Montes Taurus J. Pure Appl. Math. 8 (1), 149–163, 2026

Montes Taurus Journal of Pure and Applied Mathematics
https://www.mtjpamjournal.com
MTJPAM
ISSN: 2687-4814

Analytical expressions for the length and curved surface area of revolution of arbitrary arcs of Limacon; A hypergeometric function approach

Mohammad Idris Qureshi [a], Aarif Hussain Bhat [b], Bilal Ahmad Bhat [c]

[a]Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University), New Delhi-110025, India
[b]Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University), New Delhi-110025, India
[c]Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University), New Delhi-110025, India

Abstract

This paper aims to derive analytical expressions for the exact length of an arbitrary arc of Limacon in terms of Kampé de Fériet double hypergeometric function and Clausen function ${}_3F_2$. We have also derived the exact expressions for the curved surface areas of revolution obtained by revolving the arbitrary arc of Limacon, when axes of revolution are initial line and another line (axis of revolution, which is perpendicular to initial line and passing through the pole). Further we have also deduced the perimeters and curved surface areas of revolution for Limacon, Cardioid and a new summation theorem for Clausen function ${}_3F_2[1]$. We have verified all the results numerically using Mathematica software a general system of doing mathematics.

Keywords: Gamma function, Pochhammer symbol, generalized hypergeometric function, Kampé de Fériet double hypergeometric function, series rearrangement technique, Limacon, cardiod

2020 MSC: 33C05, 33E99, 33B10, 33C70, 33C20, 33C90

1. Introduction, preliminaries and definitions

The study of special functions is a crucial branch of mathematics. Over the past four centuries, the need to address challenges in fields such as classical and quantum physics, engineering, and applied mathematics has driven the advancement of the theory of special functions in both single and multiple variable problems. These functions serve as solutions to a wide range of functional equations that are both mathematically significant and applicable in physical contexts. They play a vital role in mathematical physics, offering formal frameworks and serving as powerful tools to develop simplified yet realistic models for physical problems. The significance of special functions has been further emphasized by their various generalizations and adaptations. Recently, the discovery of new generalized and combined special functions, as well as their applications in emerging areas of mathematics and other disciplines,

†*Article ID:* MTJPAM-D-24-00095
Email addresses: miqureshi_delhi@yahoo.co.in (Mohammad Idris Qureshi), aarifsaleem19@gmail.com (Aarif Hussain Bhat), bilalj102102@gmail.com (Bilal Ahmad Bhat)
Received:3 July 2024, Accepted:25 October 2025, Published:3 June 2026
⋆*Corresponding Author:* Aarif Hussain Bhat

has sparked considerable interest in this field. Multiple generating functions have found extensive use across many scientific and technological fields. Today, it is challenging to identify an area within applied mathematics, physics, or statistics where one does not encounter multiple generating functions related to mathematical physics, such as Bessel functions, hypergeometric functions, orthogonal polynomials, and the theory of integral transforms (e.g., Laplace, Hankel, and Mellin transforms). The study of multiple generating functions and integral transforms is explored in numerous textbooks, monographs, and scholarly articles (*cf.* [11, 12, 26, 27, 31, 33, 34]).

In the world of standard functions, the hypergeometric functions take a dominant position in mathematics both pure and applied. The generalized hypergeometric series (or function) ${}_pF_q$ $(p,\ q \in \mathbb{N}_0)$, which is a natural generalization of the Gaussian hypergeometric series ${}_2F_1$, is defined by (see, e.g., [1, 16, 24, 29, 31, 34]):

$$ {}_pF_q\left[\begin{array}{c} (\alpha_p)\,; \\ (\beta_q)\,; \end{array} z\right] = {}_pF_q\left[\begin{array}{c} \alpha_1,\ \ldots,\ \alpha_p\,; \\ \beta_1,\ \ldots,\ \beta_q\,; \end{array} z\right] = \sum_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{p}(\alpha_j)_n}{\prod\limits_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!}, \tag{1.1} $$

where $(\alpha)_p$ is the Pochhammer symbol defined for $(\alpha, p \in \mathbb{C})$ (see [31, p. 2 and p. 5]):

$$ \begin{aligned} (\alpha)_p &= \frac{\Gamma(\alpha+p)}{\Gamma(\alpha)} \\ &= \begin{cases} 1 & \left(p=0\,;\ \alpha \in \mathbb{C}\setminus\mathbb{Z}_0^-\right), \\ \alpha(\alpha+1)\cdots(\alpha+n-1) & (p=n \in \mathbb{N}\,;\ \alpha \in \mathbb{C}), \end{cases} \end{aligned} \tag{1.2} $$

it being understood that $(0)_0 = 1$ (see, e.g., [29, 34]) and assumed tacitly that the Gamma quotient exists. Here an empty product is interpreted as 1, and it is assumed that the variable z, the numerator parameters $\alpha_1, \ldots, \alpha_p$, and the denominator parameters $\beta_1, \ldots, \beta_q$ take on complex values, provided that

$$ (\beta_j \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ j = 1, \ldots, q). \tag{1.3} $$

Here and elsewhere, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers, respectively.

For more details of ${}_pF_q$ including its convergence, its various special and limiting cases, and its further diverse generalizations, one may be referred, for example see [1, 3, 4, 11, 24, 27, 29, 33, 34]. It is important to highlight that whenever the generalized hypergeometirc function ${}_pF_q(z)$ (including ${}_2F_1(z)$ with its specified argument z (for example, $z = \pm 1$ or $z = 1/2$) can be summed to be expressed in terms of the Gamma functions, the result may be very important from both theoretical and applicable points of view.

The vast appeal and wide-ranging practicality of the hypergeometric function ${}_2F_1$ and the generalized hypergeometric functions ${}_pF_q$ $(p,\ q \in \mathbb{N}_0)$ of one variable have motivated and encouraged numerous researchers to explore and analyze hypergeometric functions in two or more variables (*cf.* [2]-[4], [12, 27, 29, 33, 37, 38]). A serious, significant and systematic study of the hypergeometric functions of two variables was initiated by Appell-Kampé de Fériet [2, 22], who offered the so-called Appell functions F_1, F_2, F_3, and F_4 which are generalizations of the Gauss hypergeometric function (*cf.* [7]-[9], [33]). Here, we recall the Appell function of first kind (see, e.g., [34, p. 53, Eq. (4)]):

$$ F_1\,[a;\ b,\ c;\ d;\ x,\ y] = \sum_{m=0}^{\infty}\sum_{r=0}^{\infty} \frac{(a)_{m+r}(b)_m(c)_r x^m y^r}{(d)_{m+r}m!r!}, \tag{1.4} $$

where $\max\{|x|,\ |y|\} < 1$.

The confluent forms of the Appell functions were studied by Humbert [19, 20, 21]. We recall here the definition of a more general double-hypergeometric function (than the one defined by Kampé de Fériet) in a slightly modified notation given by Srivastava and Panda [35, p. 423, Eq. (26)]. The convenient generalization of the Kampé de Fériet function (KdF function) is defined as follows:

$$ F^{p:\ q;\ k}_{\ell:\ m;\ n}\left[\begin{array}{l} (a_p)\ \ :\ (b_q)\ ;\ (c_k)\,; \\ (\alpha_\ell)\ \ :\ (\beta_m)\ ;\ (\gamma_n)\,; \end{array} x,\ y\right] = \sum_{r,s=0}^{\infty} \frac{\prod\limits_{j=1}^{p}(a_j)_{r+s}\prod\limits_{j=1}^{q}(b_j)_r\prod\limits_{j=1}^{k}(c_j)_s}{\prod\limits_{j=1}^{\ell}(\alpha_j)_{r+s}\prod\limits_{j=1}^{m}(\beta_j)_r\prod\limits_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}. \tag{1.5} $$

To gain further insight into the convergence properties of the double series in equation (1.5), which encompasses conditional convergence as well, one can consult the research papers for example, [12]-[14], [16], [32]-[34].

