Montes Taurus J. Pure Appl. Math. 8 (2), 27–38, 2026

Montes Taurus Journal of Pure and Applied Mathematics
https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

On connected unicyclic semiregular graphs: Approach to generating functions

Hacer Ozden Ayna [a], Aysun Yurttas Gunes [b], Ismail Naci Cangul [c], Daeyeoul Kim [d], Yilmaz Simsek [e]

[a]*Department of Mathematics, Faculty of Arts and Science, Bursa Uludag University, Gorukle 16059 Bursa, Turkey*
[b]*Department of Mathematics, Faculty of Arts and Science, Bursa Uludag University, Gorukle 16059 Bursa, Turkey*
[c]*Department of Mathematics, Faculty of Arts and Science, Bursa Uludag University, Gorukle 16059 Bursa, Turkey*
[d]*Department of Mathematics and Institute of Pure and Applied Mathematics, Jeonbuk National University, Jeonju, Korea*
[e]*Department of Mathematics, Akdeniz University, Antalya, Turkey*

Abstract

Some semi-regular graphs like molecular graphs and chain-like structures cannot be regular, but still have some internal symmetry. Connected unicyclic realizations of a degree sequence having only two integers are related to the divisor function and have some specific form consisting of a cycle surrounded by pendant edges. Here tables corresponding to the divisor function in terms of several arithmetic functions are presented and shown that similar classifications can be obtained for bicyclic, tricyclic, etc. graphs. Finally two alternative generating functions are obtained for the divisor function.

Keywords: Semiregular, unicyclic, realization, degree sequence, omega invariant, generating function, divisor function

2020 MSC: 05A15, 05CXX, 11A05, 11A25, 11B73

1. Introduction

Graphs have been used frequently in recent years in both real-world problems and other branches of science because they are associated with mathematical structures that objects are related to. Similarly, generating functions of special numbers and special polynomials have vital applications in the fields of analysis and discrete mathematics, differential and difference equations, special functions, combinatorics, graph theory, probability, and other applied areas.

The motivation of this paper is to give not only many new results related to molecular graphs, chain-like structures, connected unicyclic realizations of some degree sequence consisting of two different integers involving divisor function, but also generating functions for the divisor function with their applications tables involving the Lambert series, the Stirling numbers, the triangular number, the Pyramid number, etc.

†*Article ID:* MTJPAM-D-26-00020

Email addresses: `hozden@uludag.edu.tr` (Hacer Ozden Ayna), `ayurttas@uludag.edu.tr` (Aysun Yurttas Gunes), `cangul@uludag.edu.tr` (Ismail Naci Cangul), `kdaeyeoul@jbnu.ac.kr` (Daeyeoul Kim), `ysimsek@akdeniz.edu.tr` (Yilmaz Simsek)

Received:30 January 2026, Accepted:27 April 2026, Published:15 June 2026

⋆*Corresponding Author:* Ismail Naci Cangul

Within the scope of this article, some definitions and notations are briefly explained as follows. We shall only consider finite, connected and undirected graphs in this paper. We do not put the condition to be simple as some of our graphs will have loops or multiple edges. We shall denote the vertex and edge sets of a graph G by $V(G)$ and $E(G)$, respectively. Also the order and size of G are $|V(G)| = n$ and $|E(G)| = m$, respectively. The degree of a vertex v is denoted by $d_G(v)$ or briefly by dv. A vertex with degree 1 is called a pendant vertex and the edge incident to it is called a pendant edge. A regular graph is the one with all vertices having the same degree. Regularity is an important property for mathematicians since the ancient times and regular graphs are intensively studied. Irregularity indices are used for this aim. A semi-regular graph is the one having only two different positive integers as vertex degrees. There are many semi-regular graphs which cannot be regular, but still having some internal symmetry such as some molecular graphs and chain-like structures. In a graph G, the set $D = \left\{1^{(a_1)}, 2^{(a_2)}, \ldots, \Delta^{(a_\Delta)}\right\}$ of all vertex degrees counted with multiplicities is named as the degree sequence of G. Here Δ denotes the largest vertex degree in G and $a_i \geq 0$ is the multiplicity of the degree d_i. If a set $D = \left\{1^{(a_1)}, 2^{(a_2)}, \ldots, \Delta^{(a_\Delta)}\right\}$ is randomly given, then the realizability problem is the one asking whether a graph exists with vertex degrees as given in D. If there is at least one graph, then D is called realizable and the graph is called a realization of D. There are several algorithms to decide about the realizability of a set of positive integers. Although this problem is still open, there are several partial answers for specific degree sequences.

