Montes Taurus J. Pure Appl. Math. 8 (2), 39–50, 2026

Montes Taurus Journal of Pure and Applied Mathematics
https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

Invariant submanifolds of a (κ, μ, ν)-paracontact metric manifold admitting Zamkovoy connection

Maruru Shrinivasamurthy Lakshmi [a], Hallamanavar Gangadharappa Nagaraja [b]

[a]Department of Mathematics, Jnana Bharathi Campus, Bangalore University, Bengaluru 560056, India
[b]Department of Mathematics, Jnana Bharathi Campus, Bangalore University, Bengaluru 560056, India

Abstract

This paper aims to investigate characteristics of invariant submanifolds within a (κ, μ, ν)-paracontact metric manifold with a Zamkovoy connection. We provide characterizations of such submanifolds, subject to specific curvature conditions associated with the Zamkovoy connection. Additionally, we establish necessary and sufficient criteria for such submanifolds to be totally geodesic, based on the characteristics of the functions κ, μ and ν.

Keywords: (κ, μ, ν)-paracontact metric manifold, invariant submanifold, totally geodesic, Zamkovoy connection

2020 MSC: 53C05, 53C15, 53C17, 53D10

1. Introduction

The geometry of submanifolds has emerged as a prominent area of study in recent mathematical research, primarily due to its broad applicability in fields such as applied mathematics and theoretical physics. In particular, invariant submanifolds serve as a valuable framework for examining the dynamics of nonlinear autonomous systems (*cf.* [8]). It is generally observed that these submanifolds inherit many of the geometric properties of the ambient manifold, making them essential in understanding the overall structure of the space.

The (κ, μ)-nullity distribution serves as a generalization of the κ-nullity distribution in the setting of paracontact metric manifolds $(\tilde{M}, \phi, \xi, \eta, g)$. This concept was introduced and developed in the works of Montano and Terlizzi [13], and further examined by Montano et al. [14]. For each point $p \in \tilde{M}$ and real constants $\kappa, \mu \in \mathbb{R}$, the distribution is defined as follows:

$$N_p(\kappa, \mu) = \{A_3 \in T_p\tilde{M} : \tilde{R}(A_1, A_2)A_3 = \kappa[g(A_2, A_3)A_1 - g(A_1, A_3)A_2] + \mu[g(A_2, A_3)hA_1 - g(A_1, A_3)hA_2]\}, \tag{1.1}$$

where $h = \frac{1}{2}\pounds_\xi\phi$ and $\pounds$ denotes the Lie differentiation. If the functions κ and μ are smooth and non-constant on the manifold $\tilde{M}$, then $\tilde{M}$ is termed a generalized (κ, μ)-paracontact metric manifold. This concept was first introduced and examined by Küpeli Erken and Murathan [12], and further studied by Küpeli Erken [10]. Building on this framework,

†*Article ID:* MTJPAM-D-24-00076

Email addresses: lakshmims1369@gmail.com (Maruru Shrinivasamurthy Lakshmi), hgnraj@yahoo.com (Hallamanavar Gangadharappa Nagaraja)

Received:7 June 2024, Accepted:15 December 2025, Published:4 July 2026

⋆*Corresponding Author:* Maruru Shrinivasamurthy Lakshmi

they later extended the idea to define a (κ,μ,ν)-paracontact metric manifold as a further generalization of the (κ,μ)-space. The corresponding Riemannian curvature tensor for this structure is given by (*cf.* [12]):

$$\tilde{R}(A_1,A_2)\xi = \kappa\left[\eta(A_2)A_1 - \eta(A_1)A_2\right] + \mu\left[\eta(A_2)hA_1 - \eta(A_1)hA_2\right] + \nu\left[\eta(A_2)\phi hA_1 - \eta(A_1)\phi hA_2\right], \tag{1.2}$$

for all $A_1, A_2 \in \Gamma(T\tilde{M})$ and κ,μ,ν are smooth functions on $\tilde{M}$.

The study of invariant submanifolds in various geometric structures has attracted considerable attention over the years. The foundational work in this direction was initiated by Bejancu and Papaghiuc [5], who were among the first to explore their geometric properties. Subsequently, De and Majhi [7] examined conditions under which an invariant submanifold of a Kenmotsu manifold becomes totally geodesic, while Hu and Wang [9] extended such investigations to trans-Sasakian manifolds. In the context of paracontact metric manifolds with parameters $(\tilde{\kappa} \neq -1, \tilde{\mu})$, Küpeli Erken [11] carried out a detailed study of invariant submanifolds. Further, Anitha and Bagewadi [1] explored invariant submanifolds in Kenmotsu manifolds that admit a quarter-symmetric metric connection. Numerous other contributions in this area include the works of (*cf.* [2]-[4], [6, 15, 17, 18]), which have enriched the understanding of geometric conditions and curvature properties related to such submanifolds.

Inspired by prior investigations, we examine several curvature-related identities involving the second fundamental form σ of an invariant submanifold N within a (κ,μ,ν)-paracontact metric manifold endowed with the Zamkovoy connection. In particular, we consider the conditions:

$$P(A_1,A_2)\cdot\sigma = F_1 Q(g,\sigma), \quad P(A_1,A_2)\cdot\sigma = F_2 Q(S,\sigma), \quad C(A_1,A_2)\cdot\tilde{\nabla}\sigma = F_3 Q(g,\tilde{\nabla}\sigma), \quad C(A_1,A_2)\cdot\tilde{\nabla}\sigma = F_4 Q(S,\tilde{\nabla}\sigma),$$

as well as the relations $Q(g, W_0^*\cdot\sigma) = 0$ and $Q(S, W_0^*\cdot\sigma) = 0$. Here, the symbol '$(\cdot)$' denotes the action of the curvature tensor on the corresponding tensor field. Under these assumptions, we prove that either the submanifold N is totally geodesic, or certain constraints must hold among the functions F_1, F_2, F_3, F_4 and the structure functions κ,μ,ν.

1.1. Preliminaries

A $(2m+1)$-dimensional smooth manifold $\tilde{M}$ is said to possess an almost paracontact metric structure if it admits a quadruple (ϕ,ξ,η,g), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form, and g is a pseudo-Riemannian metric, satisfying the following relations:

$$\phi^2 = I - \eta\otimes\xi, \quad \phi(\xi) = 0, \quad \eta(\xi) = 1, \quad \eta\circ\phi = 0 \tag{1.3}$$

and

$$g(\phi A_1, \phi A_2) = -g(A_1,A_2) + \eta(A_1)\eta(A_2), \tag{1.4}$$

for arbitrary vector fields $A_1, A_2 \in \Gamma(T\tilde{M})$. The corresponding fundamental 2-form Φ associated with this structure is expressed as: $\Phi(A_1,A_2) = g(A_1,\phi A_2)$. If the differential of η coincides with Φ, i.e., $d\eta = \Phi$, then the structure is said to define a paracontact metric manifold.

On a paracontact metric manifold $\tilde{M}$, the symmetric endomorphism h of type (1,1) satisfies certain structural identities (*cf.* [20]):

$$h\xi = 0, \quad h\phi = -\phi h, \quad tr\,h = tr\phi h = 0, \tag{1.5}$$

$$\tilde{\nabla}_{A_1}\xi = -\phi A_1 + \phi hA_1. \tag{1.6}$$

Following the framework established in [12], a paracontact metric manifold $\tilde{M}$ of dimension $(2m+1)$ is termed a (κ,μ,ν)-paracontact metric manifold if the curvature tensor $\tilde{R}$ satisfies a certain identity characterized by smooth functions $\kappa,\mu,\nu \in C^\infty(\tilde{M})$.

$$\begin{aligned}\tilde{R}(A_1, A_2)A_3 \;=\;& \kappa\left[g(A_2, A_3)A_1 - g(A_1, A_3)A_2\right] + \mu\left[g(A_2,A_3)hA_1 - g(A_1, A_3)hA_2\right]\\ &+\nu\left[g(A_2, A_3)\phi hA_1 - g(A_1, A_3)\phi hA_2\right],\end{aligned} \tag{1.7}$$

for all $A_1, A_2, A_3 \in \Gamma(T\tilde{M})$.