1.1. Required results

Some useful indefinite integrals are given as follows:

$$\int \sin^{2m}\theta d\theta = \left\{\frac{-(\frac{1}{2})_m \sin(\theta)\cos(\theta)}{(1)_m}\sum_{r=0}^{m-1}\frac{(1)_r \sin^{2r}(\theta)}{(\frac{3}{2})_r}\right\} + \left\{\frac{\theta(\frac{1}{2})_m}{(1)_m}\right\} + \text{Constant}, \tag{1.6}$$

where $m \geq 1$,

$$\int \cos^{2m}\theta d\theta = \left\{\frac{(\frac{1}{2})_m \sin(\theta)\cos(\theta)}{(1)_m}\sum_{r=0}^{m-1}\frac{(1)_r \cos^{2r}(\theta)}{(\frac{3}{2})_r}\right\} + \left\{\frac{\theta(\frac{1}{2})_m}{(1)_m}\right\} + \text{Constant}, \tag{1.7}$$

where $m \geq 1$,

$$\int \sin^{2m+1}\theta d\theta = \left\{\frac{-(1)_m \cos(\theta)}{(\frac{3}{2})_m}\sum_{r=0}^{m}\frac{(\frac{1}{2})_r \sin^{2r}(\theta)}{(1)_r}\right\} + \text{Constant}, \tag{1.8}$$

where $m \geq 0$ and

$$\int \cos^{2m+1}\theta d\theta = \left\{\frac{(1)_m \sin(\theta)}{(\frac{3}{2})_m}\sum_{r=0}^{m}\frac{(\frac{1}{2})_r \cos^{2r}(\theta)}{(1)_r}\right\} + \text{Constant}, \tag{1.9}$$

where $m \geq 0$ (*cf.* [28, p. 596, Eqs. (1.6), (1.7), (1.8) and (1.9)]).

The Cauchy's double series identity is given by

$$\sum_{m=0}^{\infty}\sum_{r=0}^{m}\Phi(m,r) = \sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Phi(m+r,r) \tag{1.10}$$

(*cf.* [34, p. 100, Eq. (2)]).

The decomposition identity of unilateral series are given by

$$\sum_{m=0}^{\infty}\Phi(m) = \sum_{m=0}^{\infty}\Phi(2m) + \sum_{m=0}^{\infty}\Phi(2m+1), \tag{1.11}$$

provided that both sides of identities (1.10) and (1.11) are absolutely convergent (*cf.* [34, p. 200, Eq. (1)]).

The binomial theorem is given by

$$(1-z)^{-a} = {}_1F_0\left[\begin{matrix} a; \\ -; \end{matrix}\; z\right] = \sum_{n=0}^{\infty}\frac{(a)_n z^n}{n!}, \tag{1.12}$$

where $|z| < 1, a \in \mathbb{C}$ (*cf.* [34, p. 34, Eq. (22)]).

The reduction formulas are given by

$${}_2F_1\left[\begin{matrix} -\frac{1}{2},\ 1; \\ 2; \end{matrix}\; z\right] = \frac{2}{3z}\left[1-(1-z)^{\frac{3}{2}}\right];\ \ |z| \leq 1 \tag{1.13}$$

and

$${}_2F_1\left[\begin{matrix} -\frac{1}{2},\ 2; \\ 3; \end{matrix}\; z\right] = \frac{4}{15z^2}\left[2-(2+3z)(1-z)^{\frac{3}{2}}\right];\ \ |z| \leq 1 \tag{1.14}$$

(*cf.* [27, p. 469, Entry (11) and Entry (27)]).

Definition 1.1 (Length of the Curve in polar form)**.** The length of an arc of the polar curve $r = f(\theta)$, between two points (r_1, α) and (r_2, β) is given by

$$\hat{s} = \int_{\alpha}^{\beta} \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} d\theta. \tag{1.15}$$

Definition 1.2 (Curved surface area (C.S.A.) of revolution in polar form)**.** The curved surface area of revolution, obtained by revolving the polar curve $r = G(\theta)$ (not intersecting the axis of revolution), about the initial line ($\theta = 0$), is given by

$$\text{C.S.A. (about initial line as axis of revolution)} = 2\pi \int (r \sin\theta) \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} d\theta. \tag{1.16}$$

The curved surface area of revolution, obtained by revolving the polar curve $r = G(\theta)$ (not intersecting the axis of revolution), about the perpendicular line ($\theta = \frac{\pi}{2}$), is given by

$$\text{C.S.A.(about perpendicular line as axis of revolution)} = 2\pi \int (r \cos\theta) \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} d\theta. \tag{1.17}$$

Inspired by the aforementioned papers especially by Burchnall-Chaundy [5, 6], Choi-Rathie [10], Gil [15], Hasanov et al. [17, 18], Khan [23], Maksimović-Borković [25], Nystedt [26] and Traub [36], we have mentioned the analytical expressions for the length of an arbitrary arc of the Limacons $r = a + b\cos\theta$ and $r = a + b\sin\theta$ in section 2. In Section 3, we have obtained the exact expressions for the curved surface areas of revolution by revolving the arbitrary arc of Limacon. In Section 4, we have obtained the perimeter, curved surface areas of revolution for Limacon and Cardioid and a new summation theorem for Clausen function ${}_3F_2[1]$. In this paper, any values of parameters and arguments leading to the results which do not make sense are tacitly excluded.

Remark 1.3. For the sake of convenience, we shall use the following notation for lengthy mathematical expressions

$$\sum_{\gamma\to\beta}^{\circledast} \{\Phi(\gamma)\} = \{\Phi(\gamma)\} - \{\Phi(\beta)\}. \tag{1.18}$$

2. Length of an arbitrary arc of Limacon

In this section, we derive the arc lengths of two Limacon curves by applying the standard arc length formula and necessary integrals.

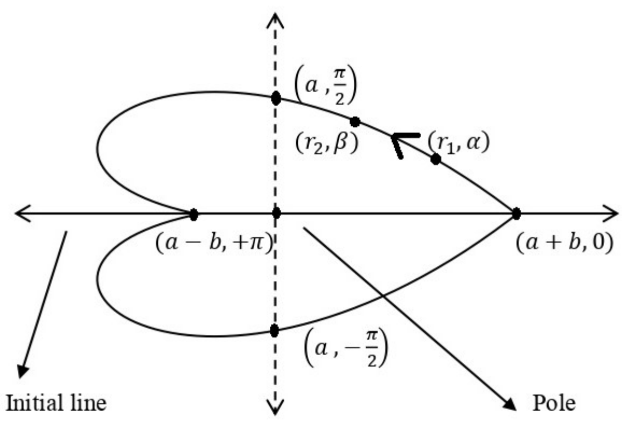

Figure 1. Limacon $r = a + b\cos\theta; a \geq b > 0$

Theorem 2.1. *The arbitrary arc length of Limacon $r = a + b\cos\theta$; $a \geq b > 0$ (see Figure 1), in the interval $(0 \leq \alpha < \theta < \beta \leq \pi)$, for the arc lying above initial line is given by*

$$\hat{s} = \sum_{\beta\to\alpha}^{\circledast}\left\{\beta\sqrt{(a^2+b^2)} - \frac{a^2b^2\sin\beta\cos\beta}{4(a^2+b^2)^{\frac{3}{2}}} F^{\,2:\,1;\,2}_{\,2:\,0;\,1}\left[\begin{array}{c} \frac{3}{4},\ \frac{5}{4}:\ 1\ ;\ 1,1; \\ \\ 2,\ 2:\ -\ ;\ \frac{3}{2}\ ; \end{array} \left(\frac{2ab}{a^2+b^2}\right)^2, \left(\frac{2ab\cos\beta}{a^2+b^2}\right)^2\right]\right. \tag{2.1}$$
$$\left. - \frac{\beta a^2b^2}{4(a^2+b^2)^{\frac{3}{2}}}\,{}_3F_2\left[\begin{array}{c} \frac{3}{4},\ \frac{5}{4},\ 1; \\ \\ 2,\ \ 2; \end{array} \left(\frac{2ab}{a^2+b^2}\right)^2\right] + \frac{ab\sin\beta}{\sqrt{(a^2+b^2)}} F^{\,2:\,1;\,1}_{\,2:\,0;\,0}\left[\begin{array}{c} \frac{1}{4},\ \frac{3}{4}:\ 1\ ;\ \frac{1}{2}\ ; \\ \\ \frac{3}{2},\ \frac{3}{2}:\ -\ ;-\ ; \end{array} \left(\frac{2ab}{a^2+b^2}\right)^2, \left(\frac{2ab\cos\beta}{a^2+b^2}\right)^2\right]\right\}.$$

Proof. We have

$$r = a + b\cos\theta;\ a \geq b > 0;\ 0 \leq \alpha \leq \theta \leq \beta \leq \pi. \tag{2.2}$$