A recently defined graph invariant called omega invariant has proven to be useful in some algebraic, graph theoretic, topological, combinatorial properties of graphs or degree sequences. A graph is called acyclic if it contains no cycle. A connected acyclic graph is named as a tree. A graph having one, two, three cycles is called as unicyclic, bicyclic, tricyclic graph, respectively. For unicyclic graphs, it is proven that omega is zero for connected graphs. All connected unicyclic realizations of some degree sequence consisting of two different integers have some specific form consisting of a cycle together with some pendant edges (as one of the two different integers in the degree sequence is 1). Similar classifications can be obtained for bicyclic, tricyclic graphs.

We define a unicyclic graph class which will be frequently used in this paper as follows: Let $U_{k,l}$ be the connected unicyclic graph where the cycle length is k and there are l pendant edges at each vertex on the cycle. See Figure 1, for some examples of $U_{k,l}$ graphs.

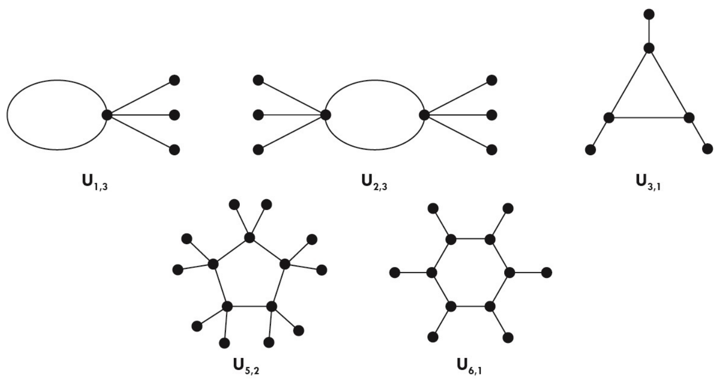

Figure 1. Some $U_{k,l}$ graphs

The graph $U_{k,l}$ has $k(l + 1)$ vertices and $k(l + 1)$ edges which is as expected for a unicyclic graph.

In [3], a graph invariant called omega has been defined to obtain information about the cyclic structure of a given graph. It is related to the cyclomatic number and also the Euler characteristics of the graph. This invariant gives several combinatorial and topological information about the realizations of a degree sequence D as above and it is denoted by $\Omega(D)$. Similarly, for a graph realization G, omega invariant of G is denoted by $\Omega(G)$. The omega invariant $\Omega(G)$ of the graph G is defined in terms of D as

$$\Omega(G) = \sum_{i=1}^{\Delta}(i-2)a_i = a_3 + 2a_4 + 3a_5 + \cdots + (\Delta - 2)a_\Delta - a_1.$$

It is shown that for a graph G, $\Omega(G) = 2(m - n)$ meaning that omega invariant is always even. Omega invariant is related to various graph parameters such as vertex degrees (*cf.* [3]), vertex and edge deletion (*cf.* [6]), average degree (*cf.* [5]), cyclicness (*cf.* [4]), connectedness (*cf.* [21]), matching number (*cf.* [18]), independence number (*cf.* [20]), nullity (*cf.* [19]) and to topological indices (*cf.* [10, 11, 29, 31, 34, 33]). Omega invariant of various families of graphs such as line graphs (*cf.* [8, 28]), total graphs (*cf.* [8]), Lucas graphs (*cf.* [9]), Fibonacci graphs (*cf.* [32]), Tribonacci graphs (*cf.* [7]), complement graphs (*cf.* [30]), graph products (*cf.* [2, 12]). Inverse problem has applications in the design of combinatorial libraries for drug discovery. Inverse problem for Bell index, Sigma index and Zagreb index (*cf.* [11, 29, 34]) was studied using omega invariant.

2. Realizations of connected unicyclic semiregular graphs

A graph is regular if all the vertex degrees are equal. Regularity is an important property in mathematics since ancient times. A graph is called semiregular if there are two different vertex degrees in the degree sequence. In this paper, we study connected unicyclic semiregular graphs and determine the degree sequences of these graphs.

Let G be a connected unicyclic semiregular graph. In [4], it was shown that $\Omega(G) = 0$. This means we have the number of vertices equal to the number of edges in such a graph. As G is semiregular, we may take its degree sequence as $\{d_1^{(a_1)}, d_2^{(a_2)}\}$ where $d_1, d_2 \geq 1$, $d_1 \neq d_2$ and $a_1, a_2 > 0$. As $\Omega(G) = 0$, we have

$$(d_1 - 2)a_1 + (d_2 - 2)a_2 = 0.$$

Now as $a_1, a_2 > 0$, only one of $d_1 - 2$ or $d_2 - 2$ could be negative. Let w.l.o.g. we may take $d_1 - 2 < 0$. Hence $d_1 < 2$ implying that $d_1 = 1$. Hence the degree sequence becomes $\{1^{(a_1)}, d_2^{(a_2)}\}$. Hence, one of the two different vertex degrees must be 1. Hence for a semiregular unicyclic graph, some vertices will have degree at least 3 and the remaining vertices will be pendant. As a pendant vertex cannot belong to a cycle, all pendant vertices must be adjacent to the vertices on the unique cycle. Hence we shall have a $U_{k,l}$ graph in each case.