In such a manifold, the following identities hold (*cf.* [12]):

$$(\tilde{\nabla}_{A_1}\phi)A_2 = -g(A_1 - hA_1, A_2)\xi + \eta(A_2)(A_1 - hA_1), \tag{1.8}$$

$$h^2 = (k+1)\phi^2, \quad \kappa \neq -1, \tag{1.9}$$

$$(\tilde{\nabla}_{A_1}\eta)A_2 = -g(A_1 - hA_1, \phi A_2), \tag{1.10}$$

$$\tilde{R}(A_1, A_2)\xi = \kappa[\eta(A_2)A_1 - \eta(A_1)A_2] + \mu[\eta(A_2)hA_1 - \eta(A_1)hA_2] + \nu[\eta(A_2)\phi hA_1 - \eta(A_1)\phi hA_2], \tag{1.11}$$

$$\begin{aligned}(\tilde{\nabla}_{A_1}h)A_2 - (\tilde{\nabla}_{A_2}h)A_1) \quad = \quad & -(1+k)(2g(A_1, \phi A_2)\xi + \eta(A_1)\phi A_2 - \eta(A_2)\phi A_1) \\ & +(1-\mu)(\eta(A_1)\phi hA_2 - \eta(A_2)\phi hA_1) - \nu(\eta(A_1)hA_2 - \eta(A_2)hA_1).\end{aligned} \tag{1.12}$$

The pseudo-projective, quasi-conformal, and W_0^* curvature tensors for $(2n+1)$-dimensional Riemannian submanifolds have been formulated in the literature [16] as follows:

$$\begin{aligned}P(A_1, A_2)A_3 \quad = \quad & a\tilde{R}(A_1, A_2)A_3 + b\,[S(A_2, A_3)A_1 - S(A_1, A_3)A_2] \\ & -\frac{r}{2n+1}\left(\frac{a}{2n} + b\right)[g(A_2, A_3)A_1 - g(A_1, A_3)A_2],\end{aligned} \tag{1.13}$$

$$\begin{aligned}C(A_1, A_2)A_3 \quad = \quad & a\tilde{R}(A_1, A_2)A_3 + b[S(A_2, A_3)A_1 - S(A_1, A_3)A_2 + g(A_2, A_3)QA_1 \\ & -g(A_1, A_3)QA_2] - \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)[g(A_2, A_3)A_1 - g(A_1, A_3)A_2]\end{aligned} \tag{1.14}$$

and

$$W_0^*(A_1, A_2)A_3 = \tilde{R}(A_1, A_2)A_3 + \frac{1}{2n}\,[S(A_2, A_3)A_1 - S(A_1, A_3)A_2], \tag{1.15}$$

where r is the scalar curvature of N and a and b are non-zero constants.

Consider a Riemannian submanifold N of dimension $(2n+1)$, embedded in a Riemannian manifold $\tilde{M}$ of dimension $(2m+1)$. The corresponding Gauss and Weingarten equations take the following form:

$$\tilde{\nabla}_{A_1}A_2 = \nabla_{A_1}A_2 + \sigma(A_1, A_2) \tag{1.16}$$

and

$$\tilde{\nabla}_{A_1}A_5 = -\mathcal{F}_{A_5}A_1 + \nabla^{\perp}_{A_1}A_5, \tag{1.17}$$

for any $A_1, A_2 \in \Gamma(TN)$ and $A_5 \in \Gamma(TN^{\perp})$, where σ represents the second fundamental form, the symbols $\tilde{\nabla}$, ∇, and $\nabla^{\perp}$ denote the Levi-Civita connection on $\tilde{M}$, the induced connection on N, and the normal connection on the normal bundle $T^{\perp}N$, respectively. The operator $\mathcal{F}$ refers to the shape operator.

The relationship between the second fundamental form and the shape operator is given by:

$$g(\mathcal{F}_{A_5}A_1, A_2) = g(\sigma(A_1, A_2), A_5). \tag{1.18}$$

Here, g denotes the Riemannian metric on the ambient manifold $\tilde{M}$, which also induces the metric on the submanifold N.

The first-order covariant derivative of the second fundamental form σ is given by:

$$(\tilde{\nabla}_{A_1}\sigma)(A_2, A_3) = \nabla^{\perp}_{A_1}\sigma(A_2, A_3) - \sigma(\nabla_{A_1}A_2, A_3) - \sigma(A_2, \nabla_{A_1}A_3), \tag{1.19}$$

for any A_1, A_2, $A_3 \in \Gamma(TN)$.

Using the Gauss and Weingarten formulas, one can derive the following expression (*cf.* [7]):

$$\tilde{R}(A_1, A_2)A_3 = R(A_1, A_2)A_3 + (\tilde{\nabla}_{A_1}\sigma)(A_2, A_3) - (\tilde{\nabla}_{A_2}\sigma)(A_1, A_3) + \mathcal{F}_{\sigma(A_1,A_3)}A_2 - \mathcal{F}_{\sigma(A_2,A_3)}A_1, \tag{1.20}$$

for any $A_1, A_2, A_3 \in \Gamma(TN)$, and R denote the Riemannian curvature tensor associated with the submanifold N.

The expressions for $\tilde{R} \cdot \sigma$ and $\tilde{R} \cdot \tilde{\nabla}\sigma$ are given by:

$$(\tilde{R}(A_1, A_2).\sigma)(A_4, A_5) = R^{\perp}(A_1, A_2)\sigma(A_4, A_5) - \sigma(R(A_1, A_2)A_4, A_5) - \sigma(A_4, R(A_1, A_2)A_5) \tag{1.21}$$

and

$$(\tilde{R}(A_1,A_2).\tilde{\nabla}\sigma)(A_3,A_4,A_5) = R^{\perp}(A_1,A_2)(\tilde{\nabla}\sigma)(A_3,A_4,A_5) - (\tilde{\nabla}\sigma)(R(A_1,A_2)A_3,A_4,A_5) - (\tilde{\nabla}\sigma)(A_3,R(A_1,A_2)A_4,A_5) - (\tilde{\nabla}\sigma)(A_3,A_4,R(A_1,A_2)A_5), \tag{1.22}$$

for all vector fields $A_1,A_2,A_3,A_4,A_5 \in \Gamma(TN)$ and $(\tilde{\nabla}\sigma)(A_3,A_4,A_5) = (\tilde{\nabla}_{A_3}\sigma)(A_4,A_5)$.

In a similar manner, the tensors $P\cdot\sigma$ and $C\cdot\tilde{\nabla}\sigma$ are defined as follows:

$$(P(A_1,A_2).\sigma)(A_4,A_5) = R^{\perp}(A_1,A_2)\sigma(A_4,A_5) - \sigma(P(A_1,A_2)A_4,A_5) - \sigma(A_4,P(A_1,A_2)A_5) \tag{1.23}$$

and

$$(C(A_1,A_2).\tilde{\nabla}\sigma)(A_3,A_4,A_5) = R^{\perp}(A_1,A_2)(\tilde{\nabla}\sigma)(A_3,A_4,A_5) - (\tilde{\nabla}\sigma)(C(A_1,A_2)A_3,A_4,A_5) - (\tilde{\nabla}\sigma)(A_3,C(A_1,A_2)A_4,A_5) - (\tilde{\nabla}\sigma)(A_3,A_4,C(A_1,A_2)A_5). \tag{1.24}$$

A submanifold is called totally geodesic if and only if its second fundamental form σ vanishes identically.