Differentiate the equation (2.2) and substitute the values of r and $\frac{dr}{d\theta}$ in the equation (1.15), we get

$$\begin{aligned}\hat{s} &= \int_\alpha^\beta \sqrt{(a+b\cos\theta)^2 + (-b\sin\theta)^2}d\theta \\ &= \sqrt{(a^2+b^2)}\int_\alpha^\beta \sqrt{\left\{1 + \frac{2ab\cos\theta}{(a^2+b^2)}\right\}}d\theta.\end{aligned}$$

Since $\frac{2ab}{a^2+b^2} < 1$, $a \neq b$, a, $b \in \mathbb{R}^+$, therefore by using binomial theorem (1.12), we come to

$$\begin{aligned}\hat{s} &= \sqrt{(a^2+b^2)}\int_\alpha^\beta {}_1F_0\left[\begin{array}{c} -\frac{1}{2}; \\ \\ -; \end{array} -\frac{2ab\cos\theta}{(a^2+b^2)}\right]d\theta \\ &= \sqrt{(a^2+b^2)}\sum_{m=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_m\left(-\frac{2ab}{a^2+b^2}\right)^m}{m!}\int_\alpha^\beta \cos^m\theta d\theta.\end{aligned}$$

Applying decomposition identity of unilateral series (1.11) in the above equation, we find

$$\hat{s} = \sqrt{(a^2+b^2)}\left\{\sum_{m=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_{2m}\left(-\frac{2ab}{a^2+b^2}\right)^{2m}}{(2m)!}\int_\alpha^\beta \cos^{2m}\theta d\theta + \sum_{m=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_{2m+1}\left(-\frac{2ab}{a^2+b^2}\right)^{2m+1}}{(2m+1)!}\int_\alpha^\beta \cos^{2m+1}\theta d\theta\right\}.$$

Now substituting the values of $\int\cos^{2m}(\theta)d\theta$ and $\int\cos^{2m+1}(\theta)d\theta$ from equations (1.7) and (1.9), we get

$$\begin{aligned}\hat{s} = \sqrt{(a^2+b^2)}\Bigg\{(\beta-\alpha) &+ \frac{\left(-\frac{1}{2}\right)_2\left(\frac{2ab}{a^2+b^2}\right)^2}{2}\sum_{m=0}^{\infty}\frac{\left(\frac{3}{2}\right)_{2m}\left(\frac{2ab}{a^2+b^2}\right)^{2m}}{(3)_{2m}}\left[\frac{\left(\frac{1}{2}\right)_{m+1}\sin\theta\cos\theta}{(m+1)!}\sum_{r=0}^{m}\frac{(1)_r\cos^{2r}\theta}{\left(\frac{3}{2}\right)_r}\right. \\ &\left.+\frac{\theta\left(\frac{1}{2}\right)_{m+1}}{(m+1)!}\right]_\alpha^\beta + \left(\frac{ab}{a^2+b^2}\right)\sum_{m=0}^{\infty}\frac{\left(\frac{1}{4}\right)_m\left(\frac{3}{4}\right)_m\left(\frac{2ab}{a^2+b^2}\right)^{2m}}{(\frac{3}{2})_m m!}\left[\frac{\sin\theta m!}{(\frac{3}{2})_m}\sum_{r=0}^{m}\frac{(\frac{1}{2})_r\cos^{2r}\theta}{r!}\right]_\alpha^\beta\Bigg\}.\end{aligned} \tag{2.3}$$

Employing the double-series manupulation formula (1.10) in the equation (2.3) and after further simplification, we acquire

$$\begin{aligned}\hat{s} = (\beta-\alpha)\sqrt{(a^2+b^2)} &- \left[\frac{a^2b^2\sin\theta\cos\theta}{4(a^2+b^2)^{\frac{3}{2}}}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\frac{\left(\frac{3}{4}\right)_{m+r}\left(\frac{5}{4}\right)_{m+r}\left(\frac{2ab}{a^2+b^2}\right)^{2m+2r}(1)_r\cos^{2r}\theta}{(2)_{m+r}(2)_{m+r}\left(\frac{3}{2}\right)_r} + \frac{a^2b^2\theta}{4(a^2+b^2)^{\frac{3}{2}}}\right. \\ &\left.\times\sum_{m=0}^{\infty}\frac{\left(\frac{3}{4}\right)_m\left(\frac{5}{4}\right)_m\left(\frac{2ab}{a^2+b^2}\right)^{2m}}{(2)_m(2)_m}\right]_\alpha^\beta + \left[\left(\frac{ab\sin\theta}{\sqrt{a^2+b^2}}\right)\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\frac{\left(\frac{1}{4}\right)_{m+r}\left(\frac{3}{4}\right)_{m+r}\left(\frac{2ab}{a^2+b^2}\right)^{2m+2r}\left(\frac{1}{2}\right)_r\cos^{2r}\theta}{(\frac{3}{2})_{m+r}(\frac{3}{2})_{m+r}\,r!}\right]_\alpha^\beta.\end{aligned}$$

Finally using the definitions of Kampé de Fériet double hypergeometric function (1.5) and generalized hypergeometric function (1.1), we arrive at the result (2.1). □

Remark 2.2. The arbitrary arc length of outer loop of Limacon $r = a + b\cos\theta;\ 0 < a \leq b$ (see Figure 2), in the interval $\left(0 \leq \alpha < \theta < \beta \leq \cos^{-1}\left(-\frac{a}{b}\right)\right)$, for the arc lying above initial line is also given by equation (2.1).

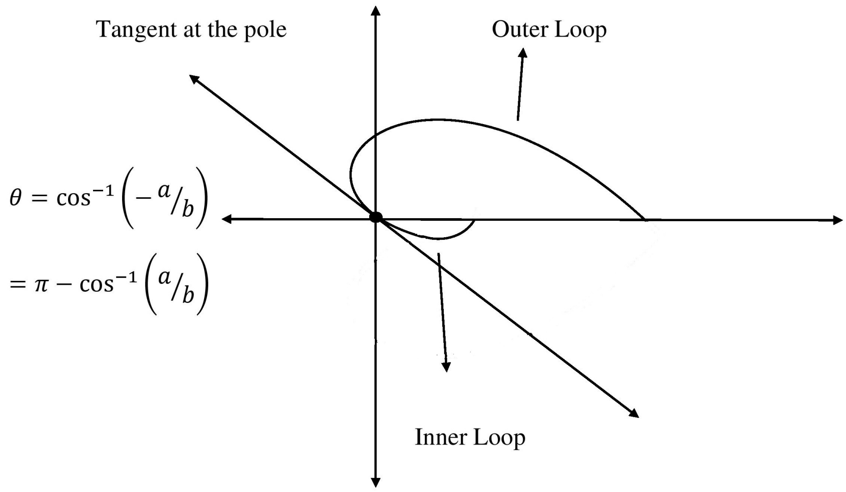

Figure 2. Limacon $r = a + b\cos\theta; 0 < a \leq b$

Remark 2.3. The arbitrary arc length of inner loop of Limacon $r = a+b\cos\theta;\ 0 < a \leq b$ (see Figure 2), in the interval $\left(\cos^{-1}\left(-\frac{a}{b}\right) \leq \alpha < \theta < \beta \leq \pi\right)$, for the arc lying below initial line is also given by equation (2.1).

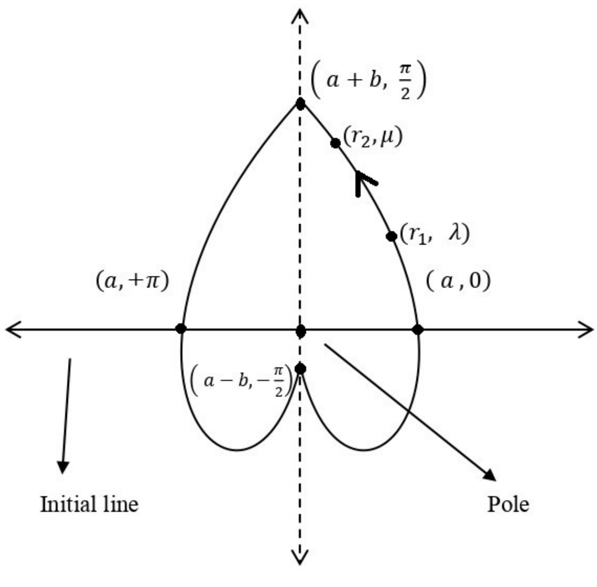

Figure 3. Limacon $r = a + b\sin\theta;\ a \geq b > 0$

Theorem 2.4. *The arbitrary arc length of Limacon $r = a + b\sin\theta;\ a \geq b > 0$ (see Figure 3), in the interval $\left(-\frac{\pi}{2} \leq \lambda < \theta < \mu \leq \frac{\pi}{2}\right)$, for the arc lying on the right hand side of the vertical line $\theta = \frac{\pi}{2}$, is given by*