Here, by the above discussion, $d_2 > 2$. By Ω invariant, we have

$$-a_1 + (d_2 - 2)a_2 = 0,$$

which implies that

$$d_2 = \frac{a_1}{a_2} + 2.$$

As $d_2 > 2$ is an integer, $a_2 | a_1$. Hence the degree sequence becomes

$$\left\{1^{(a_1)}, \left(\frac{a_1}{a_2} + 2\right)^{(a_2)}\right\}. \tag{2.1}$$

Now we first shall determine the possible degree sequences corresponding to a given value of a_1 as a_2 is determined by a_1 and d_2 is determined by a_1 and a_2. Of course at the end, we shall also try to determine the realizations of these degree sequences. We shall need to recall some number theoretical results about the divisors of an integer.

Let a_1 be given in standard form by

$$a_1 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k},$$

where $\alpha_i > 0$ for $i = 1, 2, \ldots, k$ and p_i's are different prime numbers. By the above discussion a_2 must divide a_1. That is a_2 must be a positive divisor of a_1. Then a_2 has the form

$$a_2 = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k},$$

where $0 \leq \beta_i \leq \alpha_i$. The number of positive divisors of a_1 is given by the formula

$$d(a_1) = \prod_{i=1}^{k} (1 + \alpha_i).$$

That is, there are $d(a_1)$ different degree sequences for every given a_1. Of course, we may have more then one realization for each such degree sequence. We now consider some possible cases.

Lemma 2.1. *Let a_1 be a prime. Then $d(a_1) = 2$ and only possible divisors are 1 and a_1. Hence the corresponding degree sequences are $\{1^{(a_1)}, (2 + a_1)^{(1)}\}$ for $a_2 = 1$ or $\{1^{(a_1)}, 3^{(a_1)}\}$ for $a_2 = a_1$. The former family corresponds to unicyclic graphs with one loop and the latter family corresponds to unicyclic graphs with an a_1-gon every vertex of which is having a pendant edge. In both cases, there is only one possible realization of the degree sequence.*

Therefore we proved.

Theorem 2.2. *Let a_1 be a prime. There are two realizations of $\{1^{(a_1)}, d_2^{(a_2)}\}$ which are U_{1,a_1} and $U_{a_1,1}$.*

We secondly consider composite values of a_1. Recall that $a_1 = p_1^{\alpha_1 d} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and a_2 is a positive divisor of a_1.

Theorem 2.3. *Let a_1 be a composite positive integer. There are $d(a_1)$ degree sequences $\{1^{(a_1)}, d_2^{(a_2)}\}$ and each of these degree sequences can be realized only in one way. These realizations are the graphs $U_{a_2,a_1/a_2}$ where a_2 is ranging over all positive divisors of a_1.*

Proof. We use a constructive method. Let a_1 be a composite positive integer. As above, the degree sequence $\{1^{(a_1)}, d_2^{(a_2)}\}$ becomes $\{1^{(a_1)}, (2 + \frac{a_1}{a_2})^{(a_2)}\}$ as $\Omega = 0$ for connected unicyclic graphs. Here a_2 must be a positive divisor of a_1. Now by [4], the length of the unique cycle in the corresponding unicyclic graph is a_2. Therefore we start with an a_2-gon. As there are a_1 pendant vertices in our final graph that we want to realize, these a_1 vertices of degree one must be adjacent to a_2 vertices of degree 2 each. As we have semiregular graphs, all vertices have degree either 1 or $d_2 = 2 + \frac{a_1}{a_2}$. Hence each vertex on the cycle must have degree $d_2 = 2 + \frac{a_1}{a_2}$. On the a_2-gon we already constructed, each vertex has degree 2 to begin with. So in the final graph, to each such vertex, we must add $\frac{a_1}{a_2}$ pendant edges. This gives the required realization which is $U_{a_2,\frac{a_1}{a_2}}$. This is possible for each divisor a_2 of a_1 and hence the realizations of $U_{a_2,\frac{a_1}{a_2}}$ is possible in a unique way, the total number of realizations of $\{1^{(a_1)}, (2 + \frac{a_1}{a_2})^{(a_2)}\}$ will exactly be as the number of positive divisors of a_1 which is $d(a_1)$. □

Remark 2.4. For a given a_1 and for each positive divisor d of a_1, as $U_{d,\frac{a_1}{d}}$ are all the realizations of Eq. (2.1), there is a practical way to find all possible $U_{k,l}$'s. We list all positive divisors d of a given a_1 and then the product of k and l must be a_1. For example, for $a_1 = 12$, all positive divisors are $d = 1, 2, 3, 4, 6$ or 12, and we must choose k and l such that $k \times l = 12$, That is, k and l are non-zero zero divisors in the ring $\mathbb{Z}_{12}$. Therefore for each $k = d$, l must be $l = \frac{12}{d}$ and hence all possible graph realizations are $U_{1,12}, U_{2,6}, U_{3,4}, U_{4,3}, U_{6,2}$ and $U_{12,1}$.