Let T be a tensor field of type $(0,k)$, with $k \geq 1$, and D a symmetric $(0,2)$-type tensor field on a submanifold N of a Riemannian manifold $(\tilde{M},g)$. Then, the $Q(D,T)$ tensor field is defined as [19]:

$$Q(D,T)(A_{11},A_{12},\ldots,A_{1K};A_1,A_2) = -T((A_1\Lambda_D A_2)A_{11},A_{12},\ldots,A_{1K}) - T(A_{11},(A_1\Lambda_D A_2)A_{12},\ldots,A_{1K}) \cdots - T(A_{11},A_{12},\ldots,(A_1\Lambda_D A_2)A_{1K}), \tag{1.25}$$

for all $A_{11},A_{12},\ldots,A_{1K},A_1,A_2 \in \Gamma(TN)$, where $(A_1\Lambda_D A_2)A_3$ is defined by

$$(A_1\Lambda_D A_2)A_3 = D(A_2,A_3)A_1 - D(A_1,A_3)A_2. \tag{1.26}$$

2. Invariant submanifold of a (κ,μ,ν)-paracontact metric manifold admitting Zamkovoy connection

Now, let $\tilde{M}$ be a (κ,μ,ν)-paracontact metric manifold and N be a submanifold immersed in $\tilde{M}$. If, at every point $x_1 \in N$, the image of the tangent space $T_{x_1}N$ under the structure tensor ϕ lies entirely within $T_{x_1}N$, i.e., $\phi(T_{x_1}N) \subseteq T_{x_1}N$, then N is referred to as an invariant submanifold of $\tilde{M}$ with respect to ϕ.

Zamkovoy introduced a special affine connection, known as the Zamkovoy connection, which is defined as follows (*cf.* [20]):

$$\nabla^*_{A_1}A_2 = \tilde{\nabla}_{A_1}A_2 + (\tilde{\nabla}_{A_1}\eta)(A_2)\xi - \eta(A_2)\tilde{\nabla}_{A_1}\xi + \eta(A_1)\phi A_2, \tag{2.1}$$

for all A_1, $A_2 \in \Gamma(TN)$, where ∇^* and $\tilde{\nabla}$ denote the Zamkovoy connection and Levi-Civita connection on $\tilde{M}$, respectively.

Now, making use of (1.6) and (1.10), the above equality takes the form,

$$\nabla^*_{A_1}A_2 = \tilde{\nabla}_{A_1}A_2 - g(A_1 - hA_1,\phi A_2)\xi + \eta(A_2)\phi A_1 - \eta(A_2)\phi hA_1 + \eta(A_1)\phi A_2. \tag{2.2}$$

Here, taking $A_2 = \xi$ in (2.1) and using (1.3) and (1.6), we get

$$\nabla^*_{A_1}\xi = 0. \tag{2.3}$$

In view of (1.8) and (2.2), we have

$$(\nabla^*_{A_1}\phi)A_2 = -2g(A_1 - hA_1,A_2)\xi + 2\eta(A_1)\eta(A_2)\xi. \tag{2.4}$$

We know that,

$$(\nabla^*_{A_1}\eta)A_2 = \nabla^*_{A_1}\eta(A_2) - \eta(\nabla^*_{A_1}A_2). \tag{2.5}$$

By employing (2.2) in preceding equation, we derive at

$$(\nabla^*_{A_1}\eta)A_2 = 0.$$

Similarly, we calculate $(\nabla^*_{A_1}h)A_2 = \nabla^*_{A_1}hA_2 - h(\nabla^*_{A_1}A_2)$ by using (1.9), (1.12) and (2.2), as

$$\begin{aligned}(\nabla^*_{A_1}h)A_2 - (\nabla^*_{A_2}h)A_1 &= -2(1+k)g(A_1,\phi A_2)\xi + (2-\mu)(\eta(A_1)\phi hA_2 - \eta(A_2)\phi hA_1) - \nu(\eta(A_1)hA_2 - \eta(A_2)hA_1)\\ &\quad -g(A_1 - hA_1, \phi hA_2)\xi + g(A_2 - hA_2, \phi hA_1)\xi. \end{aligned} \tag{2.6}$$

Let $\tilde{R}$ and R^* be the curvature tensors associated with the connections $\tilde{\nabla}$ and ∇^*, respectively. Based on their respective definitions,

$$R^*(A_1,A_2)A_3 = \nabla^*_{A_1}\nabla^*_{A_2}A_3 - \nabla^*_{A_2}\nabla^*_{A_1}A_3 - \nabla^*_{[A_1,A_2]}A_3. \tag{2.7}$$

Substituting equation (2.2) in equation (2.7), the expression for the Riemannian curvature tensor R^* corresponding to the Zamkovoy connection is obtained as

$$\begin{aligned}R^*(A_1,A_2)A_3 &= \tilde{R}(A_1,A_2)A_3 + \kappa[\eta(A_2)g(A_1,A_3)\xi - \eta(A_1)g(A_2,A_3)\xi + \eta(A_1)\eta(A_3)A_2\\ &\quad -\eta(A_2)\eta(A_3)A_1] - 2\eta(A_1)[g(A_2,A_3)\xi - g(hA_2,A_3)\xi]\\ &\quad +2\eta(A_2)\left[g(A_1,A_3)\xi - g(hX_1,A_3)\xi\right] - \mu[\eta(A_1)g(hA_2,A_3)\xi\\ &\quad -\eta(A_2)g(hA_1,A_3)\xi - \eta(A_1)\eta(A_3)hA_2 + \eta(A_2)\eta(A_3)hA_1]\\ &\quad -\nu[\eta(A_1)g(hA_2,\phi A_3)\xi - \eta(A_1)\eta(A_3)\phi hA_2 - \eta(A_2)g(hA_1,\phi A_3)\xi\\ &\quad +\eta(A_2)\eta(A_3)\phi hA_1] - g(A_1 - hA_1,\phi A_3)(\phi A_2 - \phi hA_2)\\ &\quad +g(A_2 - hA_2,\phi A_3)(\phi A_1 - \phi hA_1) - g(A_1 - hA_1,\phi A_2)\phi A_3 + g(A_2 - hA_2,\phi A_1)\phi A_3. \end{aligned} \tag{2.8}$$

By choosing $A_1 = \xi$, $A_2 = \xi$, and $A_3 = \xi$ in equation (2.8) and utilizing equations (1.3) and (1.5), we arrive at the following expressions:

$$R^*(A_1,A_2)\xi = 0,$$

$$R^*(\xi,A_2)A_3 = -2g(A_2,A_3)\xi + 2\eta(A_2)\eta(A_3)\xi + 2g(hA_2,A_3)\xi - \mu g(hA_2,A_3)\xi - \nu g(hA_2,\phi A_3)\xi$$

and

$$R^*(A_1,\xi)A_3 = 2g(A_1,A_3)\xi - 2\eta(A_1)\eta(A_3)\xi - 2g(hA_1,A_3)\xi + \mu g(hA_1,A_3)\xi + \nu g(hA_1,\phi A_3)\xi.$$