$$\hat{s} = \sum_{\mu\to\lambda}^{\circledast} \left\{ \mu\sqrt{(a^2+b^2)} + \frac{a^2b^2\sin\mu\cos\mu}{4(a^2+b^2)^{\frac{3}{2}}} F^{2:\,1;\,2}_{2:\,0;\,1}\left[\begin{matrix} \frac{3}{4},\ \frac{5}{4}:\ 1\ ;\ 1,1; \\ \\ 2,\ 2:\ -\ ;\ \frac{3}{2}\ ; \end{matrix}\ \left(\frac{2ab}{a^2+b^2}\right)^2, \left(\frac{2ab\sin\mu}{a^2+b^2}\right)^2\right] \right. \tag{2.4}$$

$$\left. - \frac{\mu a^2b^2}{4(a^2+b^2)^{\frac{3}{2}}}\, {}_3F_2\left[\begin{matrix} \frac{3}{4},\ \frac{5}{4},\ 1; \\ \\ 2,\ 2; \end{matrix}\ \left(\frac{2ab}{a^2+b^2}\right)^2\right] - \frac{ab\cos\mu}{\sqrt{(a^2+b^2)}} F^{2:\,1;\,1}_{2:\,0;\,0}\left[\begin{matrix} \frac{1}{4},\ \frac{3}{4}:\ 1\ ;\ \frac{1}{2}\ ; \\ \\ \frac{3}{2},\ \frac{3}{2}:\ -\ ;-\ ; \end{matrix}\ \left(\frac{2ab}{a^2+b^2}\right)^2, \left(\frac{2ab\sin\mu}{a^2+b^2}\right)^2\right]\right\}.$$

Proof. The proof of Theorem 2.4 will run parallel to Theorem 2.1 with the aid of integrals (1.6) and (1.8) and series rearrangement technique (1.10). The involved details are omitted. □

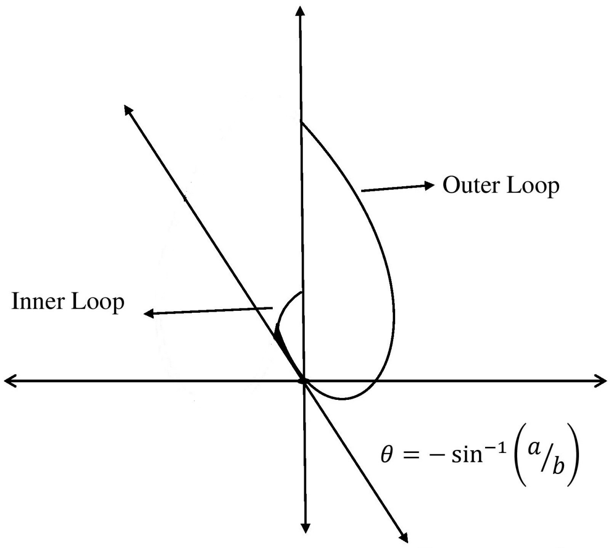

Figure 4. Limacon $r = a + b\sin\theta; 0 < a \leq b$

Remark 2.5. The arbitrary arc length of outer loop of Limacon $r = a + b\sin\theta;\ 0 < a \leq b$ (see Figure 4), in the interval $\left(-\sin^{-1}\left(\frac{a}{b}\right) \leq \lambda < \theta < \mu \leq \frac{\pi}{2}\right)$ for the arc lying on the right hand side of the vertical line $\theta = \frac{\pi}{2}$, is also given by equation (2.4).

Remark 2.6. The arbitrary arc length of inner loop of Limacon $r = a + b\sin\theta;\ 0 < a \leq b$ (see Figure 4), in the interval $\left(-\frac{\pi}{2} \leq \lambda < \theta < \mu \leq -\sin^{-1}\left(\frac{a}{b}\right)\right)$; for the arc lying on the left hand side of the vertical line $\theta = \frac{\pi}{2}$, is also given by equation (2.4).

3. Curved surface area of revolution of an arbitrary arc of Limacon

In this section, we derive the expressions for the curved surface area, when an arbitrary arc of a Limacon is revolved about initial line (axis of revolution) and about another line (axis of revolution, which is perpendicular to initial line and passing through the pole) using the standard surface area formulas in polar coordinates.

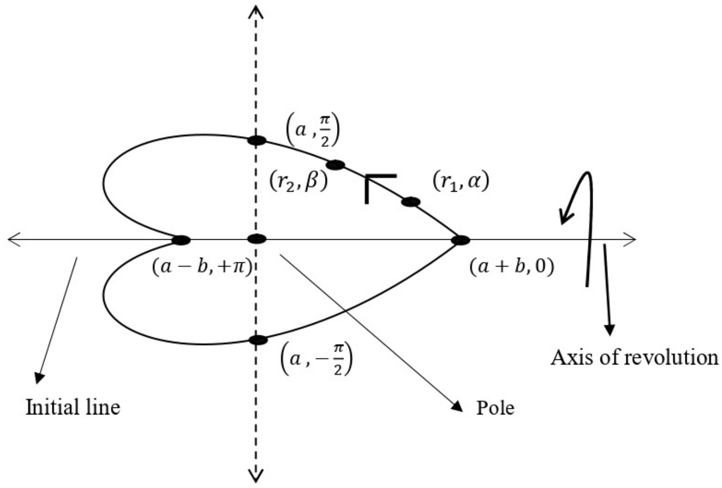

Figure 5. Limacon $r = a + b\cos\theta; a > b > 0$.

Theorem 3.1. *The exact expression for the curved surface area, obtained by revolving the arbitrary arc of Limacon $r = a + b\cos\theta;\ a \geq b > 0;\ (0 \leq \alpha < \theta < \beta \leq \pi)$ (see Figure 5), about initial line (axis of revolution), is given by*

$$\begin{aligned} C.S.A. &= \sum_{\beta\to\alpha}^{\circledast} \left\{ -2\pi a\sqrt{(a^2+b^2)}\cos(\beta)\,{}_2F_1\left[\begin{matrix} -\frac{1}{2},\ 1; & \\ & \frac{-2ab\cos(\beta)}{(a^2+b^2)} \\ 2; & \end{matrix}\right] \right. \\ &\left. - \pi b\sqrt{(a^2+b^2)}\cos^2(\beta)\,{}_2F_1\left[\begin{matrix} -\frac{1}{2},\ 2; & \\ & \frac{-2ab\cos(\beta)}{(a^2+b^2)} \\ 3; & \end{matrix}\right]\right\}, \qquad (3.1) \\ &= \sum_{\alpha\to\beta}^{\circledast} \left\{ \pi\left(\frac{a^2-b^2}{3a^2b}\right)(a^2+b^2+2ab\cos\alpha)^{\frac{3}{2}} + \frac{\pi}{5a^2b}(a^2+b^2+2ab\cos\alpha)^{\frac{5}{2}} \right\}. \qquad (3.2) \end{aligned}$$

Proof. We know that

$$\begin{aligned} \text{C.S.A.} &= 2\pi\int_\alpha^\beta y\frac{ds}{d\theta}d\theta = 2\pi\int_\alpha^\beta (r\sin(\theta))\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}d\theta \\ &= 2\pi\int_\alpha^\beta (a+b\cos(\theta))\sin(\theta)\sqrt{a^2+b^2+2ab\cos(\theta)}d\theta. \end{aligned}$$

Put $\cos(\theta) = t$ and after further simplification, we get

$$\text{C.S.A.} = -2\pi\int_{\cos\alpha}^{\cos\beta}(a+bt)\sqrt{(a^2+b^2)}\left(1+\frac{2abt}{(a^2+b^2)}\right)^{\frac{1}{2}}dt. \qquad (3.3)$$

Applying hypergeometric form of the binomial theorem (1.12) to equation (3.3), we come to

$$\begin{aligned} \text{C.S.A.} &= -2\pi\sqrt{(a^2+b^2)}\int_{\cos\alpha}^{\cos\beta}(a+bt)\,{}_1F_0\left[\begin{matrix} -\frac{1}{2}; & \\ & -\frac{2abt}{(a^2+b^2)} \\ -; & \end{matrix}\right]dt \\ &= -2\pi\sqrt{(a^2+b^2)}\int_{\cos\alpha}^{\cos\beta}(a+bt)\sum_{n=0}^{\infty}\left(\frac{1}{n!}\right)\left(-\frac{1}{2}\right)_n(-1)^n\left(\frac{2abt}{(a^2+b^2)}\right)^n dt. \end{aligned}$$

Now integrating the above equation w.r.t. t, we find

$$\begin{aligned} \text{C.S.A.} \quad = \quad & -2\pi a\sqrt{(a^2+b^2)}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_n(-1)^n(1)_n\left(\frac{2ab}{(a^2+b^2)}\right)^n}{(2)_n n!}\left[(\cos\theta)^{n+1}\right]_{\theta=\alpha}^{\theta=\beta} \\ & -\pi b\sqrt{(a^2+b^2)}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_n(-1)^n(2)_n\left(\frac{2ab}{(a^2+b^2)}\right)^n}{(3)_n n!}\left[(\cos\theta)^{n+2}\right]_{\theta=\alpha}^{\theta=\beta}. \end{aligned}$$

Finally using the definition of generalized hypergeometric function of one variable (1.1), we obtain the desired result (3.1).