3. Generating functions for divisor functions and their applications

In this section, we study on generating functions for the number of positive divisors of a_1 denoted by $d(a_1)$. In Section 2, we showed that there are $d(a_1)$ different degree sequences for every given a_1. These numbers are related to the a_1-gon at the unicyclic graphs.

In order to give the generating function, we give some definitions and notations about arithmetic functions and generating functions.

The divisor function which is an arithmetical function is defined by

$$\sigma_\beta(m) = \sum_{d|m} d^\beta,$$

where $\beta \in \mathbb{C}$ (or $\mathbb{R}$) and $m \in \mathbb{N}$. $\sigma_\beta(n)$ is called the sum of the β-th powers of the divisors of n (or called divisor function). When $\beta = 0$, $\sigma_0(m)$ is the number of divisors of m. Thus $d(m) := \sigma_0(m)$. When $\beta = 1$, $\sigma_1(m)$ is the sum of the divisors of m. This gives $\sigma(m) := \sigma_1(m)$ (*cf.* [1]).

It is well-known that $\sigma_\beta(m)$ is a multiplicative function, that is

$$\sigma_\beta(m) = \sigma_\beta(p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j}) = \prod_{v=1}^{j} \sigma_\beta(p_v^{k_v}),$$

where $v \in \{1, 2, \ldots, j\}$, p_v are prime numbers and k_v denotes the positive integers (*cf.* [1]).

The Dirichlet inverse of the divisor function σ_β which can be given by a linear combination of the β-th powers of the divisors of m, is defined as follows

$$\sigma_\beta^{-1}(m) = \sum_{d|m} d^\beta \mu(d) \mu\left(\frac{m}{d}\right),$$

where $\mu(m)$ denotes the Möbius function, [1]. In fact, σ_β^{-1} is also a multiplicative function.

Let

$$\zeta_\beta(n) = n^\beta,$$

$$\zeta := \zeta_0$$

and

$$\delta(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then we note that

$$\zeta * \mu = \mu * \zeta = \delta,$$

$$\zeta * \zeta = d$$

and

$$\zeta * \zeta_\beta = \sigma_\beta,$$

where, $*$ is the Dirichlet convolution.

Let

$$\mathfrak{Z}_0(n; x) := \zeta(n)x + \zeta * \zeta(n)x^2 + \zeta * \zeta * \zeta(n)x^3 + \cdots$$

and

$$\Delta(n; x) := \delta_0(n)x + \delta_0 * \delta_0(n)x^2 + \delta_0 * \delta_0 * \delta_0(n)x^3 + \cdots .$$

These type of functions have been studied in [13, 14]. Therefore, we can find some values of ζ_0, $\zeta * \zeta$ and $\zeta * \zeta * \zeta$ by the following Tables 1-4:

p^n	$\zeta(p^n)$	base 2
p^0	1	(1)
p^1	1	(1)
p^2	1	(1)
$\vdots$	$\vdots$	$\vdots$
p^n	1	(1)

Table 1. Numeral system of base 2 for the values of $\zeta(p^n)$

p^n	$\zeta * \zeta(p^n)$	base 2
p^0	1	(1)
p^1	2	(1)(0)
p^2	3	(1)(1)
p^3	4	(1)(0)(0)
4	5	(1)(0)(1)
$\vdots$	$\vdots$	$\vdots$
p^n	$n+1$	

Table 2. Numeral system of base 2 for the values of $\zeta * \zeta(p^n)$

n	$\zeta * \zeta * \zeta(p^n)$	base 2
0	1	(1)
1	$1 + 2$	(1)(1)
2	$1 + 2 + 3$	(1)(1)(0)
3	$1 + 2 + 3 + 4$	(1)(0)(1)(0)
4	$1 + 2 + 3 + 4 + 5$	(1)(1)(1)(1)
$\vdots$	$\vdots$	$\vdots$
n	$1 + \cdots + n + (n+1)$ $= (n+1)(n+2)/2$ T_{n+1} Triangular number	

Table 3. Numeral system of base 2 for the values of $\zeta * \zeta * \zeta(p^n)$

n	$\zeta * \zeta * \zeta * \zeta(p^n)$	base 2
0	1	(1)
1	$2 \cdot 1 + 1 \cdot 2$	(1)(0)(0)
2	$3 \cdot 1 + 2 \cdot 2 + 1 \cdot 3$	(1)(0)(1)(0)
3	$4 \cdot 1 + 3 \cdot 2 + 2 \cdot 3 + 1 \cdot 4$	(1)(0)(1)(0)(0)
4	$5 \cdot 1 + 4 \cdot 2 + 3 \cdot 3 + 2 \cdot 4 + 1 \cdot 5$	(1)(0)(0)(0)(1)(1)
$\vdots$	$\vdots$	$\vdots$
n	$(n+1) \cdot 1 + n \cdot 2 + \cdots + 1 \cdot (n+1)$ $= (n+1)(n+2)(n+3)/6$ P^3_{n+1} 3-gonal Pyramid number	

Table 4. Numeral system of base 2 for the values of $\zeta * \zeta * \zeta * \zeta(p^n)$

Note here that the dot notation $\cdot$ denotes the classical multiplication in both Table 4 and throughout this paper.

