\sqrt{5}^n F_m^{n+1}} \left(\frac{\sqrt{5} L_{mn+r} F_m}{n} - \frac{L_{m(n+1)+r}}{n+1} \right) \quad &= \quad \sqrt{5} L_r \log \left(\sqrt{5} F_m \right) \\ &- \frac{\sqrt{5}\alpha^r}{k} \log \left(5^k F_m^{2k} - \sqrt{5}^k F_m^k L_{mk} + (-1)^k \right) + \frac{5 F_r}{k} \log \left(\sqrt{5}^k F_m^k - \beta^{mk} \right) \end{aligned}$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\left\lfloor \frac{n}{k} \right\rfloor}{\sqrt{5}^n F_m^{n+1}} \left(\frac{\sqrt{5} F_{mn+r} F_m}{n} - \frac{F_{m(n+1)+r}}{n+1} \right) \quad &= \quad \sqrt{5} F_r \log(\sqrt{5} F_m) \\ &- \frac{\alpha^r}{k} \log \left(5^k F_m^{2k} - \sqrt{5}^k F_m^k L_{mk} + (-1)^k \right) + \frac{L_r}{k} \log \left(\sqrt{5}^k F_m^k - \beta^{mk} \right). \end{aligned}$$

Proof. Set $z = \frac{\alpha^m}{\sqrt{5} F_m}$ and $z = \frac{\beta^m}{\sqrt{5} F_m}$, in turn, in (2.20) and proceed as in the proof of Theorem 2.19. □

The alternating versions of (2.22) and (2.23), obtained from (2.21), are left as an exercise.

In the next lemma we present a generalization of (2.20) and (2.21) using the polylogarithm function, defined as $\mathrm{Li}_m(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^m}$, for which more details and properties can be found in [13].

Lemma 2.21. *Let $k \geq 1$ be an integer. For any integer m and $|z| < 1$, we have*

$$\sum_{n=1}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor \left(\frac{z^n}{n^m} - \frac{z^{n+1}}{(n+1)^m} \right) = \frac{1}{k^m} \mathrm{Li}_m(z^k) \tag{2.24}$$

and

$$\sum_{n=1}^{\infty} (-1)^n \left\lfloor \frac{n}{k} \right\rfloor \left(\frac{z^n}{n^m} + \frac{z^{n+1}}{(n+1)^m} \right) = \frac{1}{k^m} \mathrm{Li}_m \left((-1)^k z^k \right).$$

If $|z| = 1$, both series are valid for integer $m \geq 2$.

Proof. We prove (2.24) by induction on m for a fixed k. The identity is valid for $m = 1$, corresponding to (2.20). Assume that it is valid for an integer $m = r$. We therefore have the hypothesis:

$$P_r: \quad \sum_{n=1}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor \left(\frac{z^n}{n^r} - \frac{z^{n+1}}{(n+1)^r} \right) = \frac{1}{k^r} \operatorname{Li}_r(z^k).$$

We wish to prove that P_r implies P_{r+1}. The P_r identity is true for $z = 0$. Dividing through by $z \neq 0$ gives

$$\sum_{n=1}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor \left(\frac{z^{n-1}}{n^r} - \frac{z^n}{(n+1)^r} \right) = \frac{1}{k^r} \frac{\operatorname{Li}_r(z^k)}{z},$$

which, upon integration with respect to z from 0 to z yields

$$P_{r+1}: \quad \sum_{n=1}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor \left(\frac{z^n}{n^{r+1}} - \frac{z^{n+1}}{(n+1)^{r+1}} \right) = \frac{1}{k^{r+1}} \operatorname{Li}_{r+1}(z^k),$$

since

$$\int_0^z \frac{\operatorname{Li}_r(x^k)}{x} dx = \int_0^z \sum_{n=1}^{\infty} \frac{x^{kn-1}}{n^r} dx = \sum_{n=1}^{\infty} \frac{1}{n^r} \int_0^z x^{kn-1} dx = \frac{1}{k} \sum_{n=1}^{\infty} \frac{z^{kn}}{n^{r+1}} = \frac{\operatorname{Li}_{r+1}(z^k)}{k}.$$

Thus, P_r implies P_{r+1}, as desired. □

Remark 2.22. Since $\operatorname{Li}_m(e^{2\pi i}) = \zeta(m)$, for $m > 1$, from (2.24) we have a representation of the Riemann zeta function as follows:

$$\zeta(m) = k^m \sum_{n=1}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor \frac{(n+1)^m - n^m}{n^m (n+1)^m}, \qquad k \geq 1.$$

Lemma 2.23. *Let H_j be the j-th harmonic number and k be a positive integer. If $|z| < 1$, then*

$$\sum_{n=0}^{\infty} \frac{H_{\lfloor n/k \rfloor}}{(\lfloor n/k \rfloor + 1)^2} z^{n+k} = \frac{1-z^k}{1-z} \left(\frac{k}{2} \log z \log^2(1-z^k) + \log(1-z^k) \operatorname{Li}_2(1-z^k) - \operatorname{Li}_3(1-z^k) + \zeta(3) \right). \tag{2.25}$$

Proof. It is known that [13, p. 303, Formula (12)]:

$$\sum_{n=0}^{\infty} \frac{H_n}{(n+1)^2} z^{n+1} = \frac{1}{2} \log z \log^2(1-z) + \log(1-z) \operatorname{Li}_2(1-z) - \operatorname{Li}_3(1-z) + \zeta(3).$$

Thus, with $a_n = \frac{H_n}{(n+1)^2}$ in (2.1), the result follows. □

From (2.25) we have a series representation of $\zeta(3)$, which we present as the next theorem.

Theorem 2.24. *The following formula holds:*

$$\zeta(3) = 4(2 - \sqrt{2}) \sum_{n=1}^{\infty} \frac{H_{\lfloor \frac{n}{2} \rfloor}}{2^{\frac{n}{2}} \left\lfloor \frac{n+2}{2} \right\rfloor^2} + \frac{4}{3} \log^3 2. \tag{2.26}$$

Proof. The proof follows from (2.25) by using the formulas (*cf.* [13])

$$\operatorname{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{\log^2 2}{2} \quad \text{and} \quad \operatorname{Li}_3\left(\frac{1}{2}\right) = \frac{\log^3 2}{6} - \frac{\pi^2 \log 2}{12} + \frac{7}{8}\zeta(3).$$

□

Lemma 2.25. *Let k be a positive even integer and $|z| < 1$. Let $(a_n)_{n\geq 0}$ be an arbitrary sequence. Then*

$$(1 - z)\sum_{n=0}^{\infty} a_{\lfloor n/k\rfloor} z^n = (1 + z)\sum_{n=0}^{\infty} (-1)^n a_{\lfloor n/k\rfloor} z^n. \tag{2.27}$$

In particular, for all convergent sequences $(a_n)_{n\geq 0}$ we have

$$\sum_{n=0}^{\infty} (-1)^n a_{\lfloor n/k\rfloor} = 0, \qquad k \ \text{even}.$$

Proof. Let $h(z, k) = (1 - z)F^+(z, k)$. Clearly, from (2.1), $h(-z, k) = h(z, k)$ if k is an even integer; and hence (2.27). □

Example 2.26. Lemma 2.25 yields for $k \geq 1$:

$$\sum_{n=1}^{\infty} \frac{1 - 3(-1)^n}{2^n} F_{\lfloor \frac{n}{2k}\rfloor} = 0, \qquad \sum_{n=1}^{\infty} \frac{1 - 3(-1)^n}{2^n} L_{\lfloor \frac{n}{2k}\rfloor} = 0,$$
$$\sum_{n=1}^{\infty} \frac{1 - 2(-1)^n}{3^n} \left\lfloor \frac{n}{2k} \right\rfloor F_{\lfloor \frac{n}{2k}\rfloor} = 0, \qquad \sum_{n=1}^{\infty} \frac{1 - 2(-1)^n}{3^n} \left\lfloor \frac{n}{2k} \right\rfloor L_{\lfloor \frac{n}{2k}\rfloor} = 0,$$
$$\sum_{n=1}^{\infty} \frac{3 - 5(-1)^n}{4^n} F_{k\lfloor \frac{n}{2k}\rfloor} = 0, \qquad \sum_{n=1}^{\infty} \frac{3 - 5(-1)^n}{4^n} L_{k\lfloor \frac{n}{2k}\rfloor} = 0.$$