When we apply the reduction formulas (1.13) and (1.14), in the equation (3.1), after simplification, we get the result (3.2). □

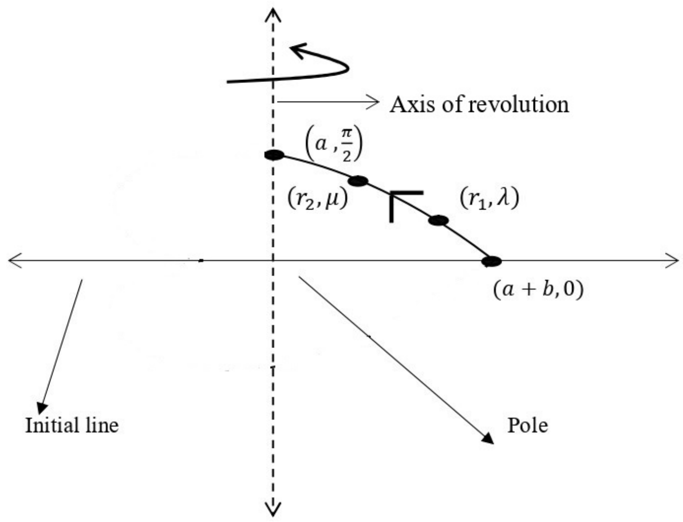

Figure 6. Limacon $r = a + b\cos\theta; a > b > 0$

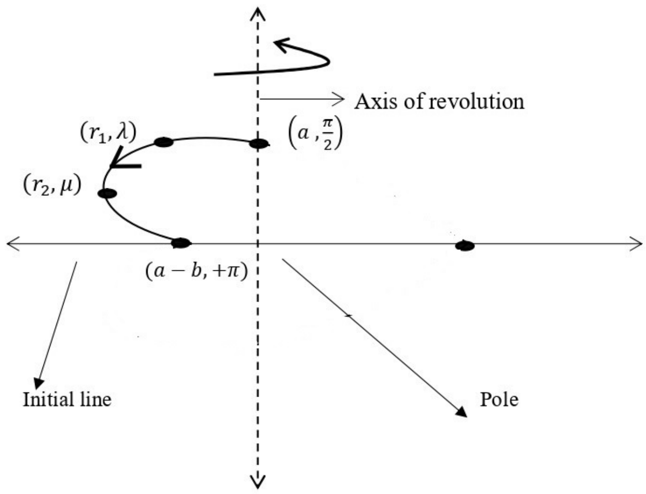

Figure 7. Limacon $r = a + b\cos\theta; a > b > 0$

Theorem 3.2. *The exact expression for the curved surface area, obtained by revolving the arbitrary arc of Limacon $r = a + b\cos\theta$; $a \geq b > 0$; $(0 \leq \lambda < \theta < \mu \leq \frac{\pi}{2})$ or $(\frac{\pi}{2} \leq \lambda < \theta < \mu \leq \pi)$ (see Figures 6 and 7), about the line (axis of revolution), which is perpendicular to initial line and passing through the pole, is given by*

$$\begin{aligned}
C.S.A. \quad = \quad & \sum_{\mu\to\lambda}^{\circledast} \Bigg\{ 2\pi a \sin(\mu)\sqrt{(a^2+b^2)}\, F^{\,2:\ 1;\ 1}_{\,2:\ 0;\ 0}\left[\begin{matrix} -\frac{1}{4},\ \frac{1}{4}\ :\ 1\ ;\ \frac{1}{2}\ ; \\ \frac{1}{2},\ \frac{3}{2}\ :\ -\ ;-\ ; \end{matrix}\ \left(\frac{2ab}{a^2+b^2}\right)^2, \left(\frac{2ab(\cos\mu)}{a^2+b^2}\right)^2\right] \\
& + \frac{a^2 b\pi \sin(\mu)\cos(\mu)}{\sqrt{(a^2+b^2)}} F^{\,2:\ 1;\ 2}_{\,2:\ 0;\ 1}\left[\begin{matrix} \frac{1}{4},\ \frac{3}{4}\ :\ 1\ ;1\ ,1\ ; \\ 1\ ,\ 2\ :\ -\ ;\frac{3}{2}\ ; \end{matrix}\ \left(\frac{2ab}{a^2+b^2}\right)^2, \left(\frac{2ab(\cos\mu)}{a^2+b^2}\right)^2\right] \\
& + \frac{a^2 b\pi(\mu)}{\sqrt{(a^2+b^2)}}\, {}_2F_1\left[\begin{matrix} \frac{1}{4},\ \frac{3}{4}; \\ 2; \end{matrix}\ \left(\frac{2ab}{a^2+b^2}\right)^2\right] + \frac{4ab^2\pi\sin(\mu)}{3\sqrt{(a^2+b^2)}}\, {}_3F_2\left[\begin{matrix} \frac{1}{4},\ \frac{3}{4},\ 2; \\ \frac{3}{2},\ \frac{5}{2}; \end{matrix}\ \left(\frac{2ab}{a^2+b^2}\right)^2\right] \\
& + b\pi\sqrt{(a^2+b^2)}\sin(\mu)\cos(\mu) F^{\,3:\ 1;\ 2}_{\,3:\ 0;\ 1}\left[\begin{matrix} -\frac{1}{4},\ \frac{1}{4},\ \frac{3}{2}\ :\ 1\ ;1\ ,1\ ; \\ \frac{1}{2}\ ,\ 1\ ,2\ :\ -\ ;\quad \frac{3}{2}\ ; \end{matrix}\ \left(\frac{2ab}{a^2+b^2}\right)^2, \left(\frac{2ab(\cos\mu)}{a^2+b^2}\right)^2\right] \\
& + \frac{2ab^2\pi\sin(\mu)\cos^2(\mu)}{3\sqrt{(a^2+b^2)}} F^{\,3:\ 1;\ 2}_{\,3:\ 0;\ 1}\left[\begin{matrix} \frac{1}{4},\ \frac{3}{4},\ 2\ :\ 1\ ;\frac{3}{2}\ ,1\ ; \\ 1,\ \frac{3}{2},\ \frac{5}{2}\ :\ -\ ;\ 2\ ; \end{matrix}\ \left(\frac{2ab}{a^2+b^2}\right)^2, \left(\frac{2ab(\cos\mu)}{a^2+b^2}\right)^2\right] \\
& + b\pi(\mu)\sqrt{(a^2+b^2)}\, {}_3F_2\left[\begin{matrix} -\frac{1}{4},\ \frac{1}{4},\ \frac{3}{2}; \\ 2,\ \ \frac{1}{2}; \end{matrix}\ \left(\frac{2ab}{a^2+b^2}\right)^2\right] \Bigg\}. \qquad (3.4)
\end{aligned}$$

Proof. We know that C.S.A. (about perpendicular line as axis of revolution) is given by

$$\text{C.S.A.} = 2\pi\int_\lambda^\mu x\frac{ds}{d\theta}d\theta = 2\pi\int_\lambda^\mu (r\cos(\theta))\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}d\theta.$$

Substituting the values of r and $\frac{dr}{d\theta}$, we acquire

$$\text{C.S.A.} = 2\pi\sqrt{(a^2+b^2)}\int_\lambda^\mu (a+b\cos(\theta))\cos(\theta)\left(1+\frac{2ab\cos\theta}{(a^2+b^2)}\right)^{\frac{1}{2}} d\theta. \qquad (3.5)$$