Taking $A_2 = \xi$ in (1.16), we get

$$\tilde{\nabla}_{A_1}\xi = \nabla_{A_1}\xi + \sigma(A_1,\xi).$$

Substituting (1.6) in above equality, we obtain

$$-\phi A_1 + \phi hA_1 = \nabla_{A_1}\xi + \sigma(A_1,\xi). \tag{2.9}$$

Now, equating the tangential and normal components of (2.9), we arrive

$$\nabla_{A_1}\xi = -\phi A_1 + \phi hA_1 \tag{2.10}$$

and

$$\sigma(A_1,\xi) = 0. \tag{2.11}$$

By virtue of (1.20) and (2.8), setting $A_3 = \xi$ and using (1.11) and (2.11), we derive

$$R(A_1,A_2)\xi = \kappa\left[\eta(A_2)A_1 - \eta(A_1)A_2\right] + \mu\left[\eta(A_2)hA_1 - \eta(A_1)hA_2\right] + \nu\left[\eta(A_2)\phi hA_1 - \eta(A_1)\phi hA_2\right]. \tag{2.12}$$

If we take $A_1 = \xi$ in (2.12), we obtain

$$R(\xi,A_2)\xi = \kappa[\eta(A_2)\xi - A_2] - \mu hA_2 - \nu\phi hA_2. \tag{2.13}$$

Taking the inner product with A_6 in (2.12), we obtain

$$\begin{aligned}g(R(A_1,A_2)\xi,A_6) &= \kappa\left[\eta(A_2)g(A_1,A_6) - \eta(A_1)g(A_2,A_6)\right] + \mu\left[\eta(A_2)g(hX_1,A_6) - \eta(A_1)g(hA_2,A_6)\right]\\ &\quad +\nu\left[\eta(A_2)g(\phi hA_1,A_6) - \eta(A_1)g(\phi hA_2,A_6)\right]. \end{aligned}$$

Now, we compute the Ricci tensor component $S(A_2,\xi)$, defined by:

$$S(A_2,\xi) = \sum_{i=1}^{2n+1} g(R(e_i,A_2)\xi,e_i),$$

where $\{e_i\}_{i=1}^{2n+1}$ is a local orthonormal frame with $e_{2n+1} = \xi$.

Substituting $A_1 = A_6 = e_i$ into the above inner product, we get:

$$\begin{aligned} g(R(e_i,A_2)\xi,e_i) &= \kappa\left[\eta(A_2)g(e_i,e_i) - \eta(e_i)g(A_2,e_i)\right] + \mu\left[\eta(A_2)g(he_i,e_i) - \eta(e_i)g(hA_2,e_i)\right] \\ &\quad +\nu\left[\eta(A_2)g(\phi he_i,e_i) - \eta(e_i)g(\phi hA_2,e_i)\right]. \end{aligned}$$

Now, summing over $i = 1$ to $(2n+1)$ and using (1.5), we have:

$$S(A_2,\xi) = 2n\kappa\eta(A_2). \tag{2.14}$$

From above, we get

$$Q\xi = 2n\kappa\xi. \tag{2.15}$$

To facilitate later computations, we consider the pseudo-projective, quasi-conformal, and W_0^* curvature tensors defined on a (κ,μ,ν)-paracontact metric manifold with respect to the Zamkovoy connection.

Taking into account (1.13), (1.14), (1.15), (2.13), (2.14) and (2.15) and setting $A_1 = A_3 = \xi$, we obtain

$$P(\xi,A_2)\xi = \left[\kappa(a+2nb) - \frac{r}{2n+1}\left(\frac{a}{2n}+b\right)\right]\left[\eta(A_2)\xi - A_2\right] - a\mu hA_2 - a\nu\phi hA_2, \tag{2.16}$$

$$C(\xi,A_2)\xi = \left[\kappa(a+4nb) - \frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]\left[\eta(A_2)\xi - A_2\right] - a\mu hA_2 - a\nu\phi hA_2 \tag{2.17}$$

and

$$W_0^*(\xi,A_2)\xi = 2\kappa\left[\eta(A_2)\xi - A_2\right] - \mu hA_2 - \nu\phi hA_2. \tag{2.18}$$

We now consider invariant submanifolds of a (κ,μ,ν)-paracontact metric manifold admitting Zamkovoy connection satisfying $P(A_1,A_2)\cdot\sigma = F_1Q(g,\sigma)$ and $P(A_1,A_2)\cdot\sigma = F_2Q(S,\sigma)$.

Theorem 2.1. *Let $\tilde{M}$ be a (κ,μ,ν)-paracontact metric manifold equipped with the Zamkovoy connection ∇^*, and let N be an invariant submanifold of $\tilde{M}$, that is, the structure vector field ξ is everywhere tangent to N. Then, the condition $P(A_1,A_2).\sigma = F_1Q(g,\sigma)$ holds on N if and only if either N is totally geodesic or the function F_1 satisfies*

$$F_1 = (a+2nb)\left(\kappa - \frac{r}{2n(2n+1)}\right) \pm a\sqrt{(\kappa+1)(\mu^2+\nu^2)}, \quad \mu\,\nu(\kappa+1) = 0.$$

Proof. Let N be an invariant submanifold of a (κ,μ,ν)-paracontact metric manifold $\tilde{M}$ admitting Zamkovoy connection. Then, there exists a smooth function F_1 such that

$$(P(A_1,A_2).\sigma)(A_4,A_5) = F_1Q(g,\sigma)(A_1,A_2;A_4,A_5), \tag{2.19}$$

for all $A_1,A_2,A_4,A_5 \in \Gamma(TN)$. Taking into account (1.23), (1.25) and (1.26) in above equality, we have

$$\begin{aligned} &R^{\perp}(A_1,A_2)\sigma(A_4,A_5) - \sigma(P(A_1,A_2)A_4,A_5) - \sigma(A_4,P(A_1,A_2)A_5) \\ &= F_1\{-g(A_2,A_4)\sigma(A_1,A_5) + g(A_1,A_4)\sigma(A_2,A_5) - g(A_2,A_5)\sigma(A_4,A_1) + g(A_1,A_5)\sigma(A_4,A_2)\}. \end{aligned} \tag{2.20}$$

Setting $A_1 = A_4 = \xi$ in (2.20) and utilizing (2.11), we derive

$$\sigma(P(\xi,A_2)\xi,A_5) = -F_1\sigma(A_2,A_5). \tag{2.21}$$

Now substituting (2.16) in last equality, we derive

$$\left\{F_1 - \left[\kappa(a+2nb) - \frac{r}{2n+1}\left(\frac{a}{2n}+b\right)\right]\right\}\sigma(A_2,A_5) - a(\mu\sigma(hA_2,A_5) + \nu\phi\sigma(hA_2,A_5)) = 0. \tag{2.22}$$

Replacing A_2 by hA_2 in (2.22) and in view of (1.9), we get

$$\left\{F_1 - \left[\kappa(a+2nb) - \frac{r}{2n+1}\left(\frac{a}{2n}+b\right)\right]\right\}\sigma(hA_2, A_5) - a(\kappa+1)(\mu\sigma(A_2, A_5) + \nu\phi\sigma(A_2, A_5)) = 0. \tag{2.23}$$

From (2.22) and (2.23), we arrive at

$$\left[\left\{F_1 - \left[\kappa(a+2nb) - \frac{r}{2n+1}\left(\frac{a}{2n}+b\right)\right]\right\}^2 - a^2(\kappa+1)(\mu^2+\nu^2)\right]\sigma(A_2, A_5) - 2a^2(\kappa+1)\mu\nu\phi\sigma(A_2, A_5) = 0. \tag{2.24}$$