Theorem 2.27. *If k is a positive even integer, m any integer, $p \geq 2$ and $(a_n)_{n\geq 0}$ an arbitrary sequence, then*

$$\sum_{n=0}^{\infty} a_{\lfloor n/k\rfloor} \frac{pL_{n+m} - L_{n+m+1}}{p^n} = \sum_{n=0}^{\infty} (-1)^n a_{\lfloor n/k\rfloor} \frac{pL_{n+m} + L_{n+m+1}}{p^n} \tag{2.28}$$

and

$$\sum_{n=0}^{\infty} a_{\lfloor n/k\rfloor} \frac{pF_{n+m} - F_{n+m+1}}{p^n} = \sum_{n=0}^{\infty} (-1)^n a_{\lfloor n/k\rfloor} \frac{pF_{n+m} + F_{n+m+1}}{p^n}. \tag{2.29}$$

Proof. Setting $z = \frac{\alpha}{p}$ and $z = \frac{\beta}{p}$, respectively, in (2.27) gives

$$(p - \alpha)\sum_{n=0}^{\infty} a_{\lfloor n/k\rfloor} \frac{\alpha^{n+m}}{p^n} = (p + \alpha)\sum_{n=0}^{\infty} (-1)^n a_{\lfloor n/k\rfloor} \frac{\alpha^{n+m}}{p^n}$$

and

$$(p - \beta)\sum_{n=0}^{\infty} a_{\lfloor n/k\rfloor} \frac{\beta^{n+m}}{p^n} = (p + \beta)\sum_{n=0}^{\infty} (-1)^n a_{\lfloor n/k\rfloor} \frac{\beta^{n+m}}{p^n},$$

where m is an arbitrary integer. Respective addition and subtraction of these two identities produce (2.28) and (2.29). □

Example 2.28. For $p = 2$ and $p = 3$ in Theorem 2.27, we have:

$$\sum_{n=0}^{\infty} \frac{a_{\lfloor n/k\rfloor}}{2^n} (L_{n+m-2} - 5(-1)^n F_{n+m+1}) = 0,$$
$$\sum_{n=0}^{\infty} \frac{a_{\lfloor n/k\rfloor}}{2^n} (F_{n+m-2} - (-1)^n L_{n+m+1}) = 0,$$
$$\sum_{n=0}^{\infty} \frac{a_{\lfloor n/k\rfloor}}{3^n} (L_{n+m-1} - (-1)^n (F_{n+m} + L_{n+m+1})) = 0$$

and

$$\sum_{n=0}^{\infty}\frac{a_{\lfloor n/k\rfloor}}{3^n}(5F_{n+m-1}-(-1)^n(L_{n+m}+5F_{n+m+1}))=0.$$

Theorem 2.29. *If k is a positive even integer, m is an even integer, r any integer and $(a_n)_{n\geq 0}$ an arbitrary sequence, then*

$$\sum_{n=1}^{\infty}\frac{a_{\lfloor n/k\rfloor}}{L_m^n}(L_{mn+r-m}-(-1)^n(L_{mn+r}L_m+L_{m(n+1)+r}))=0$$

and

$$\sum_{n=1}^{\infty}\frac{a_{\lfloor n/k\rfloor}}{L_m^n}(F_{mn+r-m}-(-1)^n(F_{mn+r}L_m+F_{m(n+1)+r}))=0.$$

Proof. Set $z=\frac{\alpha^m}{L_m}$ and $z=\frac{\beta^m}{L_m}$, in turn, in (2.27) and proceed as in the proof of Theorem 2.7. □

Theorem 2.30. *If k is a positive even integer, m is an odd integer, r is any integer and $(a_n)_{n\geq 0}$ an arbitrary sequence, then*

$$\sum_{n=1}^{\infty}\frac{a_{\lfloor 2n/k\rfloor}F_{2mn+r+m}-a_{\lfloor (2n-1)/k\rfloor}F_m(L_{2m(n-1)+r}+L_{2mn+r})}{5^nF_m^{2n}}=0$$

and

$$\sum_{n=1}^{\infty}\frac{a_{\lfloor 2n/k\rfloor}L_{2mn+r+m}-5a_{\lfloor (2n-1)/k\rfloor}F_m(F_{2m(n-1)+r}+F_{2mn+r})}{5^nF_m^{2n}}z=0.$$

Proof. Set $z=\frac{\alpha^m}{\sqrt{5}F_m}$ and $z=\frac{\beta^m}{\sqrt{5}F_m}$, in turn, in (2.27) and proceed as in the proof of Theorem 2.7. □

3. The binomial transform of $(a_{\lfloor n/k\rfloor})_{n\geq 0}$

We begin by recalling some basic facts. Let $(a_n)_{n\geq 0}$ be an arbitrary sequence of numbers with ordinary generating function $F(z)$. Let further $(s_n)_{n\geq 0}$ be the generalized binomial transform of $(a_n)_{n\geq 0}$, i.e.,

$$s_n=\sum_{j=0}^{n}\binom{n}{j}b^{n-j}c^ja_j$$

with b and c being nonzero real numbers. Then the ordinary generating function of the sequence $(s_n)_{n\geq 0}$ is given by (*cf.* [4, 18]):

$$S(z)=\frac{1}{1-bz}F\left(\frac{cz}{1-bz}\right).$$

Therefore, the ordinary generating function of the sequence $s_n^*=\sum_{j=0}^{n}\binom{n}{j}b^{n-j}c^ja_{\lfloor j/k\rfloor}$ equals

$$S^*(z)=\frac{1}{1-bz}\frac{1-\left(\frac{cz}{1-bz}\right)^k}{1-\frac{cz}{1-bz}}F\left(\left(\frac{cz}{1-bz}\right)^k\right)=\frac{1-\left(\frac{cz}{1-bz}\right)^k}{1-(b+c)z}F\left(\left(\frac{cz}{1-bz}\right)^k\right). \tag{3.1}$$

In general, such a function will be a complicated expression. Focusing on the sequence $a_n=n$, we have

$$S^*(z,k)=\frac{1-\left(\frac{cz}{1-bz}\right)^k}{1-(b+c)z}\frac{\left(\frac{cz}{1-bz}\right)^k}{\left(1-\left(\frac{cz}{1-bz}\right)^k\right)^2}=\frac{(cz)^k}{(1-(b+c)z)((1-bz)^k-(cz)^k)}.$$

The complex function $P(z) = (1 - bz)^k - (cz)^k$ is a polynomial of degree k for $b \neq \pm c$ and we have $P(z) = (z - r_1)(z - r_2)\cdots(z - r_k)$, where $r_i \in \mathbb{C}$, $i = 1, \ldots, k$, are the roots of $P(z)$; see also [20]. The partial fraction decomposition yields

$$\frac{1}{P(z)} = \sum_{i=1}^{k} \frac{1}{z - r_i} \prod_{j=1,\, j\neq i}^{k} \frac{1}{r_j - r_i}.$$

This shows that (at least theoretically) the function $S^*(z, k)$ can be expressed as a convolution. But even for small values of k this will be a challenging issue. Our first result is therefore concerned with analyzing the case $k = 2$.

Theorem 3.1. *For any numbers b, $c \neq 0$, and integer $n \geq 1$, we have*

$$\sum_{j=1}^{n} \binom{n}{j} \left\lfloor \frac{j}{2} \right\rfloor b^{n-j} c^j = \frac{cn}{2}(b + c)^{n-1} - \frac{(b + c)^n - (b - c)^n}{4}. \tag{3.2}$$

If $b = c$, then this identity becomes

$$\sum_{j=1}^{n} \binom{n}{j} \left\lfloor \frac{j}{2} \right\rfloor = 2^{n-2}(n - 1).$$

Proof. If $k = 2$ we simplify $S^*(z, 2)$ and get

$$S^*(z, 2) = \frac{(cz)^2}{(1 - (b + c)z)(1 - 2bz + (b^2 - c^2)z^2)}.$$

Obviously, near zero there exists the power series $\frac{1}{1-(b+c)z} = \sum_{n=0}^{\infty} (b+c)^n z^n$. Also, from the partial fraction decomposition

$$\frac{1}{1 - 2bz + (b^2 - c^2)z^2} = \frac{b + c}{2c(1 - (b + c)z)} - \frac{b - c}{2c(1 - (b - c)z)}$$

we get (again near zero)

$$\frac{1}{1 - 2bz + (b^2 - c^2)z^2} = \frac{1}{2c} \sum_{n=0}^{\infty} ((b + c)^{n+1} - (b - c)^{n+1}) z^n.$$

Hence, applying Cauchy's product formula for power series

$$S^*(z, 2) = \frac{c}{2} \sum_{n=2}^{\infty} \sum_{j=0}^{n-2} (b + c)^j ((b + c)^{n-1-j} - (b - c)^{n-1-j}) z^n$$

or, for $n \geq 2$,

$$\begin{aligned} \sum_{j=1}^{n} \binom{n}{j} \left\lfloor \frac{j}{2} \right\rfloor b^{n-j} c^j &= \frac{c}{2} \sum_{j=0}^{n-2} (b + c)^j \left((b + c)^{n-1-j} - (b - c)^{n-1-j} \right) \\ &= \frac{c}{2}(b + c)^{n-1}(n - 1) - \frac{c}{2}(b - c)^{n-1} \sum_{j=0}^{n-2} \left(\frac{b + c}{b - c} \right)^j. \end{aligned}$$

The statement now follows by simplifying, making use of the geometric series. □

Corollary 3.2. *For $n \geq 2$ we have*

$$\sum_{j=1}^{n} (-1)^j \binom{n}{j} \left\lfloor \frac{j}{2} \right\rfloor = 2^{n-2}$$

and

$$\sum_{j=1}^{n} \binom{n}{j} \left\lfloor \frac{j}{2} \right\rfloor 2^j = n3^{n-1} - \frac{3^n - (-1)^n}{4}.$$

We proceed with some new binomial sums involving Fibonacci and Lucas numbers.