Applying the binomial theorem of hypergeometric functions to the above equation (3.5) , we get

$$\begin{aligned}
\text{C.S.A.} \quad = \quad & 2\pi a\sqrt{(a^2+b^2)}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_n(-1)^n\left(\frac{2ab}{(a^2+b^2)}\right)^n}{n!}\int_\lambda^\mu(\cos\theta)^{n+1}d\theta \\
& +2\pi b\sqrt{(a^2+b^2)}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_n(-1)^n\left(\frac{2ab}{(a^2+b^2)}\right)^n}{n!}\int_\lambda^\mu(\cos\theta)^{n+2}d\theta. \qquad (3.6)
\end{aligned}$$

Employing decomposition identity of unilateral series (1.11) in the equation (3.6), we come to

$$\begin{aligned}\text{C.S.A.} &= 2\pi a\sqrt{(a^2+b^2)}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_{2n}\left(\frac{2ab}{(a^2+b^2)}\right)^{2n}}{(2n)!}\int_{\lambda}^{\mu}(\cos\theta)^{2n+1}d\theta\\ &\quad -2\pi a\sqrt{(a^2+b^2)}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_{2n+1}\left(\frac{2ab}{(a^2+b^2)}\right)^{2n+1}}{(2n+1)!}\int_{\lambda}^{\mu}(\cos\theta)^{2n+2}d\theta\\ &\quad +2\pi b\sqrt{(a^2+b^2)}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_{2n}\left(\frac{2ab}{(a^2+b^2)}\right)^{2n}}{(2n)!}\int_{\lambda}^{\mu}(\cos\theta)^{2n+2}d\theta\\ &\quad -2\pi b\sqrt{(a^2+b^2)}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_{2n+1}\left(\frac{2ab}{(a^2+b^2)}\right)^{2n+1}}{(2n+1)!}\int_{\lambda}^{\mu}(\cos\theta)^{2n+3}d\theta. \end{aligned} \tag{3.7}$$

Substituting the values of integrals from (1.7) and (1.9) into the equation (3.7), we find

$$\begin{aligned}\text{C.S.A.} &= 2\pi a\sqrt{(a^2+b^2)}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_{2n}\left(\frac{2ab}{(a^2+b^2)}\right)^{2n}}{(2n)!}\left\{\frac{(1)_n\sin(\theta)}{\left(\frac{3}{2}\right)_n}\sum_{r=0}^{n}\frac{\left(\frac{1}{2}\right)_r\cos^{2r}(\theta)}{(1)_r}\right\}_{\lambda}^{\mu}\\ &\quad -2\pi a\sqrt{(a^2+b^2)}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_{2n+1}\left(\frac{2ab}{(a^2+b^2)}\right)^{2n+1}}{(2n+1)!}\left\{\frac{\left(\frac{3}{2}\right)_n\sin(\theta)\cos(\theta)}{2\,(2)_n}\sum_{r=0}^{n}\frac{(1)_r\cos^{2r}(\theta)}{\left(\frac{3}{2}\right)_r}+\frac{(\theta)\left(\frac{3}{2}\right)_n}{2\,(2)_n}\right\}_{\lambda}^{\mu}\\ &\quad +2\pi b\sqrt{(a^2+b^2)}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_{2n}\left(\frac{2ab}{(a^2+b^2)}\right)^{2n}}{(2n)!}\left\{\frac{\left(\frac{3}{2}\right)_n\sin(\theta)\cos(\theta)}{2\,(2)_n}\sum_{r=0}^{n}\frac{(1)_r\cos^{2r}(\theta)}{\left(\frac{3}{2}\right)_r}+\frac{(\theta)\left(\frac{3}{2}\right)_n}{2\,(2)_n}\right\}_{\lambda}^{\mu}\\ &\quad -2\pi b\sqrt{(a^2+b^2)}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_{2n+1}\left(\frac{2ab}{(a^2+b^2)}\right)^{2n+1}}{(2n+1)!}\left\{\frac{2(2)_n\sin(\theta)}{3\left(\frac{5}{2}\right)_n}+\frac{(2)_n\sin(\theta)\cos^2(\theta)}{3\left(\frac{5}{2}\right)_n}\sum_{r=0}^{n}\frac{\left(\frac{3}{2}\right)_r\cos^{2r}(\theta)}{(2)_r}\right\}_{\lambda}^{\mu}. \end{aligned} \tag{3.8}$$

Finally using Cauchy double series identity (1.10) and definition of double hypergeometric function of Kampé de Fériet (1.5), after simplification, we prove the desired result (3.4). □

4. Evaluation, reduction for perimeter and curved surface area of revolution related with Limacon and Cardioid

Building upon the foundational arbitrary arc length and curved surface area formulations established in the preceding sections, Section 4 systematically addresses the global geometric evaluations and parameter reductions for both standard and special cases of the Limacon and cardioid. By enforcing specific boundary limits such as evaluating the definite integral space over the complete period $[0,\pi]$—we derive closed-form expressions for total perimeters and total curved surfaces of revolution.

i) Putting $\alpha = 0$, $\beta = \pi$ in the equation (2.1) and multiplying by 2, we get the perimeter of the Limacon $r = a + b\cos\theta;\ a > b > 0$

$$\text{Perimeter} = 2\pi\sqrt{(a^2+b^2)} - \frac{\pi a^2 b^2}{2(a^2+b^2)^{\frac{3}{2}}}\,{}_3F_2\left[\begin{matrix}\frac{3}{4},\ \frac{5}{4},\ 1;\\ 2,\ 2;\end{matrix}\ \left(\frac{2ab}{a^2+b^2}\right)^2\right]. \tag{4.1}$$

ii) Substituting $\beta = \cos^{-1}(-\frac{a}{b})$, $\alpha = 0$ in the equation (2.1) and multiplying by 2, we get the perimeter of outer loop of Limacon $r = a + b\cos\theta;\ 0 < a < b$

$$\begin{aligned}\text{Perimeter} &= 2\sqrt{(a^2+b^2)}\left(\pi - \cos^{-1}\left(\frac{a}{b}\right)\right) + \frac{a^3 b\sqrt{\left(1-\frac{a^2}{b^2}\right)}}{2(a^2+b^2)^{\frac{3}{2}}} F^{\,2:\,1;\,2}_{\,2:\,0;\,1}\left[\begin{matrix}\frac{3}{4},\ \frac{5}{4}:\ 1\ ;\ 1,1;\\ \\ 2,\ 2:\ -\ ;\ \frac{3}{2}\ ;\end{matrix}\quad \left(\frac{2ab}{a^2+b^2}\right)^2, \left(\frac{2a^2}{a^2+b^2}\right)^2\right]\\
&-\frac{\left(\pi - \cos^{-1}\left(\frac{a}{b}\right)\right)a^2b^2}{2(a^2+b^2)^{\frac{3}{2}}}\,{}_3F_2\left[\begin{matrix}\frac{3}{4},\ \frac{5}{4},\ 1;\\ \\ 2,\ 2;\end{matrix}\quad \left(\frac{2ab}{a^2+b^2}\right)^2\right] + \frac{2ab\sqrt{\left(1-\frac{a^2}{b^2}\right)}}{\sqrt{(a^2+b^2)}}\\
&\times F^{\,2:\,1;\,1}_{\,2:\,0;\,0}\left[\begin{matrix}\frac{1}{4},\ \frac{3}{4}:\ 1\ ;\ \frac{1}{2}\ ;\\ \\ \frac{3}{2},\ \frac{3}{2}:\ -\ ;-\ ;\end{matrix}\quad \left(\frac{2ab}{a^2+b^2}\right)^2, \left(\frac{2a^2}{a^2+b^2}\right)^2\right]. \qquad (4.2)\end{aligned}$$

iii) Replacing β by π, α by $\cos^{-1}\left(-\frac{a}{b}\right)$ in the equation (2.1) and multiplying by 2, we get the perimeter of inner loop of Limacon $r = a + b\cos\theta;\ 0 < a < b$