Since the vectors $\sigma(A_2, A_5)$ and $\phi\sigma(A_2, A_5)$ are orthogonal, we conclude that N is totally geodesic provided

$$F_1 \neq (a+2nb)\left(\kappa - \frac{r}{2n(2n+1)}\right) \pm a\sqrt{(\kappa+1)(\mu^2+\nu^2)}, \quad \mu.\nu(\kappa+1) \neq 0,$$

which completes the proof. □

Theorem 2.2. *Let $\tilde{M}$ be a (κ, μ, ν)-paracontact metric manifold equipped with the Zamkovoy connection ∇^*, and let N be an invariant submanifold of $\tilde{M}$, that is, the structure vector field ξ is everywhere tangent to N. Then, the condition $P(A_1, A_2).\sigma = F_2 Q(S, \sigma)$ holds on N if and only if either N is totally geodesic submanifold or the function F_2 satisfies*

$$F_2 = \frac{1}{2nk}\left[(a+2nb)\left(\kappa - \frac{r}{2n(2n+1)}\right) \pm a\sqrt{(\kappa+1)(\mu^2+\nu^2)}\right], \quad \mu\,\nu(\kappa+1) = 0.$$

Proof. Let us suppose that

$$(P(A_1, A_2).\sigma)(A_4, A_5) = F_2 Q(S, \sigma)(A_1, A_2; A_4, A_5),$$

for all $A_1, A_2, A_4, A_5 \in (\Gamma N)$. Then we have

$$\begin{aligned} &R^{\perp}(A_1, A_2)\sigma(A_4, A_5) - \sigma(P(A_1, A_2)A_4, A_5) - \sigma(A_4, P(A_1, A_2)A_5) \\ &= F_2\{-S(A_2, A_4)\sigma(A_1, A_5) + S(A_1, A_4)\sigma(A_2, A_5) - S(A_2, A_5)\sigma(A_4, A_1) + S(A_1, A_5)\sigma(A_4, A_2)\}. \end{aligned} \tag{2.25}$$

Substituting $A_1 = A_4 = \xi$ in (2.25) and employing (2.11), we get

$$\sigma(P(\xi, A_2)\xi, A_5) = -F_2 S(\xi, \xi)\sigma(A_2, A_5). \tag{2.26}$$

If we use (2.14) and (2.16) in (2.26), we get

$$\left\{2n\kappa F_2 - \left[\kappa(a+2nb) - \frac{r}{2n+1}\left(\frac{a}{2n}+b\right)\right]\right\}\sigma(A_2, A_5) - a(\mu\sigma(hA_2, A_5) + \nu\phi\sigma(hA_2, A_5)) = 0. \tag{2.27}$$

If hA_2 is taken instead of A_2 in above equality, we arrive at

$$\left\{2n\kappa F_2 - \left[\kappa(a+2nb) - \frac{r}{2n+1}\left(\frac{a}{2n}+b\right)\right]\right\}\sigma(hA_2, A_5) - a(\kappa+1)(\mu\sigma(A_2, A_5) + \nu\phi\sigma(A_2, A_5)) = 0. \tag{2.28}$$

From (2.27) and (2.28), one can easily see that

$$\left[\left\{2n\kappa F_2 - \left[\kappa(a+2nb) - \frac{r}{2n+1}\left(\frac{a}{2n}+b\right)\right]\right\}^2 - a^2(\kappa+1)(\mu^2+\nu^2)\right]\sigma(A_2, A_5) - 2a^2(\kappa+1)\mu\nu\phi\sigma(A_2, A_5) = 0. \tag{2.29}$$

This leads to the conclusion that N is either a totally geodesic submanifold or

$$F_2 = \frac{1}{2nk}\left[(a+2nb)\left(\kappa - \frac{r}{2n(2n+1)}\right) \pm a\sqrt{(\kappa+1)(\mu^2+\nu^2)}\right], \ \mu\nu(\kappa+1) = 0.$$

Thus, proof is completed. □

Next we deal with invariant submanifolds of a (κ,μ,ν)-paracontact metric manifold admitting Zamkovoy connection satisfying $C(A_1,A_2).\tilde{\nabla}\sigma = F_3 Q(g,\tilde{\nabla}\sigma)$ and $C(A_1,A_2).\tilde{\nabla}\sigma = F_4 Q(S,\tilde{\nabla}\sigma)$.

Theorem 2.3. *Let $\tilde{M}$ be a (κ,μ,ν)-paracontact metric manifold equipped with the Zamkovoy connection ∇^*, and let N be an invariant submanifold of $\tilde{M}$, that is, the structure vector field ξ is everywhere tangent to N. Then, the condition $C(A_1,A_2).\tilde{\nabla}\sigma = F_3 Q(g,\tilde{\nabla}\sigma)$ holds on N if and only if either N is totally geodesic or the function F_3 satisfies the condition*

$$F_3 = (a+4nb)\left(\kappa - \frac{r}{2n(2n+1)}\right) \pm a\sqrt{(\kappa+1)(\mu^2+\nu^2)}, \quad \mu\nu(\kappa+1) = 0.$$

Proof. Let N be an invariant submanifold of a (κ,μ,ν)-paracontact metric manifold $\tilde{M}$ with Zamkovoy connection. Then, there exists a function F_3 such that

$$(C(A_1,A_2).\tilde{\nabla}\sigma)(A_4,A_5,A_6) = F_3 Q(g,\tilde{\nabla}\sigma)(A_1,A_2;A_4,A_5,A_6), \tag{2.30}$$

for all $A_1,A_2,A_4,A_5,A_6 \in \Gamma(TN)$. In view of (1.24), (1.25) and (1.26) in (2.30), we have

$$\begin{aligned} &R^{\perp}(A_1,A_2)(\tilde{\nabla}\sigma)(A_4,A_5,A_6) - (\tilde{\nabla}\sigma)(C(A_1,A_2)A_4,A_5,A_6) - (\tilde{\nabla}\sigma)(A_4,C(A_1,A_2)A_5,A_6) - (\tilde{\nabla}\sigma)(A_4,A_5,C(A_1,A_2)A_6) \\ &= F_3\{-(\tilde{\nabla}\sigma)((A_1\Lambda_g A_2)A_4,A_5,A_6) - (\tilde{\nabla}\sigma)((A_4,(A_1\Lambda_g A_2)A_5,\ A_6)) - (\tilde{\nabla}\sigma)((A_4,A_5,(A_1\Lambda_g A_2)A_6))\}. \end{aligned} \tag{2.31}$$