Theorem 3.3. *For m an integer, we have*

$$\sum_{j=1}^{n}\binom{n}{j}\left\lfloor\frac{j}{2}\right\rfloor F_{j+m}=\frac{n}{2}F_{2n+m-1}-\frac{1}{4}\left(F_{2n+m}+(-1)^{m}F_{n-m}\right) \tag{3.3}$$

and

$$\sum_{j=1}^{n}\binom{n}{j}\left\lfloor\frac{j}{2}\right\rfloor L_{j+m}=\frac{n}{2}L_{2n+m-1}-\frac{1}{4}\left(L_{2n+m}-(-1)^{m}L_{n-m}\right). \tag{3.4}$$

Proof. Work with $(b,c)=(1,\alpha)$ and $(b,c)=(1,\beta)$ in Theorem 3.1. When simplifying use $F_{-n}=(-1)^{n-1}F_n$ and $L_{-n}=(-1)^nL_n$, respectively. □

Identities (3.3) and (3.4) should be compared to the classical results (*cf.* [5, 27]):

$$\sum_{j=0}^{n}\binom{n}{j}F_{j+m}=F_{2n+m}\quad\text{and}\quad\sum_{j=0}^{n}\binom{n}{j}L_{j+m}=L_{2n+m}.$$

Theorem 3.4. *For m an integer and p an odd integer, we have*

$$\sum_{j=1}^{n}\binom{n}{j}\left\lfloor\frac{j}{2}\right\rfloor F_{2pj+m}=\begin{cases}\frac{n\sqrt{5}^{n-1}}{2}F_p^{n-1}F_{p(n+1)+m}-\frac{\sqrt{5}^{n-1}}{4}F_p^nL_{pn+m}-\frac{1}{4}L_p^nF_{pn+m}, & n\text{ odd};\\ \frac{n\sqrt{5}^{n-2}}{2}F_p^{n-1}L_{p(n+1)+m}-\frac{\sqrt{5}^{n}}{4}F_p^nF_{pn+m}+\frac{1}{4}L_p^nF_{pn+m}, & n\text{ even}\end{cases} \tag{3.5}$$

and

$$\sum_{j=1}^{n}\binom{n}{j}\left\lfloor\frac{j}{2}\right\rfloor L_{2pj+m}=\begin{cases}\frac{n\sqrt{5}^{n-1}}{2}F_p^{n-1}L_{p(n+1)+m}-\frac{\sqrt{5}^{n+1}}{4}F_p^nF_{pn+m}-\frac{1}{4}L_p^nL_{pn+m}, & n\text{ odd};\\ \frac{n\sqrt{5}^{n}}{2}F_p^{n-1}F_{p(n+1)+m}-\frac{\sqrt{5}^{n}}{4}F_p^nL_{pn+m}+\frac{1}{4}L_p^nL_{pn+m}, & n\text{ even}.\end{cases} \tag{3.6}$$

Proof. Work with $(b,c)=(1,\alpha^p)$ and $(b,c)=(1,\beta^p)$ in Theorem 3.1, using, for odd p, the relations $1+\alpha^{2p}=\sqrt{5}F_p\alpha^p$, $1+\beta^{2p}=-\sqrt{5}F_p\beta^p$, $1-\alpha^{2p}=-L_p\alpha^p$, and $1-\beta^{2p}=-L_p\beta^p$. □

Theorem 3.5. *For m an integer and p an even integer, we have*

$$\sum_{j=0}^{n}\binom{n}{j}\left\lfloor\frac{j}{2}\right\rfloor F_{2pj+m}=\begin{cases}\frac{n}{2}L_p^{n-1}F_{p(n+1)+m}-\frac{1}{4}L_p^nF_{pn+m}-\frac{5^{(n-1)/2}}{4}F_p^nL_{pn+m}, & n\text{ odd};\\ \frac{n}{2}L_p^{n-1}F_{p(n+1)+m}-\frac{1}{4}L_p^nF_{pn+m}+\frac{5^{n/2}}{4}F_p^nF_{pn+m}, & n\text{ even}\end{cases}$$

and

$$\sum_{j=0}^{n}\binom{n}{j}\left\lfloor\frac{j}{2}\right\rfloor L_{2pj+m}=\begin{cases}\frac{n}{2}L_p^{n-1}L_{p(n+1)+m}-\frac{1}{4}L_p^nL_{pn+m}-\frac{5^{(n+1)/2}}{4}F_p^nF_{pn+m}, & n\text{ odd};\\ \frac{n}{2}L_p^{n-1}L_{p(n+1)+m}-\frac{1}{4}L_p^nL_{pn+m}+\frac{5^{n/2}}{4}F_p^nL_{pn+m}, & n\text{ even}.\end{cases}$$

Proof. Work with $(b,c)=(1,\alpha^p)$ and $(b,c)=(1,\beta^p)$ in Theorem 3.1, using, for even p, the relations $1+\alpha^{2p}=L_p\alpha^p$, $1+\beta^{2p}=L_p\beta^p$, $1-\alpha^{2p}=-\sqrt{5}F_p\alpha^p$, and $1-\beta^{2p}=\sqrt{5}F_p\beta^p$. □

The classical counterparts of identities (3.5) and (3.6) are

$$\sum_{j=0}^{n}\binom{n}{j}F_{2j+m}=\begin{cases}5^{(n-1)/2}L_{n+m}, & n\text{ odd};\\ 5^{n/2}F_{n+m}, & n\text{ even}\end{cases}$$

and

$$\sum_{j=0}^{n}\binom{n}{j}L_{2j+m}=\begin{cases}5^{(n+1)/2}F_{n+m}, & n\text{ odd};\\ 5^{n/2}L_{n+m}, & n\text{ even}\end{cases}$$

(*cf.* [5]).

Theorem 3.6. *For m an integer, we have*

$$\sum_{j=1}^{n}\binom{n}{j}\left\lfloor\frac{j}{2}\right\rfloor F_{3j+m}=2^{n-2}(nF_{2n+m+1}-F_{2n+m}+(-1)^nF_{n+m})$$

and

$$\sum_{j=1}^{n}\binom{n}{j}\left\lfloor\frac{j}{2}\right\rfloor L_{3j+m}=2^{n-2}(nL_{2n+m+1}-L_{2n+m}+(-1)^nL_{n+m}).$$

Proof. Work with $(b,c)=(1,\alpha^3)$ and $(b,c)=(1,\beta^3)$ in Theorem 3.1. When simplifying use $1-\alpha^3=-2\alpha$, $1-\beta^3=-2\beta$, $1+\alpha^3=2\alpha^2$ and $1+\beta^3=2\beta^2$. □

Theorem 3.7. *For any integer m, we have*

$$\sum_{j=1}^{n}\binom{n}{j}\left\lfloor\frac{j}{2}\right\rfloor 2^jF_{j+m}=\begin{cases}nF_{3n+m-2}-\frac{1}{4}F_{3n+m}+\frac{5^{n/2}}{4}F_m, & n \text{ even};\\ nF_{3n+m-2}-\frac{1}{4}F_{3n+m}-\frac{5^{(n-1)/2}}{4}L_m, & n \text{ odd}\end{cases}$$

and

$$\sum_{j=1}^{n}\binom{n}{j}\left\lfloor\frac{j}{2}\right\rfloor 2^jL_{j+m}=\begin{cases}nL_{3n+m-2}-\frac{1}{4}L_{3n+m}+\frac{5^{n/2}}{4}L_m, & n \text{ even};\\ nL_{3n+m-2}-\frac{1}{4}L_{3n+m}-\frac{5^{(n+1)/2}}{4}F_m, & n \text{ odd}.\end{cases}$$

Proof. Work with $(b,c)=(1,2\alpha)$ and $(b,c)=(1,2\beta)$ in Theorem 3.1. When simplifying use the relations $1-2\alpha=-\sqrt{5}$, $1-2\beta=\sqrt{5}$, $1+2\alpha=\alpha^3$ and $1+2\beta=\beta^3$. □

The classical counterparts of the identities from Theorem 3.7 are

$$\sum_{j=0}^{n}\binom{n}{j}2^jF_{j+m}=F_{3n+m}\quad\text{and}\quad\sum_{j=0}^{n}\binom{n}{j}2^jL_{j+m}=L_{3n+m}$$

(*cf.* [5]).