$$\begin{aligned}\text{Perimeter} &= 2\pi\sqrt{(a^2+b^2)} - \frac{\pi a^2b^2}{2(a^2+b^2)^{\frac{3}{2}}}\,{}_3F_2\left[\begin{matrix}\frac{3}{4},\ \frac{5}{4},\ 1;\\ \\ 2,\ 2;\end{matrix}\quad \left(\frac{2ab}{a^2+b^2}\right)^2\right] - 2\sqrt{a^2+b^2}\left(\pi - \cos^{-1}\frac{a}{b}\right)\\
&-\frac{a^3 b\sqrt{\left(1-\frac{a^2}{b^2}\right)}}{2(a^2+b^2)^{\frac{3}{2}}} F^{\,2:\,1;\,2}_{\,2:\,0;\,1}\left[\begin{matrix}\frac{3}{4},\ \frac{5}{4}:\ 1\ ;\ 1,1;\\ \\ 2,\ 2:\ -\ ;\ \frac{3}{2}\ ;\end{matrix}\quad \left(\frac{2ab}{a^2+b^2}\right)^2, \left(\frac{2a^2}{a^2+b^2}\right)^2\right]\\
&+\frac{\left(\pi - \cos^{-1}\frac{a}{b}\right)a^2b^2}{2(a^2+b^2)^{\frac{3}{2}}}\,{}_3F_2\left[\begin{matrix}\frac{3}{4},\ \frac{5}{4},\ 1;\\ \\ 2,\ 2;\end{matrix}\quad \left(\frac{2ab}{a^2+b^2}\right)^2\right] - \frac{2ab\sqrt{\left(1-\frac{a^2}{b^2}\right)}}{\sqrt{(a^2+b^2)}}\\
&\times F^{\,2:\,1;\,1}_{\,2:\,0;\,0}\left[\begin{matrix}\frac{1}{4},\ \frac{3}{4}:\ 1\ ;\ \frac{1}{2}\ ;\\ \\ \frac{3}{2},\ \frac{3}{2}:\ -\ ;-\ ;\end{matrix}\quad \left(\frac{2ab}{a^2+b^2}\right)^2, \left(\frac{2a^2}{a^2+b^2}\right)^2\right]. \qquad (4.3)\end{aligned}$$

iv) Setting $\lambda = -\frac{\pi}{2},\ \mu = \frac{\pi}{2}$ in the equation (2.4) and multiplying by 2, we get the perimeter of another Limacon $r = a + b\sin\theta; a > b > 0$ in the form of equation (4.1).

v) Substituting $\mu = \frac{\pi}{2}$, $\lambda = \left(-\sin^{-1}\frac{a}{b}\right)$ in the equation (2.4) and multiplying by 2, we get the perimeter of outer loop of Limacon $r = a + b\sin\theta;\ 0 < a < b$

$$\begin{aligned}\text{Perimeter} &= \pi\sqrt{(a^2+b^2)} - \frac{\pi a^2b^2}{4(a^2+b^2)^{\frac{3}{2}}}\,{}_3F_2\left[\begin{matrix}\frac{3}{4},\ \frac{5}{4},\ 1;\\ \\ 2,\ 2;\end{matrix}\quad \left(\frac{2ab}{a^2+b^2}\right)^2\right] + 2\sqrt{a^2+b^2}\left(\sin^{-1}\frac{a}{b}\right)\\
&+\frac{a^3 b\sqrt{\left(1-\frac{a^2}{b^2}\right)}}{2(a^2+b^2)^{\frac{3}{2}}} F^{\,2:\,1;\,2}_{\,2:\,0;\,1}\left[\begin{matrix}\frac{3}{4},\ \frac{5}{4}:\ 1\ ;\ 1,1;\\ \\ 2,\ 2:\ -\ ;\ \frac{3}{2}\ ;\end{matrix}\quad \left(\frac{2ab}{a^2+b^2}\right)^2, \left(\frac{2a^2}{a^2+b^2}\right)^2\right]\\
&-\frac{\left(\sin^{-1}\frac{a}{b}\right)a^2b^2}{2(a^2+b^2)^{\frac{3}{2}}}\,{}_3F_2\left[\begin{matrix}\frac{3}{4},\ \frac{5}{4},\ 1;\\ \\ 2,\ 2;\end{matrix}\quad \left(\frac{2ab}{a^2+b^2}\right)^2\right] + \frac{2ab\sqrt{\left(1-\frac{a^2}{b^2}\right)}}{\sqrt{(a^2+b^2)}}\\
&\times F^{\,2:\,1;\,1}_{\,2:\,0;\,0}\left[\begin{matrix}\frac{1}{4},\ \frac{3}{4}:\ 1\ ;\ \frac{1}{2}\ ;\\ \\ \frac{3}{2},\ \frac{3}{2}:\ -\ ;-\ ;\end{matrix}\quad \left(\frac{2ab}{a^2+b^2}\right)^2, \left(\frac{2a^2}{a^2+b^2}\right)^2\right]. \qquad (4.4)\end{aligned}$$

vi) Putting $\mu = \left(-\sin^{-1}\frac{a}{b}\right),\ \lambda = -\frac{\pi}{2}$ in the equation (2.4) and multiplying by 2, we get the perimeter of inner loop of Limacon $r = a + b\sin\theta;\ 0 < a < b$

$$\begin{aligned}\text{Perimeter} &= -2\sqrt{(a^2+b^2)}\left(\sin^{-1}\frac{a}{b}\right) - \frac{a^3b\sqrt{\left(1-\frac{a^2}{b^2}\right)}}{2(a^2+b^2)^{\frac{3}{2}}} F^{2:\,1;\,2}_{2:\,0;\,1}\left[\begin{matrix}\frac{3}{4},\ \frac{5}{4}:\ 1\ ;\ 1,1; \\ 2,\ 2:\ -\ ;\ \frac{3}{2}\ ;\end{matrix}\ \left(\frac{2ab}{a^2+b^2}\right)^2, \left(\frac{2a^2}{a^2+b^2}\right)^2\right] \\ &+ \frac{\left(\sin^{-1}\frac{a}{b}\right)a^2b^2}{2(a^2+b^2)^{\frac{3}{2}}}\,{}_3F_2\left[\begin{matrix}\frac{3}{4},\ \frac{5}{4},\ 1; \\ 2,\ 2;\end{matrix}\ \left(\frac{2ab}{a^2+b^2}\right)^2\right] - \frac{2ab\sqrt{\left(1-\frac{a^2}{b^2}\right)}}{\sqrt{(a^2+b^2)}} \\ &\times F^{2:\,1;\,1}_{2:\,0;\,0}\left[\begin{matrix}\frac{1}{4},\ \frac{3}{4}:\ 1\ ;\ \frac{1}{2}\ ; \\ \frac{3}{2},\ \frac{3}{2}:\ -\ ;-\ ;\end{matrix}\ \left(\frac{2ab}{a^2+b^2}\right)^2, \left(\frac{2a^2}{a^2+b^2}\right)^2\right] \\ &+ \pi\sqrt{(a^2+b^2)} - \frac{\pi a^2b^2}{4(a^2+b^2)^{\frac{3}{2}}}\,{}_3F_2\left[\begin{matrix}\frac{3}{4},\ \frac{5}{4},\ 1; \\ 2,\ 2;\end{matrix}\ \left(\frac{2ab}{a^2+b^2}\right)^2\right].\end{aligned} \tag{4.5}$$

vii) Put $b = a$ in the equation (4.1), we get the perimeter of cardioid, $r = a(1+\cos\theta)$ or $r = a(1+\sin\theta)$ in the form

$$\text{Perimeter} = 2a\pi\sqrt{2} - \frac{\pi a}{4\sqrt{2}}\,{}_3F_2\left[\begin{matrix}\frac{3}{4},\ \frac{5}{4},\ 1; \\ 2,\ 2;\end{matrix}\ 1\right]. \tag{4.6}$$

But we know that the perimeter of Cardioid $r = a(1+\cos\theta)$ or $r = a(1+\sin\theta)$ is given by

$$\text{Perimeter} = 8a. \tag{4.7}$$

Comparing the equations (4.6) and (4.7), we get a new summation theorem given by

$${}_3F_2\left[\begin{matrix}\frac{3}{4},\ \frac{5}{4},\ 1; \\ 2,\ 2;\end{matrix}\ 1\right] = 16 - \frac{32\sqrt{2}}{\pi}. \tag{4.8}$$

viii) The curved surface area obtained by revolving the semi-Limacon (lying above initial line) about initial line (axis of revolution), is given by equation (4.9), when we set $\beta = \pi$ and $\alpha = 0$, in the general result (3.2)

$$C.S.A. = \pi\left(\frac{a^2-b^2}{3a^2b}\right)\left[(a+b)^3 - (a-b)^3\right] + \frac{\pi}{5a^2b}\left[(a+b)^5 - (a-b)^5\right]. \tag{4.9}$$

ix) The curved surface area obtained by revolving the semi-Cardiod about initial line (axis of revolution), is given by equation (4.10), when we put $b = a$ in the equation (4.9)

$$\text{C.S.A.} = \frac{32\pi a^2}{5}. \tag{4.10}$$

Remark 4.1. The results (4.7), (4.9) and (4.10) are known results available in the chapter of Rectification in any text book of Integral Calculus at graduate level.