In particular, expressing the ζ_k-function as a p-adic number leads to some very interesting numbers. In other words, it is very useful to write the ζ_k-function as a p-adic number if it is a divisor function. See Tables 5-10.

n	$\zeta_1(p^n)$	base p
0	1	(1)
1	p	(1)(0)
2	p^2	(1)(0)(0)
$\vdots$	$\vdots$	$\vdots$
n	p^n	$(1)\underbrace{(0)\cdots(0)}_{n}$

Table 5. Numeral system of base p for the values of $\zeta_1(p^n)$

n	$\zeta_1 * \mu(p^n)$	base p
0	1	(1)
1	$p - 1$	(1)(−1)
2	$p^2 - p$	(1)(−1)(0)
$\vdots$	$\vdots$	$\vdots$
n	$p^n - p^{n-1}$	$(1)(-1)\underbrace{(0)\cdots(0)}_{n-1}$

Table 6. Numeral system of base p for the values of $\zeta_1 * \mu(p^n)$

n	$\zeta_1 * \zeta(p^n)$	base p
0	1	(1)
1	$p + 1$	(1)(1)
2	$p^2 + p + 1$	(1)(1)(1)
$\vdots$	$\vdots$	$\vdots$
n	$p^n + p^{n-1} + \cdots + p + 1$	$\underbrace{(1)\cdots(1)}_{n+1}$

Table 7. Numeral system of base p for the values of $\zeta_1 * \zeta(p^n)$

n	$\zeta_1 * \zeta * \zeta(p^n)$	base p
0	1	(1)
1	$p + 2$	(1)(2)
2	$p^2 + 2p + 3$	(1)(2)(3)
$\vdots$	$\vdots$	$\vdots$
n	$p^n + 2p^{n-1} + \cdots + np + (n + 1) \cdot 1$	$(1)(2)(3)\cdots(n)(n + 1)$

Table 8. Numeral system of base p for the values of $\zeta_1 * \zeta * \zeta(p^n)$

n	$\zeta_1 * \zeta * \zeta * \zeta(p^n)$	base p
0	1	(1)
1	$p + 3$	(1)(3)
2	$p^2 + 3p + 6$	(1)(3)(6)
3	$p^3 + 3p^2 + 6p + 10$	(1)(3)(6)(10)
$\vdots$	$\vdots$	$\vdots$
n	$p^n + 3p^{n-1} + \cdots + T_{n-1}p + T_n \cdot 1$	$(T_1)\cdots(T_n)$

Table 9. Numeral system of base p for the values of $\zeta_1 * \zeta * \zeta * \zeta(p^n)$

n	$\zeta_1 * \zeta * \zeta * \zeta * \zeta(p^n)$	base p
0	1	(1)
1	$p + 4$	(1)(4)
2	$p^2 + 4p + 10$	(1)(4)(10)
3	$p^3 + 4p^2 + 10p + 20$	(1)(4)(10)(20)
$\vdots$	$\vdots$	$\vdots$
n	$p^n + P_2p^{n-1} + \cdots + P_{n-1}p + P_n \cdot 1$	$(P_1)\cdots(P_n)$

Table 10. Numeral system of base p for the values of $\zeta_1 * \zeta * \zeta * \zeta * \zeta(p^n)$

See Table 11 and Table 12 for the values of the ζ-function for triangular number and 3-gonal Pyramid number.

n	$\zeta * \zeta * \zeta(p^n)$	base 2
0	1	(1)
1	$1+2$	(1)(1)
2	$1+2+3$	(1)(1)(0)
3	$1+2+3+4$	(1)(0)(1)(0)
4	$1+2+3+4+5$	(1)(1)(1)(1)
$\vdots$	$\vdots$	$\vdots$
n	$1+\cdots+n+(n+1)$ $=(n+1)(n+2)/2$ T_{n+1} : triangular number	