Although it is possible to state some more binomial identities we conclude with the following sums.

Theorem 3.8. *For any integers m and q, we have*

$$\sum_{j=0}^{n}(-1)^{q(n-j)}\binom{n}{j}\left\lfloor\frac{j}{2}\right\rfloor F_{2qj+m}=\begin{cases}\frac{n}{2}F_{q(n+1)+m}L_q^{n-1}-\frac{1}{4}F_{qn+m}L_q^n+\frac{5^{n/2}}{4}F_{qn+m}F_q^n, & n \text{ even};\\ \frac{n}{2}F_{q(n+1)+m}L_q^{n-1}-\frac{1}{4}F_{qn+m}L_q^n-\frac{5^{(n-1)/2}}{4}L_{qn+m}F_q^n, & n \text{ odd}\end{cases}$$

and

$$\sum_{j=0}^{n}(-1)^{q(n-j)}\binom{n}{j}\left\lfloor\frac{j}{2}\right\rfloor L_{2qj+m}=\begin{cases}\frac{n}{2}L_{q(n+1)+m}L_q^{n-1}-\frac{1}{4}L_{qn+m}L_q^n+\frac{5^{n/2}}{4}L_{qn+m}F_q^n, & n \text{ even};\\ \frac{n}{2}L_{q(n+1)+m}L_q^{n-1}-\frac{1}{4}L_{qn+m}L_q^n-\frac{5^{(n+1)/2}}{4}F_{qn+m}F_q^n, & n \text{ odd}.\end{cases}$$

Proof. Work with $(b,c)=((-1)^q,\alpha^{2q})$ and $(b,c)=((-1)^q,\beta^{2q})$ in Theorem 3.1, respectively. When simplifying use the relations $(-1)^q+\alpha^{2q}=\alpha^qL_q$, $(-1)^q-\alpha^{2q}=-\alpha^qF_q\sqrt{5}$, $(-1)^q+\beta^{2q}=\beta^qL_q$, and $(-1)^q-\beta^{2q}=\beta^qF_q\sqrt{5}$. □

Corollary 3.9. *Let n, r and s be non-negative integers such that $n\geq r+s$. Let b and c be nonzero numbers. Then*

$$\sum_{j=0}^{n}\binom{n-s-r}{j-s}\left\lfloor\frac{j}{2}\right\rfloor b^{n-j-r}c^{j-s}=\frac{(n-r)c+bs}{2}(b+c)^{n-r-s-1}-\frac{1}{4}\left((b+c)^{n-r-s}-(-1)^s(b-c)^{n-r-s}\right).\tag{3.7}$$

Proof. Differentiate (3.2) r times with respect to b and s times with respect to c. □

Given (3.7), it is obvious that Theorems 3.3 to 3.8 allow further generalizations.

Theorem 3.10. *Let n, r and s be non-negative integers. Then*

$$\sum_{j=1}^{n} \binom{n-s-r}{j-s} \left\lfloor \frac{j}{2} \right\rfloor = 2^{n-s-r-2}(n+s-r-1), \qquad n > r+s$$

and

$$\sum_{j=1}^{n} (-1)^j \binom{n-s-r}{j-s} \left\lfloor \frac{j}{2} \right\rfloor = 2^{n-s-r-2}, \qquad n \geq r+s+2.$$

Proof. Evaluate (3.7) at $b = c$ and $b = -c$, respectively. □

Corollary 3.11. *Let n, r and s be non-negative integers such that $n \geq r+s+2$. Then*

$$\sum_{j=1}^{\lfloor n/2 \rfloor} \binom{n-s-r}{2j-s} j = 2^{n-s-r-3}(n+s-r)$$

and

$$\sum_{j=1}^{\lfloor n/2 \rfloor} \binom{n-s-r}{2j-s+1} j = 2^{n-s-r-3}(n+s-r-2).$$

Proof. Writing the sums in Theorem 3.10 according to the parity of the index, we obtain $A+B = 2^{n-s-r-2}(n+s-r-1)$ and $A - B = 2^{n-s-r-2}$, where

$$A = \sum_{j=1}^{\lfloor n/2 \rfloor} \binom{n-s-r}{2j-s} j, \qquad B = \sum_{j=1}^{\lfloor n/2 \rfloor} \binom{n-s-r}{2j-s+1} j.$$

The result follows immediately by solving this linear system. □

Lemma 3.12. *If p, q, r and s are rational numbers, then $p + q\sqrt{5} = r + s\sqrt{5}$ if and only if $p = r$ and $q = s$.*

Proof. From $p + q\sqrt{5} = r + s\sqrt{5}$ we obtain $(p-r) + (q-s)\sqrt{5} = 0$. If $q \neq s$, then $\sqrt{5} = -\frac{p-r}{q-s} \in \mathbb{Q}$, a contradiction. Hence $q = s$, and then $p = r$.

The converse is obvious. □

Theorem 3.13. *Let n, r and s be non-negative integers with $n \geq r \geq s$. If s is odd, then*

$$\sum_{j=1}^{\lfloor n/2 \rfloor} \binom{n-r}{2j-s} 5^j j = 2^{n-r-3} 5^{\frac{s+1}{2}} \left((n-r)L_{n-r-1} + 2sF_{n-r}\right), \tag{3.8}$$

$$\sum_{j=1}^{\lfloor n/2 \rfloor} \binom{n-r}{2j+1-s} 5^j j = 2^{n-r-3} 5^{\frac{s-1}{2}} \left(5(n-r)F_{n-r-1} + 2(s-1)L_{n-r}\right), \tag{3.9}$$

while if s is even, then

$$\sum_{j=1}^{\lfloor n/2 \rfloor} \binom{n-r}{2j-s} 5^j j = 2^{n-r-3} 5^{\frac{s}{2}} (5(n-r)F_{n-r-1} + 2sL_{n-r}), \tag{3.10}$$

$$\sum_{j=1}^{\lfloor n/2 \rfloor} \binom{n-r}{2j+1-s} 5^j j = 2^{n-r-3} 5^{\frac{s}{2}} ((n-r)L_{n-r-1} + 2(s-1)F_{n-r}). \tag{3.11}$$

Proof. Choosing $b = \frac{1}{2}$ and $c = \frac{\sqrt{5}}{2}$ in (3.7), and assuming s is an odd integer, gives

$$\frac{\sqrt{5}}{2^{n-r-s}}\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n-s-r}{2j-s}5^j j + \frac{1}{2^{n-r-s}}\sum_{j=1}^{\lceil n/2\rceil}\binom{n-s-r}{2j-1-s}5^j(j-1) = \frac{\sqrt{5}(n-s-r)}{4}\alpha^{n-r-s-1} + \frac{s}{2}\alpha^{n-r-s} - \frac{1}{4}L_{n-r-s}.$$

The use of $2\alpha^m = L_m + F_m\sqrt{5}$ in the above identity, along with the application of Lemma 3.12 yields (3.8) and (3.9). The proof of (3.10) and (3.11) is similar. □

4. More series and identities

From the general relations (2.1) and (3.1) it is obvious that many particular examples can be studied. For instance, by choosing $a_n = n^2$ we get the next corollary.

Corollary 4.1. *For $k \geq 1$ and $z \in \mathbb{C}$ with $|z| < 1$, the following expressions are valid:*

$$\sum_{n=0}^{\infty}\left\lfloor\frac{n}{k}\right\rfloor^2 z^n = \frac{z^k(1+z^k)}{(1-z)(1-z^k)^2}$$

and

$$\sum_{n=0}^{\infty}(-1)^n\left\lfloor\frac{n}{k}\right\rfloor^2 z^n = \frac{(-1)^k z^k(1+(-1)^k z^k)}{(1+z)(1+(-1)^{k+1}z^k)^2}.$$

Proof. Use Lemma 2.1 with $a_n = n^2$ in conjunction with $F(z) = \sum_{n=0}^{\infty} n^2 z^n = \frac{z(1+z)}{(1-z)^3}$. The second identity follows from the first by replacing z with $-z$. □

Corollary 4.1 also leads to new Fibonacci (Lucas) series evaluations, one of which is stated in the next theorem.