Remark 4.2. Similar type expressions for length and curved surface area of revolution of arbitrary arc can be obtained, when we consider the Limacons $r = a - b\cos\theta$ and $r = a - b\sin\theta$.

5. Conclusion

In our present investigation, with the help of Kampé de Fériet double hypergeometric function (1.5), series rearrangement technique and binomial theorem, we have obtained the analytical expressions for the length of an arbitrary arc of Limacons lying between two arbitrary points. We have also derived the formulas for the curved surface areas of revolution, obtained by revolving the arbitrary arc of Limacon. Furthermore, we provided formulas for curved surface area of revolution, perimeter of Limacons, Cardioids and a new summation theorem for Clausen function. We conclude this investigation by emphasizing the possibility of further research involving length and curved surface areas of revolution for arbitrary arc of new curve using hypergeometric functions. Our main results and their special cases, given in this paper, are potentially useful in scientific problems, engineering mathematics and mathematical physics.

Acknowledgments

The authors are very grateful to the anonymous referees for their constructive encouraging comments and suggestions which improved this paper.

Author Contributions: Writing-original draft, M. I. Qureshi, A. H. Bhat and B. A. Bhat; writing-review and editing, M. I. Qureshi, A. H. Bhat and B. A. Bhat. All authors have read and agreed to the published version of the manuscript.

Conflict of Interest: The authors have no conflict of interest.

Funding (Financial Disclosure): Not applicable.

References

[1] G. E. Andrews, R. Askey and R. Roy, *Special functions*, Cambridge University Press, Cambridge, U.K., 1999.
[2] P. Appell and J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques-polynômes d'Hermite*, Gauthier-Villars, Paris, 1926.
[3] W. N. Bailey, *Generalized hypergeometric series*, Cambridge University Press, London, 1935.
[4] Y. A. Brychkov, *Handbook of special functions, derivatives, integrals, series and other formulas*, CRC Press, Taylor and Francis Group, Boca Raton, London, New York, 2008.
[5] J. L. Burchnall and T. W. Chaundy, *Expansions of Appell's double hypergeometric functions (I)*, Quart. J. Math. Oxford Ser. **11**, 249–270, 1940; https://doi.org/10.1093/qmath/os-11.1.249.
[6] J. L. Burchnall and T. W. Chaundy, *Expansions of Appell's double hypergeometric functions (II)*, Quart. J. Math. Oxford Ser. **12**, 112–128, 1941.
[7] L. Carlitz, *A Saalschützian theorem for double series*, J. Lond. Math. Soc. **38**, 415–418, 1963; https://doi.org/10.1112/jlms/s1-38.1.415.
[8] L. Carlitz, *Another Saalschützian theorem for double series*, Rend. Semin. Mat. Univ. Padova **34**, 200–203, 1964.
[9] L. Carlitz, *A summation theorem for double hypergeometric series*, Rend. Semin. Mat. Univ. Padova **37**, 230–233, 1967.
[10] J. Choi and A. K. Rathie, *General summation formulas for the Kampé de Fériet function*, Montes Taurus J. Pure Appl. Math. **1 (1)**, 107–128, 2019.
[11] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher transcendental functions* (Volume I), McGraw-Hill Book Company, New York, Toronto and London, 1955.
[12] H. Exton, *Multiple hypergeometric functions*, Halsted Press, New York, 1976.
[13] H. Exton, *On the reducibility of the Kampé de Fériet function*, J. Comput. Appl. Math. **83 (1)**, 119–121, 1997.
[14] H. Exton, *New hypergeometric identities arising from Gauss's second summation theorem*, J. Comput. Appl. Math. **88 (2)**, 269–274, 1998.
[15] J. Gil, *On the arc length parametrization problem*, Int. J. Pure Appl. Math. **31**, 401–419, 2006.
[16] N. T. Hài, O. I. Marichev and H. M. Srivastava, *A note on the convergence of certain families of multiple hypergeometric series*, J. Math. Anal. Appl. **164**, 104–115, 1992.
[17] A. Hasanov and T. G. Ergashev, *New decomposition formulas associated with the Lauricella multivariable hypergeometric functions*, Montes Taurus J. Pure Appl. Math. **3 (3)**, 317–326, 2021; Article ID: MTJPAM-D-20-00049.
[18] A. Hasanov, A. Ryskan and J. Choi, *Decomposition formulas for second-order quadruple Gaussian hypergeometric series by means of operators H(A, B)*, Montes Taurus J. Pure Appl. Math. **4 (3)**, 41–60, 2022.
[19] P. Humbert, *La fonction* $W_{k,\mu_1,\mu_2,...,\mu_n}(x_1, x_2, ..., x_n)$, C. R. Acad. Sci. Paris **171**, 428–430, 1920.
[20] P. Humbert, *The confluent hypergeometric functions of two variables*, Proc. Roy. Soc. Edinburgh Sect. A **41**, 73–96, 1920-21.
[21] P. Humbert, *The confluent hypergeometriques d'ordre superieur a deux variables*, C. R. Acad. Sci. Paris **173**, 73–96, 1921.
[22] J. Kampé de Fériet, *Les fonctions hypergéometriques d'ordre supérieur à deux variables*, C. R. Acad. Sci. Paris **173**, 401–404, 1921.
[23] M. F. Khan, *Arc length of an elliptical curve*, Int. J. Sci. Res. Publ. **3 (8)**, 1–5, 2013.
[24] N. N. Lebedev, *Special functions and their applications* (Translated by Richard A. Silverman), Prentice-Hall, Englewood Cliffs, New Jersey, 1965.

[25] S. Maksimović and A. Borković, *A new class of plane curves with arc length parametrization and its application to linear analysis of curved beams*, Mathematics **9**, 1–9, 2021; Article ID: 1778.

[26] P. Nystedt, *Arc length of function graphs via Taylor's formula*, Int. J. Math. Educ. Sci. Technol. **52 (2)**, 310–323, 2021.

[27] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and series*, Volume 3: More special functions (Translated from Russian by G. G. Gould), Nauka, Moscow, 1986; Gordon and Breach Science Publishers, New York, Philadelphia, London, Paris, Montreux, Tokyo, Melbourne, 1990.

[28] M. I. Qureshi and K. A. Quraishi, *Generalizations of incomplete elliptic integrals of first and second kinds*, Tamkang J. Math. **43 (4)**, 595–601, 2012.

[29] E. D. Rainville, *Special functions*, The Macmillan Co. Inc., New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.

[30] U. P. Singh, R. J. Srivastava and N. H. Siddiqui, *A text book of integral calculus*, Wisdom Press, 2011.

[31] H. M. Srivastava and J. Choi, *Zeta and q-zeta functions and associated series and integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[32] H. M. Srivastava and M. C. Daoust, *A note on the convergence of Kampé de Fériet's double hypergeometric series*, Math. Nachr. **53**, 151–159, 1972.

[33] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian hypergeometric series*, In: Ellis Horwood Series in Mathematics and its Applications, Ellis Horwood Ltd., Chichester, Halsted Press, John Wiley and Sons, Inc., New York, 1985.

[34] H. M. Srivastava and H. L. Manocha, *A treatise on generating functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, Toronto, 1984.

[35] H. M. Srivastava and R. Panda, *An integral representation for the product of two Jacobi polynomials*, J. Lond. Math. Soc. **12 (2)**, 419–425, 1976.

[36] G. Traub, *The development of the mathematical analysis of curve length from Archimedes to Lebesgue*, New York University, Ph.D. Thesis, 1984.

[37] E. M. Wright, *The asymptotic expansion of the generalized hypergeometric function*, J. Lond. Math. Soc. **10**, 286–293, 1935; https://doi.org/10.1112/jlms/s1-10.40.286.

[38] E. M. Wright, *The asymptotic expansion of the generalized hypergeometric function*, Proc. Lond. Math. Soc. **46 (2)**, 389–408, 1940; https://doi.org/10.1112/plms/s2-54.3.254-s.

How to cite this article: M. I. Qureshi, A. H. Bhat and B. A. Bhat, *Analytical expressions for the length and curved surface area of revolution of arbitrary arcs of Limacon; a hypergeometric function approach*, Montes Taurus J. Pure Appl. Math. **8 (1)**, 149–163, 2026; Article ID: MTJPAM-D-24-00095.