Table 11. Values of ζ_0 and triangular number

n	$\zeta * \zeta * \zeta * \zeta(p^n)$	base 2
0	1	(1)
1	$2\cdot 1+1\cdot 2$	(1)(0)(0)
2	$3\cdot 1+2\cdot 2+1\cdot 3$	(1)(0)(1)(0)
3	$4\cdot 1+3\cdot 2+2\cdot 3+1\cdot 4$	(1)(0)(1)(0)(0)
4	$5\cdot 1+4\cdot 2+3\cdot 3+2\cdot 4+1\cdot 5$	(1)(0)(0)(0)(1)(1)
$\vdots$	$\vdots$	$\vdots$
n	$(n+1)\cdot 1+n\cdot 2+\cdots+1\cdot(n+1)$ $=(n+1)(n+2)(n+3)/6$ P^3_{n+1} 3-gonal Pyramid number	

Table 12. Values of ζ_0 and 3-gonal Pyramid number

If p is a prime, then we obtain the following results:

$$\mathfrak{z}_0(1;x)=\Delta(1;x)=x+x^2+x^3+\cdots=\frac{x}{1-x},$$

$$\mathfrak{z}_0(p;x):=x+2x^2+3x^3+\cdots=\frac{x}{(1-x)^2},$$

$$\mathfrak{z}_0(p^2;x):=x+(1+2)x^2+(1+2+3)x^3+\cdots=\sum_{n=1}T_nx^n,$$

$$\mathfrak{z}_0(p^3;x):=\sum_{n=1}P^3_nx^n$$

and

$$\Delta(p;x)=\Delta(p^2;x)=\Delta(p^3;x)=0,$$

where

$$T_n=1+2+\cdots+n$$

is n-th triangular number and

$$P^3_n=n\cdot 1+2\cdot(n-1)+\cdots+1\cdot n$$

is n-th 3-gonal Pyramid number. For the Pyramid number and the triangular number (*cf.* [13, 14, 26]).

To understand these type arithmetic functions, it is very important to understand the following generating functions, so we state the following:

Generating function for the σ_k is given by the following the Lambert series

$$F(y;k) = \sum_{n=1}^{\infty} \frac{n^k y^n}{1-y^n}.$$

That is, in [1],

$$F(y;k) = \sum_{m=1}^{\infty} \sigma_k(m) y^m \tag{3.1}$$

(*cf.* [1, 17, 23]). Taking derivative of $F(y,k)$ with respect to y, we get

$$\frac{d}{dy} F(y,k) = \sum_{m=1}^{\infty} m\sigma_k(m) y^{m-1}$$

and

$$\begin{aligned} \frac{d}{dy}\{F(y,k)y\} &= \sum_{m=1}^{\infty} (m+1)\sigma_k(m) y^m \\ &= \sum_{m=1}^{\infty} m\sigma_k(m) y^m + F(m,k). \end{aligned}$$

By applying the Euler operator $\varphi = y\frac{d}{dy}$ to the above equation, we get the following partial differential equation:

Theorem 3.1. *For $F(y;k)$ as in Eqn.* (3.1)*, we have*

$$\frac{d}{dy}\{F(y;k)y\} - F(y;k) = \varphi\{F(y,k)\}.$$

Generating function for $d(m)$ is given by

$$\sum_{m=1}^{\infty} d(m) q^m = \sum_{m=1}^{\infty} \frac{q^m}{1-q^m}, \tag{3.2}$$

where $|q| < 1$. Here we assume that $d(m) = 0$ for $m \leq 0$, see [16]. Since the Lambert series $\sum_{m=1}^{\infty} \frac{q^m}{1-q^m}$ are related to infinite and finite products, and also to the partition function, we also have the following results:

$$\sum_{m=1}^{\infty} d(m) q^m = \frac{1}{(q;q)_\infty} \sum_{m=1}^{\infty} \frac{(-1)^{m-1} m q^{\frac{m(m+1)}{2}}}{(q;q)_m}, \tag{3.3}$$

where

$$(b;q)_n = \prod_{j=0}^{n-1} (1 - by^j),\ (b;y)_0 = 1,\ |y| < 1.$$

The Eq. (3.3) can also be deduced with the aid of Theorem 1 in [16].

Next we deal with the finite sums involving $d(m)$: For $0 < q < 1$, in [16], Merca showed that

$$\sum_{v=1}^{m} d(v) q^v < \sum_{v=1}^{m} \frac{q^v}{1-q^v}.$$

By using geometric series in the above equation, we get

Theorem 3.2. *For the divisor function, we have*

$$\sum_{v=1}^{m} d(v) q^v < \sum_{v=1}^{m} q^v \sum_{l=0}^{\infty} q^{lv},$$

where $|q| < 1$.