Theorem 4.2. *For $k \geq 1$ and m integers, we have*

$$\sum_{n=0}^{\infty}\left\lfloor\frac{n}{k}\right\rfloor^2\frac{F_{n+m-2}}{2^{n+1}} = \frac{4^k F_{m+2k} + 2^{k+1}(2^{2k-1}-(-1)^k)F_{m+k} + 2^k F_{m-k} + (1-2(-4)^k)F_m}{\left(4^k - 2^k L_k + (-1)^k\right)^2}$$

and

$$\sum_{n=0}^{\infty}\left\lfloor\frac{n}{k}\right\rfloor^2\frac{L_{n+m-2}}{2^{n+1}} = \frac{4^k L_{m+2k} + 2^{k+1}(2^{2k-1}-(-1)^k)L_{m+k} + 2^k L_{m-k} + (1-2(-4)^k)L_m}{\left(4^k - 2^k L_k + (-1)^k\right)^2}.$$

For the binomial transform of $a_{\lfloor n/k\rfloor} = \lfloor\frac{n}{k}\rfloor^2$, i.e., the sum $u_n = \sum_{j=0}^{n}\binom{n}{j}b^{n-j}c^j\lfloor\frac{n}{k}\rfloor^2$ with b and c real, the ordinary generating function is given by

$$S_u^*(z,k) = \frac{1-(\frac{cz}{1-bz})^k}{1-(b+c)z}\,\frac{(\frac{cz}{1-bz})^k\left(1+(\frac{cz}{1-bz})^k\right)}{\left(1-(\frac{cz}{1-bz})^k\right)^3} = \frac{(\frac{cz}{1-bz})^k\left(1+(\frac{cz}{1-bz})^k\right)}{(1-(b+c)z)\left(1-(\frac{cz}{1-bz})^k\right)^2}.$$

To highlight the increasing algebraic complexity, we again consider only the case $k = 2$. In this particular case, the ordinary generating function reduces to

$$S_u^*(z,2) = \frac{(cz)^2((1-bz)^2+(cz)^2)}{(1-(b+c)z)((1-bz)^2-(cz)^2)^2}.$$

From this expression, we can prove the following result.

Theorem 4.3. *For nonzero real numbers b and c and $n \geq 1$, we have*

$$\sum_{j=1}^{n} \binom{n}{j} \left\lfloor \frac{j}{2} \right\rfloor^2 b^{n-j} c^j = \frac{c^2 n(n-1)(b+c)^{n-2}}{4} + \frac{(b+c)^n - (b-c)^n}{8} - \frac{cn(b-c)^{n-1}}{4}. \tag{4.1}$$

Proof. We write

$$S_u^*(z,2) = \frac{(cz)^2((1-bz)^2 + (cz)^2)}{(1-(b+c)z)^3(1-(b-c)z)^2}$$

and use the power series

$$\frac{1}{(1-wz)^2} = \sum_{n=0}^{\infty}(n+1)w^n z^n \quad \text{and} \quad \frac{2}{(1-wz)^3} = \sum_{n=0}^{\infty}(n+1)(n+2)w^n z^n$$

to get

$$\frac{2}{(1-(b+c)z)^3(1-(b-c)z)^2} = \sum_{n=0}^{\infty}\sum_{j=0}^{n}(j+1)(b-c)^j(n-j+1)(n-j+2)(b+c)^{n-j}z^n.$$

This shows that

$$\begin{aligned}
S_u^*(z,2) &= \frac{c^2}{2}\sum_{n=0}^{\infty}\sum_{j=0}^{n}(j+1)(b-c)^j(n-j+1)(n-j+2)(b+c)^{n-j}z^{n+2} \\
&\quad -bc^2\sum_{n=0}^{\infty}\sum_{j=0}^{n}(j+1)(b-c)^j(n-j+1)(n-j+2)(b+c)^{n-j}z^{n+3} \\
&\quad +\frac{b^2c^2+c^4}{2}\sum_{n=0}^{\infty}\sum_{j=0}^{n}(j+1)(b-c)^j(n-j+1)(n-j+2)(b+c)^{n-j}z^{n+4} \\
&= c^2z^2 + (3bc^2+c^3)z^3 + \sum_{n=4}^{\infty}(A(n)+B(n))z^n
\end{aligned}$$

with

$$\begin{aligned}
A(n) &= c^2(n-1)(b-c)^{n-2} + 3c^2(n-2)(b-c)^{n-3}(b+c) - 2bc^2(n-2)(b-c)^{n-3}, \\
B(n) &= \sum_{j=0}^{n-4}(j+1)(b-c)^j(b+c)^{n-4-j}\Big(\frac{c^2}{2}(b+c)^2(n-j)(n-1-j) \\
&\quad -bc^2(b+c)(n-1-j)(n-2-j) + \frac{1}{2}(b^2c^2+c^4)(n-2-j)(n-3-j)\Big) \\
&= \frac{c^2}{2}(b+c)^{n-2}\sum_{j=0}^{n-4}(j+1)(n-j)(n-1-j)q^j \\
&\quad -bc^2(b+c)^{n-3}\sum_{j=0}^{n-4}(j+1)(n-1-j)(n-2-j)q^j \\
&\quad +\frac{1}{2}(b^2c^2+c^4)(b+c)^{n-4}\sum_{j=0}^{n-4}(j+1)(n-2-j)(n-3-j)q^j,
\end{aligned}$$

where $q = \frac{b-c}{b+c}$. After simplifying $A(n)$ this shows that

$$\sum_{j=1}^{n}\binom{n}{j}\left\lfloor\frac{j}{2}\right\rfloor^2 b^{n-j}c^j = \begin{cases} c^2, & n=2; \\ 3bc^2+c^3, & n=3; \\ c^2(b-c)^{n-3}(b+c)(2n-3) - 2c^3(b-c)^{n-3} + B(n), & n \geq 4. \end{cases}$$

From here we use the convolution identities

$$\begin{aligned}\sum_{j=0}^{n-4}(j+1)(n-j)(n-1-j)z^j &= \frac{1}{(z-1)^4z^3}\Big((n^2+3n+2)z^5-2(n^2+n-2)z^4\\&\quad+(n^2-n)z^3-6(n-2)z^n+(22n-46)z^{n+1}-4(7n-16)z^{n+2}+12(n-3)z^{n+3}\Big),\\ \sum_{j=0}^{n-4}(j+1)(n-1-j)(n-2-j)z^j &= \frac{1}{(z-1)^4z^3}\Big((n^2+n)z^5-2(n^2-n-2)z^4\\&\quad+(n^2-3n+2)z^3-2(n-2)z^n+8(n-2)z^{n+1}-12(n-2)z^{n+2}+6(n-3)z^{n+3}\Big)\end{aligned}$$

and

$$\sum_{j=0}^{n-4}(j+1)(n-2-j)(n-3-j)z^j=\frac{1}{(z-1)^4z^3}\Big((n^2-5n+6)z-2(n^2-3n)z^2+(n^2-n)z^3-2nz^n+2(n-3)z^{n+1}\Big),$$

insert into $B(n)$ and simplify to get the intimidating expression

$$\sum_{j=1}^{n}\binom{n}{j}\Big\lfloor\frac{j}{2}\Big\rfloor^2 b^{n-j}c^j=\begin{cases}c^2, & n=2;\\ 3bc^2+c^3, & n=3;\\ S(n,b,c,q), & n\geq 4,\end{cases}$$

where

$$\begin{aligned}S(n,b,c,q) &= c^2(b-c)^{n-3}(b+c)(2n-3)-2c^3(b-c)^{n-3}\\&\quad+8c^6(b+c)^{n-9}(b-c)^3((n^2+3n+2)q^5-(2n^2+2n-4)q^4\\&\quad+(n^2-n)q^3-(6n-12)q^n+(22n-46)q^{n+1}-(28n-64)q^{n+2}+(12n-36)q^{n+3})\\&\quad-16bc^6(b+c)^{n-10}(b-c)^3((n^2+n)q^5-(2n^2-2n-4)q^4+(n^2-3n+2)q^3\\&\quad-(2n-4)q^n+(8n-16)q^{n+1}-(12n-24)q^{n+2}+(6n-18)q^{n+3})\\&\quad+8c^4(b^2c^2+c^4)(b+c)^{n-11}(b-c)^3((n^2-5n+6)q-(2n^2-6n)q^2\\&\quad+(n^2-n)q^3-2nq^n+(2n-6)q^{n+1})\end{aligned}$$

with $q=\frac{b-c}{b+c}$. Additional simplifications reduce the above expression to the stated form. □

Remark 4.4. A much shorter and more elegant proof of Theorem 4.3 not relying on generating functions goes as follows. Let

$$h_1(x;b,c,n)=\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j}b^{n-2j}c^{2j}x^{2j}=\frac{(b+cx)^n+(b-cx)^n}{2}$$

and

$$h_2(x;b,c,n)=\sum_{j=1}^{\lceil n/2\rceil}\binom{n}{2j-1}b^{n-2j+1}c^{2j-1}x^{2j-2}=\frac{(b+cx)^n-(b-cx)^n}{2x}.$$

Then

$$4\sum_{j=1}^{n}\binom{n}{j}\Big\lfloor\frac{j}{2}\Big\rfloor^2 b^{n-j}c^j=\frac{d}{dx}\left(x\frac{dh_1}{dx}\right)\bigg|_{x=1}+\frac{d}{dx}\left(x\frac{dh_2}{dx}\right)\bigg|_{x=1},$$

from which the result follows. This procedure also gives Theorem 3.1 as

$$2\sum_{j=1}^{n}\binom{n}{j}\Big\lfloor\frac{j}{2}\Big\rfloor b^{n-j}c^j=\frac{dh_1}{dx}\bigg|_{x=1}+\frac{dh_2}{dx}\bigg|_{x=1}.$$

Of course, sums involving higher powers of the floor function can be computed in this manner.