In view of $\tilde{\nabla}\sigma(A_4,A_5,A_6) = \tilde{\nabla}_{A_4}\sigma(A_5,A_6)$ and using (1.19) the above equation reduces to

$$\begin{aligned} &R^{\perp}(A_1,A_2)\left[(\nabla^{\perp}_U\sigma)(A_5,A_6) - \sigma(\nabla_{A_4}A_5,A_6) - \sigma(A_5,\nabla_{A_4}A_6)\right] \\ &- \left[(\nabla^{\perp}_{C(A_1,A_2)A_4}\sigma)(A_5,A_6) + \sigma(\nabla_{C(A_1,A_2)A_4}A_5,A_6) + \sigma(A_5,\nabla_{C(A_1,A_2)A_4}A_6)\right] \\ &- \left[(\nabla^{\perp}_{A_4}\sigma)(C(A_1,A_2)A_5,A_6) + \sigma(\nabla_{A_4}(C(A_1,A_2)A_5,A_6) + \sigma(C(A_1,A_2)A_5,\nabla_{A_4}A_6)\right] \\ &- \left[(\nabla^{\perp}_{A_4}\sigma)(A_5,C(A_1,A_2)A_6) + \sigma(\nabla_{A_4}A_5,C(A_1,A_2)A_6) + \sigma(A_5,\nabla_{A_4}C(A_1,A_2)A_6)\right] \\ = \ & F_3\{-(\nabla^{\perp}_{(A_1\Lambda_g A_2)A_4}\sigma)(A_5,A_6) + \sigma(\nabla_{(A_1\Lambda_g A_2)A_4}A_5,A_6) + \sigma(A_5,\nabla_{(A_1\Lambda_g A_2)A_4}A_6) \\ &-(\nabla^{\perp}_{A_4}\sigma)((A_1\Lambda_g A_2)A_5,A_6) + \sigma(\nabla_{A_4}(A_1\Lambda_g A_2)A_5,A_6) + \sigma((A_1\Lambda_g A_2)A_5,\nabla_{A_4}A_6) \\ &-(\nabla^{\perp}_{A_4}\sigma)(A_5,(A_1\Lambda_g A_2)A_6) + \sigma(\nabla_{A_4}A_5,(A_1\Lambda_g A_2)A_6) + \sigma(A_5,\nabla_{A_4}(A_1\Lambda_g A_2)A_6)\}. \end{aligned} \tag{2.32}$$

Inserting $A_1 = A_5 = \xi$ in (2.32) and utilizing (2.11), we obtain

$$\begin{aligned} &-R^{\perp}(\xi,A_2)\sigma(\nabla_{A_4}\xi,A_6) + \sigma(\nabla_{C(\xi,A_2)A_4}\xi,A_6) - (\nabla^{\perp}_{A_4}\sigma)(C(\xi,A_2)\xi,A_6) \\ &+\sigma(\nabla_{A_4}(C(\xi,A_2)\xi,A_6) + \sigma(C(\xi,A_2)\xi,\nabla_{A_4}A_6) + \sigma(\nabla_{A_4}\xi,C(\xi,A_2)A_6) \\ = \ & F_3\{\sigma(\nabla_{(\xi\Lambda_g A_2)A_4}\xi,A_6) - (\nabla^{\perp}_{A_4}\sigma)((\xi\Lambda_g A_2)\xi,A_6) + \sigma(\nabla_{A_4}(\xi\Lambda_g A_2)\xi,A_6) \\ &+\sigma((\xi\Lambda_g A_2)\xi,\nabla_{A_4}A_6) + \sigma(\nabla_{A_4}\xi,(\xi\Lambda_g A_2)A_6)\}. \end{aligned} \tag{2.33}$$

Again plugging $A_6 = \xi$, (2.33) reduces to

$$\sigma((C(\xi,A_2)\xi,\nabla_{A_4}\xi) = F_3\sigma((\xi\Lambda_g A_2)\xi,\nabla_{A_4}\xi). \tag{2.34}$$

If (1.26), (2.10) and (2.17) are put in (2.34), we reach at

$$\begin{aligned} &\left\{F_3 - \left[\kappa(a+4nb) - \frac{r}{(2n+1)}\left(\frac{a}{2n}+2b\right)\right]\right\}(\sigma(A_2,\phi A_4) - \sigma(A_2,\phi hA_4)) \\ &- a\left[\mu(\sigma(hA_2,\phi A_4) - \sigma(hA_2,\phi hA_4)) + \nu\phi(\sigma(hA_2,\phi A_4) - \sigma(hA_2,\phi hA_4))\right] = 0. \end{aligned} \tag{2.35}$$

Substituting hA_2 instead of A_2 in (2.35) and using (1.9), we observe

$$\begin{aligned} &\left\{F_3 - \left[\kappa(a+4nb) - \frac{r}{(2n+1)}\left(\frac{a}{2n}+2b\right)\right]\right\}(\sigma(hA_2,\phi A_4) - \sigma(hA_2,\phi hA_4)) \\ &- a(\kappa+1)\left[\mu(\sigma(A_2,\phi A_4) - \sigma(A_2,\phi hA_4)) + \nu\phi(\sigma(A_2,\phi A_4) - \sigma(A_2,\phi hA_4))\right] = 0. \end{aligned} \tag{2.36}$$

From common solutions of (2.35) and (2.36), we can infer

$$\left[\left\{F_3 - \left[\kappa(a+4nb) - \frac{r}{(2n+1)}\left(\frac{a}{2n}+2b\right)\right]\right\}^2 - a^2(\kappa+1)(\mu^2+\nu^2)\right](\sigma(A_2,\phi A_4) - \sigma(A_2,\phi h A_4))$$
$$- 2a^2(\kappa+1)\mu\nu\phi(\sigma(A_2,\phi A_4) - \sigma(A_2,\phi h A_4)) = 0. \tag{2.37}$$

We note that $\sigma(A_2,\phi A_4) - \sigma(A_2,\phi h A_4) = 0$ if and only if $\kappa\sigma(A_2,A_4) = 0$.
Therefore, N is either totally geodesic or

$$F_3 = (a+4nb)\left(\kappa - \frac{r}{2n(2n+1)}\right) \pm a\sqrt{(\kappa+1)(\mu^2+\nu^2)}, \quad \mu\nu(\kappa+1) = 0,$$

and thus, our result is established. □

Theorem 2.4. *Let $\tilde{M}$ be a (κ,μ,ν)-paracontact metric manifold equipped with the Zamkovoy connection ∇^*, and let N be an invariant submanifold of $\tilde{M}$, that is, the structure vector field ξ is everywhere tangent to N. Then, the condition $C(A_1,A_2).\tilde{\nabla}\sigma = F_4 Q(S,\tilde{\nabla}\sigma)$ holds on N if and only if N is either totally geodesic or the function F_4 satisfies*

$$F_4 = \frac{1}{2nk}\left((a+4nb)\left(\kappa - \frac{r}{2n(2n+1)}\right) \pm a\sqrt{(\kappa+1)(\mu^2+\nu^2)}\right), \quad \mu\nu(\kappa+1) = 0.$$

Proof. Let us suppose that $C(A_1,A_2).\tilde{\nabla}\sigma = F_4 Q(S,\tilde{\nabla}\sigma)$ holds on N. Then we have

$$(C(A_1,A_2).\tilde{\nabla}\sigma)(A_4,A_5,A_6) = F_4 Q(S,\tilde{\nabla}\sigma)(A_1,A_2;A_4,A_5,A_6), \tag{2.38}$$

for all $A_1,A_2,A_4,A_5,A_6 \in \Gamma(TN)$. This leads to

$$R^\perp(A_1,A_2)(\tilde{\nabla}\sigma)(A_4,A_5,A_6) - (\tilde{\nabla}\sigma)(C(A_1,A_2)A_4,A_5,A_6) - (\tilde{\nabla}\sigma)(A_4,C(A_1,A_2)A_5,A_6) - (\tilde{\nabla}\sigma)(A_4,A_5,C(A_1,A_2)A_6)$$
$$= F_4\{-(\tilde{\nabla}\sigma)((A_1\Lambda_S A_2)A_4,A_5,A_6) - (\tilde{\nabla}\sigma)((A_4,(A_1\Lambda_S A_2)A_5,A_6)) - (\tilde{\nabla}\sigma)((A_4,A_5,(A_1\Lambda_S A_2)A_6))\}. \tag{2.39}$$