In [16], Merca gave the following result for $d(m)$:

$$\sum_{v=1}^{m} d(v)q^{v} < \frac{1}{(q;q)_m}\sum_{v=1}^{m}(-1)^{v+1}vq^{\frac{v(v+1)}{2}}\begin{bmatrix} m \\ v \end{bmatrix}, \tag{3.4}$$

where

$$\begin{bmatrix} m \\ v \end{bmatrix} = \begin{cases} \frac{(q;q)_m}{(q;q)_v(q;q)_{m-v}}, & v \in \{0,1,2,\ldots,m\} \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$q^{v} = \sum_{j=0}^{v} S_2(v,j)q(q-1)(q-2)\cdots(q-j+1), \tag{3.5}$$

where

$$x(x-1)\cdots(x-j+1) = x^{(j)},\ j \neq 0,\ x \neq 0,\ x^{(0)} = 1.$$

Here, $x^{(j)}$ denotes the falling factorial function (*cf.* [15, 25, 24, 27]). Thus we get the following theorem:

Theorem 3.3. *The divisor function satisfies the following inequality:*

$$\sum_{v=1}^{m}\sum_{k=0}^{v} d(v)S_2(v,k)q^{(k)} < \frac{1}{(q;q)_m}\sum_{v=1}^{m}(-1)^{v+1}v\begin{bmatrix} m \\ v \end{bmatrix}\sum_{k=0}^{(v+1)v} S_2(v^2+v,k)q^{(k)}.$$

The well-known Lambert series generating function for arithmetic function f is given by

$$\sum_{n=1}^{\infty}\frac{f(n)q^n}{1+q^n} = \sum_{n=1}^{\infty}\frac{f(n)q^n}{1-q^n} - 2\sum_{n=1}^{\infty}\frac{f(n)q^{2n}}{1-q^{2n}} \tag{3.6}$$

(*cf.* [1, 22, 27]). Substituting $f(n) = n$ into Eq. (3.6), we get

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{nq^n}{1+q^n} &= \sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} - 2\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}} \\ &= F(q;1) - F(q^2;1).\end{aligned}$$

Here, $|q| < 1$. Combining the above function with Eq. (3.1), we get

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{nq^n}{1+q^n} &= \sum_{n=1}^{\infty} d(n)q^n - 2\sum_{n=1}^{\infty} d(n)q^{2n} \\ &= \sum_{n=1}^{\infty} d(n)q^n - 2\sum_{n=1}^{\infty} d(n)\sum_{m,m=2n} q^m.\end{aligned}$$

Thus the series

$$G(q) = \sum_{n=1}^{\infty}\frac{nq^n}{1+q^n}$$

is also a generating function for $d(n)$.

4. Conclusion

In this work, we study the connected, unicyclic and semiregular graph realizations of a given degree sequence. We do classify them into two classes according to the primeness of the number of pendant vertices. We obtain all possible connected, unicyclic and semiregular degree sequences and their realizations as a graph. They correspond to some specific class of graphs which we denote by $U_{k,l}$. As the divisor function is an essential part of this study, we also gave two different alternative generating functions for this function.

Author Contributions: The authors contributed equally to this work.

Conflict of Interest: The authors report there are no competing interests to declare.

Funding (Financial Disclosure): There is no funding for the authors.

References

[1] T. M. Apostol, *Introduction to analytic number theory*, Springer, New York, Berlin, 1976.

[2] M. Ascioglu, M. Demirci and I. N. Cangul, *Omega invariant of union, join and corona product of two graphs*, Adv. Stud. Contemp. Math. **30 (3)**, 297–306, 2020.

[3] S. Delen and I. N. Cangul, *A new graph invariant*, Turk. J. Anal. Number Theory **6 (1)**, 30–33, 2018.

[4] S. Delen, A. Yurttas, M. Togan and I. N. Cangul, *Omega invariant of graphs and cyclicness*, Appl. Sci. **21**, 91–95, 2019.

[5] S. Delen, M. Demirci, A. S. Cevik and I. N. Cangul, *On omega index and average degree of graphs*, J. Math. **2021**, 1–6, 2021; Article ID: 5565246.

[6] S. Delen, M. Togan, A. Yurttas, U. Ana et al., *The effect of edge and vertex deletion on omega invariant*, Appl. Anal. Discrete Math. **14 (3)**, 685–696, 2020.

[7] M. Demirci and I. N. Cangul, *Tribonacci graphs*, ITM Web Conf. **34**, 1–9, 2020; Article ID: 01002.

[8] M. Demirci, S. Delen, A. S. Cevik and I. N. Cangul, *Omega index of line and total graphs*, J. Math. **2021**, 1–6, 2021; Article ID: 5552202.

[9] M. Demirci, A. Ozbek, O. Akbayrak and I. N. Cangul, *Lucas graphs*, J. Appl. Math. Comput. **65 (1-2)**, 93–106, 2021.

[10] M. Demirci, A. Yurttas Gunes, S. Delen and I. N. Cangul, *The effect of omega invariant on some topological graph indices*, Creat. Math. Inform. **30 (2)**, 175–180, 2021.

[11] I. Gutman, M. Togan, A. Yurttas, A. S. Cevik et al., *Inverse problem for sigma index*, MATCH Commun. Math. Comput. Chem. **79 (2)**, 491–508, 2018.