As an example of a binomial sum with Fibonacci (Lucas) numbers and $\lfloor \frac{j}{2} \rfloor^2$ we can state the next theorem. We invite interested readers to derive many more identities of this kind.

Theorem 4.5. *For m an integer and n a non-negative integer, we have*

$$\sum_{j=1}^{n}\binom{n}{j}\left\lfloor\frac{j}{2}\right\rfloor^2 F_{j+m} = \frac{n(n-1)}{4}F_{2n+m-2} + \frac{1}{8}(F_{2n+m} + (-1)^m F_{n-m}) - \frac{(-1)^m n}{4}F_{n-2-m}$$

and

$$\sum_{j=1}^{n}\binom{n}{j}\left\lfloor\frac{j}{2}\right\rfloor^2 L_{j+m} = \frac{n(n-1)}{4}L_{2n+m-2} + \frac{1}{8}(L_{2n+m} - (-1)^m L_{n-m}) + \frac{(-1)^m n}{4}L_{n-2-m}.$$

Corollary 4.6. *We have*

$$\sum_{j=1}^{n}\binom{n}{j}\left\lfloor\frac{j}{2}\right\rfloor^2 = \begin{cases} 0, & n = 1; \\ 2^{n-4}(n^2 - n + 2), & n \geq 2 \end{cases}$$

and

$$\sum_{j=1}^{n}(-1)^j\binom{n}{j}\left\lfloor\frac{j}{2}\right\rfloor^2 = \begin{cases} 0, & n = 1; \\ 1, & n = 2; \\ 2^{n-3}(n-1), & n \geq 3. \end{cases}$$

Proof. Set $b = c$ in (4.1) in conjunction with the convention $0^0 = 1$. For the second identity, work with $b = 1$ and $c = -1$ in (4.1). □

Corollary 4.7. *Let n, r and s be non-negative integers such that $n \geq r \geq s$. Let b and c be nonzero real numbers. Then*

$$\begin{aligned}\sum_{j=1}^{n}\binom{n-r}{j-s}b^{n-j-r+s}c^{j-s}\left\lfloor\frac{j}{2}\right\rfloor^2 &= \frac{(n-r-1)(n-r)}{4}c^2(b+c)^{n-r-2} \\ &+\frac{(n-r)c}{4}\left(2s(b+c)^{n-r-1} - (-1)^s(b-c)^{n-r-1}\right) \\ &+\frac{2s(s-1)+1}{8}(b+c)^{n-r} + \frac{(-1)^s(2s-1)}{8}(b-c)^{n-r}. \end{aligned} \tag{4.2}$$

Proof. Differentiate (4.1) r times with respect to b and s times with respect to c. □

In particular, we have

$$\sum_{j=1}^{n}\binom{n-r}{j-s}\left\lfloor\frac{j}{2}\right\rfloor^2 = 2^{n-r-4}\left(\left(n + 2s - r - \frac{1}{2}\right)^2 + \frac{7}{4} - 2s\right), \qquad n-2 \geq r \geq s$$

and

$$\sum_{j=1}^{n}(-1)^j\binom{n-r}{j-s}\left\lfloor\frac{j}{2}\right\rfloor^2 = 2^{n-r-3}\left(n + 2s - r - 1\right), \qquad n-2 > r \geq s,$$

with the special values, for $n \geq 2s + 2$,

$$\sum_{j=1}^{2n-s}\binom{n-2s}{j-s}\left\lfloor\frac{j}{2}\right\rfloor^2 = 2^{n-2s-4}\left(n^2 - n - 2s + 2\right)$$

and

$$\sum_{j=1}^{n}(-1)^j\binom{n-2s+1}{j-s}\left\lfloor\frac{j}{2}\right\rfloor^2 = 2^{n-2s-2}n.$$

Corollary 4.8. *Let n, r and s be non-negative integers such that $n-2>r\geq s$. Then*

$$\sum_{j=1}^{\lfloor n/2\rfloor}\binom{n-r}{2j-s}j^2=2^{n-r-5}((n-r+2s)^2+n-r)$$

and

$$\sum_{j=1}^{\lfloor n/2\rfloor}\binom{n-r}{2j-s+1}j^2=2^{n-r-5}((n-r+2s)^2-3(n-r)-8s+4).$$

Proof. Proceeding as in the proof of Corollary 3.11 and separating the sums according to the parity of the index, we obtain the required identities from Corollary 4.7 with $b=c=1$. □

Theorem 4.9. *Let n, r and s be non-negative integers such that $n\geq r\geq s$. If s is odd, then*

$$\sum_{j=1}^{\lfloor n/2\rfloor}\binom{n-r}{2j-s}5^j j^2=2^{n-r-5}\,5^{(s+1)/2}\left(5(n-r-1)(n-r)F_{n-r-2}+2(2s+1)(n-r)L_{n-r-1}+4s^2F_{n-r}\right)$$

and

$$\sum_{j=1}^{\lfloor n/2\rfloor}\binom{n-r}{2j-s+1}5^j j^2=2^{n-r-5}\,5^{(s-1)/2}\left(5(n-r-1)(n-r)L_{n-r-2}+10\,(2s-1)(n-r)F_{n-r-1}+4(s-1)^2L_{n-r}\right);$$

while if s is even, then

$$\sum_{j=1}^{\lfloor n/2\rfloor}\binom{n-r}{2j-s}5^j j^2=2^{n-r-5}\,5^{s/2}\left(5(n-r-1)(n-r)L_{n-r-2}+10\,(2s+1)(n-r)F_{n-r-1}+4s^2L_{n-r}\right)$$

and

$$\sum_{j=1}^{\lfloor n/2\rfloor}\binom{n-r}{2j-s+1}5^j j^2=2^{n-r-5}\,5^{s/2}\left(5(n-r-1)(n-r)F_{n-r-2}+2(2s-1)(n-r)L_{n-r-1}+4(s-1)^2F_{n-r}\right).$$

Proof. Set $b=\frac{1}{2}$ and $c=\frac{\sqrt{5}}{2}$ in (4.2) and proceed as in the proof of Theorem 3.13. □

5. A final example

Another interesting family of infinite series comes from choosing $a_n=(-1)^n$ or $a_{\lfloor n/k\rfloor}=(-1)^{\lfloor n/k\rfloor}$. As in this case $F(z)=\frac{1}{1+z}$, $|z|<1$, we get immediately from Lemma 2.1

$$F^+(z,k)=\sum_{n=0}^{\infty}(-1)^{\lfloor\frac{n}{k}\rfloor}z^n=\frac{1-z^k}{(1-z)(1+z^k)},\qquad k\geq 1. \tag{5.1}$$

This leads to the next theorem.