If we substitute (1.19) in (2.39) and in view of $\tilde{\nabla}\sigma(A_4,A_5,A_6) = \tilde{\nabla}_{A_4}\sigma(A_5,A_6)$, we can derive

$$\begin{aligned}
& R^\perp(A_1,A_2)\left[(\nabla^\perp_{A_4}\sigma)(A_5,A_6) - \sigma(\nabla_{A_4}A_5,A_6) - \sigma(A_5,\nabla_{A_4}A_6)\right] \\
& - \left[(\nabla^\perp_{C(A_1,A_2)A_4}\sigma)(A_5,A_6) + \sigma(\nabla_{C(A_1,A_2)A_4}A_5,A_6) + \sigma(A_5,\nabla_{C(A_1,A_2)A_4}A_6)\right] \\
& - \left[(\nabla^\perp_{A_4}\sigma)(C(A_1,A_2)A_5,A_6) + \sigma(\nabla_{A_4}(C(A_1,A_2)A_5,A_6) + \sigma(C(A_1,A_2)A_5,\nabla_{A_4}A_6)\right] \\
& - \left[(\nabla^\perp_{A_4}\sigma)(A_5,C(A_1,A_2)A_6) + \sigma(\nabla_{A_4}A_5,C(A_1,A_2)A_6) + \sigma(A_5,\nabla_{A_4}C(A_1,A_2)A_6)\right] \\
= \; & F_4\{-(\nabla^\perp_{(A_1\Lambda_S A_2)A_4}\sigma)(A_5,\ A_6) + \sigma(\nabla_{(A_1\Lambda_S A_2)A_4}A_5,A_6) + \sigma(A_5,\nabla_{(A_1\Lambda_S A_2)A_4}A_6) \\
& -(\nabla^\perp_{A_4}\sigma)((A_1\Lambda_S A_2)A_5,A_6) + \sigma(\nabla_{A_4}(A_1\Lambda_g A_2)A_5,A_6) + \sigma((A_1\Lambda_S A_2)A_5,\nabla_{A_4}A_6) \\
& -(\nabla^\perp_{A_4}\sigma)(A_5,(A_1\Lambda_S A_2)A_6) + \sigma(\nabla_{A_4}A_5,(A_1\Lambda_S A_2)A_6) + \sigma(A_5,\nabla_{A_4}(A_1\Lambda_S A_2)A_6)\}.
\end{aligned} \tag{2.40}$$

If we choose $A_1 = A_5 = \xi$ in (2.40) and applying (2.11), we get

$$\begin{aligned}
& -R^\perp(\xi,A_2)\sigma(\nabla_{A_4}\xi,A_6) + \sigma(\nabla_{C(\xi,A_2)A_4}\xi,A_6) - (\nabla^\perp_{A_4}\sigma)(C(\xi,A_2)\xi,A_6) \\
& +\sigma(\nabla_{A_4}(C(\xi,A_2)\xi,A_6) + \sigma(C(\xi,A_2)\xi,\nabla_{A_4}A_6) + \sigma(\nabla_{A_4}\xi,C(\xi,A_2)A_6) \\
= \; & F_4\{\sigma(\nabla_{(\xi\Lambda_g A_2)A_4}\xi,A_6) - (\nabla^\perp_{A_4}\sigma)((\xi\Lambda_S A_2)\xi,A_6) + \sigma(\nabla_{A_4}(\xi\Lambda_S A_2)\xi,A_6) \\
& +\sigma((\xi\Lambda_S A_2)\xi,\nabla_{A_4}A_6) + \sigma(\nabla_{A_4}\xi,(\xi\Lambda_S A_2)A_6)\}.
\end{aligned} \tag{2.41}$$

Again inserting $A_6 = \xi$, (2.41) reduces to

$$\sigma((C(\xi,A_2)\xi,\nabla_{A_4}\xi) = F_4\sigma((\xi\Lambda_S A_2)\xi,\nabla_{A_4}\xi). \tag{2.42}$$

If (1.26), (2.10) and (2.17) are put in previous equation, we reach at

$$\left\{2nkF_4 - \left[\kappa(a+4nb) - \frac{r}{(2n+1)}\left(\frac{a}{2n}+2b\right)\right]\right\}(\sigma(A_2,\phi A_4) - \sigma(A_2,\phi hA_4))$$
$$- a\,(\mu(\sigma(hA_2,\phi A_4) - \sigma(hA_5,\phi hA_4)) + \nu\phi(\sigma(hA_2,\phi A_4) - \sigma(hA_2,\phi hA_4))) = 0. \tag{2.43}$$

Substituting hA_2 for A_2 in (2.43) and by view of (1.9), we observe

$$\left\{2nkF_4 - \left[\kappa(a+4nb) - \frac{r}{(2n+1)}\left(\frac{a}{2n}+2b\right)\right]\right\}(\sigma(hA_2,\phi A_4) - \sigma(hA_2,\phi hA_4))$$
$$- a(\kappa+1)\left[\mu(\sigma(A_2,\phi A_4) - \sigma(A_2,\phi hA_4)) + \nu\phi(\sigma(A_2,\phi A_4) - \sigma(A_2,\phi hA_4))\right] = 0. \tag{2.44}$$

From (2.43) and (2.44), we finally conclude that

$$\left[\left\{2nkF_4 - \left[\kappa(a+4nb) - \frac{r}{(2n+1)}\left(\frac{a}{2n}+2b\right)\right]\right\}^2 - a^2(\kappa+1)(\mu^2+\nu^2)\right]$$
$$(\sigma(A_2,\phi A_4) - \sigma(A_2,\phi hA_4)) - 2a^2(\kappa+1)\mu\nu\phi(\sigma(A_2,\phi A_4) - \sigma(A_2,\phi hA_4)) = 0, \tag{2.45}$$

which proves our assertion. We note that $\sigma(A_2,\phi A_4) - \sigma(A_2,\phi hA_4) = 0$ if and only if $\kappa\sigma(A_2,A_4) = 0$. □

We now consider invariant submanifolds of a (κ,μ,ν)-paracontact metric manifold admitting Zamkovoy connection satisfying $Q(g, W_0^*.\sigma) = 0$ and $Q(S, W_0^*.\sigma) = 0$.

Theorem 2.5. *Let $\tilde{M}$ be a (κ,μ,ν)-paracontact metric manifold equipped with the Zamkovoy connection ∇^*, and let N be an invariant submanifold of $\tilde{M}$, that is, the structure vector field ξ is everywhere tangent to N. Then $Q(g, W_0^*.\sigma) = 0$ holds on N if and only if N is totally geodesic provided*

$$\kappa \neq \pm\frac{1}{2}\sqrt{(\kappa+1)(\mu^2+\nu^2)}, \quad \mu\nu(\kappa+1) \neq 0.$$

Proof. Assume that the condition $Q(g, W_0^* \cdot \sigma) = 0$ holds. This implies that

$$0 = Q(g, W_0^*(A_1,A_2).\sigma)(A_6,A_7;A_4,A_5), \tag{2.46}$$

for $A_1,A_2,A_4,A_5,A_6,A_7 \in \Gamma(TN)$. Now making use of (1.25) and (1.26) in above equation, we have

$$\begin{aligned} 0 \;=\;& (W_0^*(A_1,A_2).\sigma)(g(A_5,A_6)A_4,A_7) - (W_0^*(A_1,A_2).\sigma)(g(A_4,A_6)A_5,A_7) \\ &+ (W_0^*(A_1,A_2).\sigma)(A_6,g(A_5,A_7)A_4) - (W_0^*(A_1,A_2).\sigma)(A_6,g(A_4,A_7)A_5). \end{aligned} \tag{2.47}$$