[12] M. I. Huilgol, G. D. D'Souza and I. N. Cangul, *Omega indices of strong and lexicographic products of graphs*, Curr. Org. Synth. **22 (2)**, 143–158, 2025.

[13] N. Y. Ikikardes and D. Kim, *A study of restricted divisor functions with coprime conditions*, Montes Taurus J. Pure Appl. Math. **6 (3)**, 284–298, 2024.

[14] S. Y. Kong and D. Kim, *Inverse divisor functions using dirichlet convolution*, Montes Taurus J. Pure Appl. Math. **6 (3)**, 169–185, 2024.

[15] I. Kucukoglu and Y. Simsek, *Unified presentations of the generating functions for a comprehensive class of numbers and polynomials*, Montes Taurus J. Pure Appl. Math. **6 (1)**, 40–63, 2024; Article ID: MTJPAM-D-24-00033.

[16] M. Merca, *A new look on the generating function for the divisors*, J. Number Theory **149**, 57–69, 2015.

[17] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark, *Nist handbook of mathematical functions*, Cambridge Univ. Press, Cambridge, 2010.

[18] M. S. Oz and I. N. Cangul, *Bounds for matching number of fundamental realizations according to new graph invariant omega*, Proc. Jangjeon Math. Soc. **23 (1)**, 23–37, 2020.

[19] M. S. Oz and I. N. Cangul, *A survey of the maximal and the minimal nullity in terms of omega invariant on graphs*, Acta Univ. Sapientiae Math. **15 (2)**, 337–353, 2023.

[20] H. Ozden, F. E. Zihni, F. O. Erdogan, I. N. Cangul et al., *Independence number of graphs and line graphs of trees by means of omega invariant*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. **114**, 1–14, 2020.

[21] U. Sanli, F. Celik, S. Delen and I. N. Cangul, *Connectedness criteria for graphs by means of omega invariant*, Filomat **34 (2)**, 647–652, 2020.

[22] M. D. Schmidt, *Factorization theorems for hadamard products and higher-order derivatives of the lambert series generating function*, 2017; https://doi.org/10.48550/arXiv.1712.00608.

[23] M. D. Schmidt, *Combinatorial sums and identities involving generalized sum-of-divisors functions with bounded divisors*, 2020; ArXiv: 1704.05595.

[24] Y. Simsek, *New families of special numbers of computing negative order euler numbers and related numbers and polynomials*, Appl. Anal. Discrete Math. **12 (1)**, 1–36, 2018.

[25] Y. Simsek, *Explicit formulas for p-adic integrals: approach to p-adic distributions and some families of special numbers and polynomials*, Montes Taurus J. Pure Appl. Math. **1 (1)**, 1–76, 2019.

[26] Y. Simsek, *Construction of general forms of ordinary generating functions for more families of numbers and multiple variables polynomials*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. **117**, 1–26, 2023.

[27] Y. Simsek, *Families of generating functions and their applications with some open questions*, In: Convex and Variational Analysis with Applications: In Honor of Terry Rockafellar's 90th Birthday (Ed. by T. M. Rassias and P. M. Pardalos), Springer Optim. Appl., Springer Nature Switzerland, 2026.

[28] M. Togan, A. Yurttas Gunes, S. Delen and I. N. Cangul, *Omega invariant of the line graphs of unicyclic graphs*, Montes Taurus J. Pure Appl. Math. **2** (**2**), 45–48, 2020.
[29] M. Togan, A. Yurttas, U. Sanli, F. Celik et al., *Inverse problem for bell index*, Filomat **34** (**2**), 615–621, 2020.
[30] A. Yurttas Gunes, *Omega invariant of complement graphs and nordhaus-gaddum type results*, Curr. Org. Synth. **21** (**3**), 298–302, 2024.
[31] A. Yurttas Gunes, *New relations between zagreb indices and omega invariant*, Curr. Org. Synth. **21** (**3**), 257–262, 2024.
[32] A. Yurttas Gunes, S. Delen, M. Demirci, A. S. Cevik et al., *Fibonacci graphs*, Symmetry **12**, 1383, 2020.
[33] A. Yurttas, M. Togan, S. Delen and I. N. Cangul, *Omega invariant and its applications in graph theory*, In: Proceedings Book of the Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2018) (Ed. by Y. Simsek), Antalya, Turkey, October 26–29, 2018, pp. 86–90; ISBN: 978-86-6016-036-4.
[34] A. Yurttas, M. Togan, V. Lokesha, I. N. Cangul et al., *Inverse problem for zagreb indices*, J. Math. Chem. **57**, 609–615, 2019.

How to cite this article: H. O. Ayna, A. Y. Gunes, I. N. Cangul, D. Kim and Y. Simsek, *On connected unicyclic semiregular graphs: Approach to generating functions*, Montes Taurus J. Pure Appl. Math. **8** (**2**), 27–38, 2026; Article ID: MTJPAM-D-26-00020.