Theorem 5.1. *For integers $k\geq 1$, $p\geq 2$ and any integer m, we have*

$$\sum_{n=0}^{\infty}(-1)^{\lfloor\frac{n}{k}\rfloor}\frac{F_{n+m}}{p^{n+1}}=\frac{(p^{2k}-(-1)^k)(pF_m+F_{m-1})-p^k(pF_{k+m}+F_{k+m-1})+(-1)^m p^k(F_{k-m+1}-pF_{k-m})}{(p^2-p-1)(p^{2k}+p^kL_k+(-1)^k)} \tag{5.2}$$

and

$$\sum_{n=0}^{\infty}(-1)^{\lfloor\frac{n}{k}\rfloor}\frac{L_{n+m}}{p^{n+1}}=\frac{(p^{2k}-(-1)^k)(pL_m+L_{m-1})-p^k(pL_{k+m}+L_{k+m-1})-(-1)^m p^k(L_{k-m+1}-pL_{k-m})}{(p^2-p-1)(p^{2k}+p^kL_k+(-1)^k)}. \tag{5.3}$$

Proof. Use of $z = \frac{\alpha}{p}$ and $z = \frac{\beta}{p}$ with $p \geq 2$, in turn, in (5.1). Then, difference and addition of the resulting identities give (5.2) and (5.3), respectively. □

Example 5.2. For integers $k \geq 1$ and any integer m,

$$\sum_{n=0}^{\infty}(-1)^{\lfloor \frac{n}{k} \rfloor}\frac{F_{n+m}}{2^{n+1}} = \frac{(4^k-(-1)^k)F_{m+2}-2^kF_{k+m+2}-(-1)^m2^kF_{k-m-2}}{4^k+2^kL_k+(-1)^k},$$

$$\sum_{n=0}^{\infty}(-1)^{\lfloor \frac{n}{k} \rfloor}\frac{L_{n+m}}{2^{n+1}} = \frac{(4^k-(-1)^k)L_{m+2}-2^kL_{k+m+2}+(-1)^m2^kL_{k-m-2}}{4^k+2^kL_k+(-1)^k},$$

$$\sum_{n=0}^{\infty}(-1)^{\lfloor \frac{n}{k} \rfloor}\frac{F_{n+m}}{3^{n+1}} = \frac{(9^k-(-1)^k)L_{m+1}-3^kL_{k+m+1}-(-1)^m3^kL_{k-m-1}}{5(9^k+3^kL_k+(-1)^k)}$$

and

$$\sum_{n=0}^{\infty}(-1)^{\lfloor \frac{n}{k} \rfloor}\frac{L_{n+m}}{3^{n+1}} = \frac{(9^k-(-1)^k)F_{m+1}-3^kF_{k+m+1}+(-1)^m3^kF_{k-m-1}}{9^k+3^kL_k+(-1)^k}.$$

For the finite binomial sum

$$v_n = \sum_{j=0}^{n}(-1)^{\lfloor \frac{j}{k} \rfloor}\binom{n}{j}b^{n-j}c^j$$

with b and c real, the ordinary generating function is given by

$$S_v^*(z,k) = \frac{1-(\frac{cz}{1-bz})^k}{1-(b+c)z}\frac{1}{1+(\frac{cz}{1-bz})^k} = \frac{1}{1-(b+c)z}\left(1-\frac{2(cz)^k}{(1-bz)^k+(cz)^k}\right).$$

Now the same analysis as above applies.

Theorem 5.3. *For nonzero real numbers b and c and $n \geq 2$, we have*

$$\sum_{j=0}^{n}(-1)^{\lfloor \frac{j}{2} \rfloor}\binom{n}{j}b^{n-j}c^j = \frac{1}{2}(1-i)(b+ic)^n + \frac{1}{2}(1+i)(b-ic)^n, \tag{5.4}$$

where i is the imaginary unit. Equivalently, using the polar form, we get the expression

$$\sum_{j=0}^{n}(-1)^{\lfloor \frac{j}{2} \rfloor}\binom{n}{j}b^{n-j}c^j = (\operatorname{sgn} b)\ \sqrt{2(b^2+c^2)^n}\cos\left(n\arctan\frac{c}{b}-\frac{\pi}{4}\right).$$

Proof. As

$$\frac{1}{(1-bz)^2+(cz)^2} = \frac{1}{2c}\sum_{n=0}^{\infty}((b+ic)^n(c-ib)+(b-ic)^n(c+ib))z^n,$$

we have

$$\begin{aligned}S_v^*(z,2) &= \sum_{n=0}^{\infty}(b+c)^nz^n - cz^2\sum_{n=0}^{\infty}(b+c)^nz^n\sum_{n=0}^{\infty}((b+ic)^n(c-ib)+(b-ic)^n(c+ib))z^n \\ &= (b+c)^0+(b+c)^1z+\sum_{n=2}^{\infty}\Big((b+c)^n - c\sum_{j=0}^{n-2}(b+c)^j((b+ic)^{n-2-j}(c-ib)+(b-ic)^{n-2-j}(c+ib))\Big)z^n.\end{aligned}$$

This shows that

$$\sum_{j=0}^{n}(-1)^{\lfloor \frac{j}{2} \rfloor}\binom{n}{j}b^{n-j}c^j = (b+c)^n - c\sum_{j=0}^{n-2}(b+c)^j((b+ic)^{n-2-j}(c-ib)+(b-ic)^{n-2-j}(c+ib)).$$

The last sum can be rewritten as

$$\begin{aligned}
&\sum_{j=0}^{n-2}(b+c)^j\left((b+ic)^{n-2-j}(c-ib)+(b-ic)^{n-2-j}(c+ib)\right)\\
&\qquad=(b+ic)^{n-2}(c-ib)\sum_{j=0}^{n-2}\left(\frac{b+c}{b+ic}\right)^j+(b-ic)^{n-2}(c+ib)\sum_{j=0}^{n-2}\left(\frac{b+c}{b-ic}\right)^j\\
&\qquad=\frac{c-ib}{c(1-i)}\left((b+c)^{n-1}-(b+ic)^{n-1}\right)+\frac{c+ib}{c(1+i)}\left((b+c)^{n-1}-(b-ic)^{n-1}\right)\\
&\qquad=\frac{(b+c)^n}{c}-\frac{1}{2}(b+ic)^{n-1}\left(1+\frac{b}{c}+i\left(1-\frac{b}{c}\right)\right)-\frac{1}{2}(b-ic)^{n-1}\left(1+\frac{b}{c}-i\left(1-\frac{b}{c}\right)\right)\\
&\qquad=\frac{(b+c)^n}{c}-\frac{b+c}{2c}\left((b+ic)^{n-1}+(b-ic)^{n-1}\right)+i\frac{b-c}{2c}\left((b+ic)^{n-1}-(b-ic)^{n-1}\right).
\end{aligned}$$

This gives

$$\sum_{j=0}^{n}(-1)^{\lfloor\frac{j}{2}\rfloor}\binom{n}{j}b^{n-j}c^j=\frac{b+c}{2}\Big((b+ic)^{n-1}+(b-ic)^{n-1}\Big)-i\frac{b-c}{2}\left((b+ic)^{n-1}-(b-ic)^{n-1}\right),$$

and from here

$$\begin{aligned}
\sum_{j=0}^{n}(-1)^{\lfloor\frac{j}{2}\rfloor}\binom{n}{j}b^{n-j}c^j&=\frac{b}{2}(1-i)(b+ic)^{n-1}+\frac{c}{2}(1+i)(b+ic)^{n-1}+\frac{b}{2}(1+i)(b-ic)^{n-1}+\frac{c}{2}(1-i)(b-ic)^{n-1}\\
&=\frac{(b+ic)^n}{b^2+c^2}\left(\frac{b}{2}(1-i)(b-ic)+\frac{c}{2}(1+i)(b-ic)\right)+\frac{(b-ic)^n}{b^2+c^2}\left(\frac{b}{2}(1+i)(b+ic)+\frac{c}{2}(1-i)(b+ic)\right),
\end{aligned}$$

which is the stated formula. □

Remark 5.4. It is worth noting that a very short proof of Theorem 5.3 goes as follows. Recall

$$h_1(x;b,c,n)=\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j}b^{n-2j}c^{2j}x^{2j}=\frac{(b+cx)^n+(b-cx)^n}{2}$$

and

$$h_2(x;b,c,n)=\sum_{j=1}^{\lceil n/2\rceil}\binom{n}{2j-1}b^{n-2j+1}c^{2j-1}x^{2j-2}=\frac{(b+cx)^n-(b-cx)^n}{2x}.$$

Therefore,

$$\sum_{j=0}^{n}(-1)^{\lfloor\frac{j}{2}\rfloor}\binom{n}{j}b^{n-j}c^j=h_1(i;b,c,n)+h_2(i;b,c,n)=\frac{1}{2}\left((1-i)(b+ic)^n+(1+i)(b-ic)^n\right).$$

Corollary 5.5. *For $n\geq 2$, we have*

$$\sum_{j=0}^{n}(-1)^{\lfloor\frac{j}{2}\rfloor}\binom{n}{j}=2^{\frac{n+1}{2}}\cos\frac{(n-1)\pi}{4}$$

and

$$\sum_{j=0}^{n}(-1)^{\lfloor\frac{3j}{2}\rfloor}\binom{n}{j}=-2^{\frac{n+1}{2}}\sin\frac{(n-1)\pi}{4}.$$

Proof. Set $b = c$ in Theorem 5.3 to get

$$\begin{aligned}\sum_{j=0}^{n}(-1)^{\lfloor \frac{j}{2}\rfloor}\binom{n}{j} &= (1+i)^{n-1} + (1-i)^{n-1}\\ &= 2^{\frac{n-1}{2}}\left(e^{\frac{\pi i(n-1)}{4}} + e^{-\frac{\pi i(n-1)}{4}}\right) = 2^{\frac{n-1}{2}}2\cos\frac{(n-1)\pi}{4},\end{aligned}$$

as desired. Equivalently, one can use $\arctan 1 = \frac{\pi}{4}$. The second identity is proved in the same manner working with $(b, c) = (1, -1)$ and using the definition of the complex sine function. □

Corollary 5.6. *Let n, r and s be non-negative integers such that $n \geq r \geq s$. Let b and c be nonzero real or complex numbers. Then*

$$\sum_{j=0}^{n}(-1)^{\lfloor \frac{j}{2}\rfloor}\binom{n-r}{j-s}b^{n-j-r+s}c^{j-s} = \frac{1-i}{2}i^s(b+ic)^{n-r} + \frac{1+i}{2}(-i)^s(b-ic)^{n-r}. \tag{5.5}$$

Proof. Differentiate (5.4) r times with respect to b and s times with respect to c. □

The polar form of (5.5) is

$$\sum_{j=0}^{n}(-1)^{\lfloor \frac{j}{2}\rfloor}\binom{n-r}{j-s}b^{n-j-r+s}c^{j-s} = (\operatorname{sgn} b)\ \sqrt{2(b^2+c^2)^{n-r}}\cos\left((n-r)\arctan\frac{c}{b} + \frac{2s-1}{4}\pi\right).$$

Substituting $(b, c) = (1, 1)$ and $(b, c) = (1, -1)$ into the above formula, we obtain the following identities.