Hence, a simple calculation gives that

$$\begin{aligned} 0 \;=\;& g(A_5,A_6)\left[R^\perp(A_1,A_2)\sigma(A_4,A_7) - \sigma(W_0^*(A_1,A_2)A_4,A_7) - \sigma(A_4,W_0^*(A_1,A_2)A_7)\right] \\ &- g(A_4,A_6)\left[R^\perp(A_1,A_2)\sigma(A_5,A_7) - \sigma(W_0^*(A_1,A_2)A_5,A_7) - \sigma(A_5,W_0^*(A_1,A_2)A_7)\right] \\ &+ g(A_5,A_7)\left[R^\perp(A_1,A_2)\sigma(A_6,A_4) - \sigma(W_0^*(A_1,A_2)A_6,A_4) - \sigma(A_6,W_0^*(A_1,A_2)A_4)\right] \\ &- g(A_4,A_7)\left[R^\perp(A_1,A_2)\sigma(A_6,A_5) - \sigma(W_0^*(A_1,A_2)A_6,A_5) - \sigma(A_6,W_0^*(A_1,A_2)A_5)\right]. \end{aligned} \tag{2.48}$$

If we choose $A_1 = A_6 = A_7 = A_5 = \xi$ in (2.48) and by view of (2.11), we get

$$\sigma(W_0^*(\xi,A_2)\xi,A_4) = 0. \tag{2.49}$$

Making use of (2.18) in (2.49), we obtain

$$2\kappa\sigma(A_2,A_4) + \mu\sigma(hA_2,A_4) + \nu\phi\sigma(hA_2,A_4) = 0. \tag{2.50}$$

If hA_2 is written instead of A_2 in last equality and make use of (1.6), we observe

$$2\kappa\sigma(hX_2, A_4) + (\kappa + 1)(\mu\sigma(A_2, A_4) + \nu\phi\sigma(A_2, A_4)) = 0. \tag{2.51}$$

From common solutions of (2.50) and (2.51), we can infer

$$(4\kappa^2 - (\kappa + 1)(\mu^2 + \nu^2))\sigma(A_2, A_4) - 2(\kappa + 1)\mu\nu\phi\sigma(A_2, A_4) = 0. \tag{2.52}$$

Since the vectors $\sigma(A_2, A_4)$ and $\phi\sigma(A_2, A_4)$ are orthogonal, we conclude that M is totally geodesic submanifold provided

$$\kappa \neq \pm\frac{1}{2}\sqrt{(\kappa + 1)(\mu^2 + \nu^2)}, \quad \mu\nu(\kappa + 1) \neq 0,$$

which completes the proof. □

Theorem 2.6. *Let $\tilde{M}$ be a (κ, μ, ν)-paracontact metric manifold equipped with the Zamkovoy connection ∇^*, and let N be an invariant submanifold of $\tilde{M}$, that is, the structure vector field ξ is everywhere tangent to N. Then $Q(S, W_0^*.\sigma) = 0$ holds on N if and only if N is totally geodesic provided*

$$\kappa \neq \pm\frac{1}{2}\sqrt{(\kappa + 1)(\mu^2 + \nu^2)}, \quad \kappa \neq 0, \quad \mu\nu(\kappa + 1) \neq 0.$$

Proof. Assume that $Q(S, W_0^* \cdot \sigma) = 0$. Consequently, we obtain

$$0 = Q(S, W_0^*(A_1, A_2).\sigma)(A_6, A_7; A_4, A_5), \tag{2.53}$$

for $A_1, A_2, A_4, A_5, A_6, A_7 \in \Gamma(TN)$. Substituting (1.25) and (1.26) in (2.53), we obtain

$$\begin{aligned} 0 \;=\;& (W_0^*(A_1, A_2).\sigma)(S(A_5, A_6)A_4, A_7) - (W_0^*(A_1, A_2).\sigma)(S(A_4, A_6)A_5, A_7) \\ &+(W_0^*(A_1, A_2).\sigma)(A_6, S(A_5, A_7)A_4) - (W_0^*(A_1, A_2).\sigma)(A_6, S(A_4, A_7)A_5). \end{aligned} \tag{2.54}$$

Easily from here, we can write

$$\begin{aligned} 0 \;=\;& S(A_5, A_6)\left[R^\perp(A_1, A_2)\sigma(A_4, A_7) - \sigma(W_0^*(A_1, A_2)A_4, A_7) - \sigma(A_4, W_0^*(A_1, A_2)A_7)\right] \\ &-S(A_4, A_6)\left[R^\perp(A_1, A_2)\sigma(A_5, A_7) - \sigma(W_0^*(A_1, A_2)A_5, A_7) - \sigma(A_5, W_0^*(A_1, A_2)A_7)\right] \\ &+S(A_5, A_7)\left[R^\perp(A_1, A_2)\sigma(A_6, A_4) - \sigma(W_0^*(A_1, A_2)A_6, A_4) - \sigma(A_6, W_0^*(A_1, A_2)A_4)\right] \\ &-S(A_4, A_7)\left[R^\perp(A_1, A_2)\sigma(A_6, A_5) - \sigma(W_0^*(A_1, A_5)A_6, A_5) - \sigma(A_6, W_0^*(A_1, A_2)A_5)\right]. \end{aligned} \tag{2.55}$$

If we put $A_1 = A_6 = A_7 = A_5 = \xi$ in above equality and using (2.11), we find that

$$S(\xi, \xi)\sigma(W_0^*(\xi, A_2)\xi, A_4) = 0. \tag{2.56}$$

In view of (2.18) and (2.14) in (2.56), we obtain

$$2n\kappa\{2\kappa\sigma(A_2, A_4) + \mu\sigma(hA_2, A_4) + \nu\phi\sigma(hA_2, A_4)\} = 0. \tag{2.57}$$

If hX_2 is written instead of A_2 in last equality and make use of (1.9), we observe

$$2n\kappa\{2\kappa\sigma(hA_2, A_4) + (\kappa + 1)(\mu\sigma(A_2, A_4) + \nu\phi\sigma(A_2, A_4))\} = 0. \tag{2.58}$$

From (2.57) and (2.58), we conclude that

$$2n\kappa\{(4\kappa^2 - (\kappa + 1)(\mu^2 + \nu^2))\sigma(A_2, A_4) - 2(\kappa + 1)\mu\nu\phi\sigma(A_2, A_4)\} = 0, \tag{2.59}$$

which proves our assertion. □

3. Conclusion

In this paper, we studied invariant submanifolds of a (κ, μ, ν)-paracontact metric manifold admitting a Zamkovoy connection. We established several characterizations involving the projective, concircular, and Weyl projective curvature tensors acting on the second fundamental form and its covariant derivative.

Our results show that such curvature conditions hold if and only if the submanifold is totally geodesic or the associated scalar functions satisfy specific relations involving the structure functions κ, μ, and ν. These findings provide new insights into the geometry of invariant submanifolds and generalize earlier results in contact geometry.

Author Contributions: All authors contributed equally to the manuscript. All authors carefully read and approved the final version of the manuscript.

Conflict of Interest: The authors declare that there is no conflict of interest.

Funding (Financial Disclosure): Not applicable.

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How to cite this article: M. S. Lakshmi and H. G. Nagaraja, *Invariant submanifolds of a (κ, μ, ν)-paracontact metric manifold admitting Zamkovoy connection*, Montes Taurus J. Pure Appl. Math. **8 (2)**, 39–50, 2026; Article ID: MTJPAM-D-24-00076.