Corollary 5.7. *Let n, r and s be non-negative integers such that $n \geq r \geq s$. Then*

$$\sum_{j=0}^{n}(-1)^{\lfloor \frac{j}{2}\rfloor}\binom{n-r}{j-s} = 2^{\frac{n-r+1}{2}}\cos\frac{(n+2s-r-1)\pi}{4}$$

and

$$\sum_{j=0}^{n}(-1)^{\lfloor \frac{3j}{2}\rfloor}\binom{n-r}{j-s} = (-1)^s\, 2^{\frac{n-r+1}{2}}\cos\frac{(n-2s-r+1)\pi}{4}.$$

Corollary 5.8. *Let n, r and s be non-negative integers such that $n \geq r \geq s$. Let b and c be non-zero real numbers. Then*

$$\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n-r}{2j-s}b^{n-2j-r+s}c^{2j-s} = \frac{1}{2}\left((b+c)^{n-r} + (-1)^s(b-c)^{n-r}\right) \tag{5.6}$$

and

$$\sum_{j=1}^{\lfloor n/2\rfloor}\binom{n-r}{2j-s+1}b^{n-2j-1-r+s}c^{2j+1-s} = \frac{1}{2}\left((b+c)^{n-r} - (-1)^s(b-c)^{n-r}\right). \tag{5.7}$$

Proof. Assume that b and c are real numbers. Write ic for c in (5.5) and compare real and imaginary parts of both sides. □

Example 5.9. Choosing $b = \frac{1}{2}$, $c = \frac{\sqrt{5}}{2}$ in (5.6) and (5.7) gives for $n \geq r \geq s$:

$$\sum_{j=1}^{\lfloor n/2\rfloor}\binom{n-r}{2j-s}5^j = 2^{n-r-1}\,5^{\frac{s+1}{2}}F_{n-r}$$

and

$$\sum_{j=1}^{\lfloor n/2\rfloor}\binom{n-r}{2j+1-s}5^j = 2^{n-r-1}\,5^{\frac{s-1}{2}}L_{n-r}.$$

6. Concluding comments

This paper was about studying infinite series and binomial sums involving a general floor sequence $(a_{\lfloor n/k\rfloor})_{n\geq 0}$ which we termed the dual sequence corresponding to $(a_n)_{n\geq 0}$. In the first part, we derived some properties of infinite series involving $(a_{\lfloor n/k\rfloor})_{n\geq 0}$ and evaluated these series in closed form at certain Fibonacci (Lucas) arguments. As a byproduct, we have also established an apparently new expression for the polylogarithm. In the second part of the study we have studied finite binomial sums, hereby focusing on three particular classes of sequences, namely, $a_{\lfloor n/k\rfloor} = \lfloor \frac{n}{k}\rfloor$, $a_{\lfloor n/k\rfloor} = \lfloor \frac{n}{k}\rfloor^2$ and $a_{\lfloor n/k\rfloor} = (-1)^{\lfloor n/k\rfloor}$. Based on three general theorems (Theorems 3.1, 4.3 and 5.3) a range of new Fibonacci (Lucas) binomial sums were evaluated exactly. As was pointed out in the main text there are still gaps to fill which we leave for the readers' exploration.

We cannot resist to close the article with another obvious class of sequences. This class comes from choosing $a_n = F_n$ and $a_n = L_n$, respectively. Using well-known generating functions $\sum_{n=0}^{\infty} F_n z^n = \frac{z}{1-z-z^2}$ and $\sum_{n=0}^{\infty} L_n z^n = \frac{2-z}{1-z-z^2}$, we get from Lemma 2.1 for integers $k \geq 1$ and $|z| < |\beta|$ as follows

$$\sum_{n=0}^{\infty} F_{\lfloor n/k\rfloor} z^n = \frac{(1-z^k)z^k}{(1-z)(1-z^k-z^{2k})}, \qquad \sum_{n=0}^{\infty} L_{\lfloor n/k\rfloor} z^n = \frac{(1-z^k)(2-z^k)}{(1-z)(1-z^k-z^{2k})}.$$

This produces the next results.

Theorem 6.1. *For integers $k \geq 1$, $p \geq 2$ and any integer m, we have*

$$\begin{aligned}
D_k(p)\sum_{n=0}^{\infty}\frac{F_{\lfloor n/k\rfloor}F_{n+m}}{p^{n+1}} &= pF_m + F_{m-1} + p^{3k+1}F_{k+m} + p^{3k}F_{k+m-1} - p^{2k+1}((-1)^kF_m + F_{2k+m})\\
&\quad -p^{2k}((-1)^kF_{m-1} + F_{2k+m-1}) + (-1)^kp^{k+1}((-1)^mF_{k-m} + F_{k+m})\\
&\quad -(-1)^kp^k((-1)^mF_{k-m+1} - F_{k+m-1})\\
D_k(p)\sum_{n=0}^{\infty}\frac{F_{\lfloor n/k\rfloor}L_{n+m}}{p^{n+1}} &= pL_m + L_{m-1} + p^{3k+1}L_{k+m} + p^{3k}L_{k+m-1} - p^{2k+1}((-1)^kL_m + L_{2k+m})\\
&\quad -p^{2k}((-1)^kL_{m-1} + L_{2k+m-1}) - (-1)^kp^{k+1}((-1)^mL_{k-m} - L_{k+m})\\
&\quad +(-1)^kp^k((-1)^mL_{k-m+1} + L_{k+m-1})\\
D_k(p)\sum_{n=0}^{\infty}\frac{L_{\lfloor n/k\rfloor}F_{n+m}}{p^{n+1}} &= 2p^{4k+1}F_m + 2p^{4k}F_{m-1} + p^{3k+1}(2(-1)^mF_{k-m} - 3F_{k+m})\\
&\quad -p^{3k}(3F_{k+m-1} + 2(-1)^mF_{k-m+1}) + p^{2k+1}(F_{2k+m} + 2(-1)^mF_{2k-m} + 3(-1)^kL_m)\\
&\quad +p^{2k}(F_{2k+m-1} + 3(-1)^kF_{m-1} - 2(-1)^mF_{2k-m+1})\\
&\quad +(-1)^kp^{k+1}(3(-1)^mF_{k-m} + F_{k+m}) - (-1)^kp^k(F_{k+m-1} - 3(-1)^mF_{k-m+1}) - pF_m - F_{m-1}
\end{aligned}$$

and

$$\begin{aligned}
D_k(p)\sum_{n=0}^{\infty}\frac{L_{\lfloor n/k\rfloor}L_{n+m}}{p^{n+1}} &= 2p^{4k+1}L_m + 2p^{4k}L_{m-1} - p^{3k+1}(3L_{k+m} + 2(-1)^mL_{k-m})\\
&\quad -p^{3k}(3L_{k+m-1} - 2(-1)^mL_{k-m+1}) + p^{2k+1}(L_{2k+m} - 2(-1)^mL_{2k-m} + 3(-1)^kL_m)\\
&\quad +p^{2k}(L_{2k+m-1} + 3(-1)^kL_{m-1} + 2(-1)^mL_{2k-m+1})\\
&\quad +(-1)^kp^{k+1}(3(-1)^mL_{k-m} - L_{k+m}) - (-1)^kp^k(3(-1)^mL_{k-m+1} + L_{k+m-1}) - pL_m - L_{m-1},
\end{aligned}$$

where $D_k(p) = (p^2-p-1)\left(p^{4k} - p^{3k}L_k - p^{2k}(L_{2k} - (-1)^k) + (-1)^kp^kL_k + 1\right)$.

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