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Tri-sequential statistical convergence controlled by triplet of parameters in topological spaces

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Abstract

In the framework of topological spaces, the concept of statistical convergence of triple order for triple sequences is investigated. The classical notion of statistical convergence is extended by incorporating three parameters, where parameters lie between zero and one, allowing for a more precise analysis of the convergence behavior of triple sequences. Additionally, the properties of this mode of convergence are examined within the setting of product spaces, with a particular focus on the influence of the topological structure of such spaces on statistical convergence. The main results include the derivation of necessary and adequate conditions for statistical convergence of triple order, the establishment of relationships between this generalized form of convergence and existing notions in the literature, and the analysis of the impact of product spaces on these convergence properties.

Keywords: Asymptotic density, statistical convergence of order α, statistical convergence of order (α, β), statistical convergence of order (α, β, γ), product space

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1. Introduction

The notion of statistical convergence originated in 1951 when Schoenberg [20] and Fast [13] both independently used the principle of statistical density to expand the conventional idea of convergence in sequences. This form of convergence weakens the classical definition by allowing sequences to converge based on the density of terms that satisfy the convergence criterion, rather than requiring all terms beyond a certain point to behave consistently. Additional research on this concept was conducted by a number of individuals [11, 17], Connor [8], Fridy [14], and Šalát [21]. As a measure of the size of the set A relative to the set of all natural numbers, the natural density or asymptotic density of the set $A \subseteq \mathbb{N}$, where $\mathbb{N}$ represents the set of natural numbers, is obtained. The definition is as follows:

$$\delta(A) = \lim_{n\to\infty} \frac{|\{k \leq n : k \in A\}|}{n}$$

if the limit is present (*cf.* [12]). A sequence $\{x_n : n \in \mathbb{N}\}$ in a topological space (X, τ) is considered statistically convergent to a point ℓ if, for each neighborhood U of ℓ, the natural density of the set of indices where the sequence does not belong to U is zero, that is, $\delta(\{n \in \mathbb{N} : x_n \notin U\}) = 0$. Stated otherwise, the sequence x_n is considered statistically

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convergent to ℓ if the proportion of terms in the series that are outside of U becomes negligible as $n \to \infty$ for each neighborhood U of ℓ. The German mathematician Alfred Pringsheim [18] was the first to propose the concept of convergence of double sequences in 1900.

Definition 1.1. Pringsheim states that a double sequence x_{mn} converges to a limit L if, for any $\epsilon > 0$, there is an integer M such that, for all $m, n \geq M$, the inequality $|x_{mn} - L| < \epsilon$ is obtained. That is, given both indices m and n are higher than or equal to M, we may discover a point M beyond which all terms x_{mn} of the double sequence lie within a distance ϵ of the limit L, for any arbitrarily small positive value ϵ.

While it applies to sequences indexed by two variables, this definition is similar to the traditional definition of convergence for single sequences. Mursaleen [16] explored the idea of statistical convergence for double sequences in 2003, expanding the concept of statistical convergence from single sequences to double sequences. The development expanded the usefulness of statistical convergence by offering a framework for assessing convergence in more complex circumstances, such as those in which sequences are indexed by two variables. Further studies on this subject are being carried out in [22, 23].

The concept of statistical convergence was strengthened for real sequences by Bhunia et al. [6], who imposed an upper bound on asymptotic density up to a specified order α, where $0 < \alpha < 1$. By adding the parameter α, this new type of convergence broadens the traditional understanding of s-convergence and offers a more adaptable framework for examining the behavior of real sequences. By using the percentage of terms that satisfy specific criteria in relation to the selected value of α, the inclusion of α enables a more sophisticated evaluation of the behavior of terms in a sequence. Then he introduced the concept of statistical convergence of order (α, β) and showed that the same technique applies to double sequences.

Statistical convergence is then extended to triple sequences from single and double sequences, which are indexed by three natural numbers. Further investigation into the statistical convergence of triple sequences is underway in 2007 in [19]. In order to comprehend the behaviour of increasingly complicated sequences in higher dimensions, this form of convergence has been researched. In this study, we present and investigate the notion of statistical convergence for triple sequences up to order (α, β, γ), where $0 < \alpha, \beta, \gamma < 1$.

1.1. Preliminaries

Throughout this work, a space is considered to be a topological space X endowed with a topology τ. Unless explicitly stated otherwise, no separation axioms are assumed. Standard terminology, notation, and fundamental concepts are adopted from [12]. The parameters $\alpha, \beta, \gamma, \delta, \mu, \eta$ are assumed to take values within the open interval $(0, 1)$. Also throughout this paper, the notation $\mathbb{N}^3$ is consistently represented as $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$. For the reader's convenience, a compilation of essential preliminary concepts is provided in the subsequent section.

Definition 1.2 (*cf.* [19])**.** A real triple sequence $\{x_{nk\ell} : (n, k, \ell) \in \mathbb{N}^3\}$ is considered statistically convergent to the number L if, for any $\epsilon > 0$,

$$\lim_{p,q,r \to \infty} \frac{|(n, k, \ell) \leq (p, q, r) : |x_{nk\ell} - L| \geq \epsilon|}{pqr} = 0.$$

Definition 1.3 (*cf.* [16])**.** Let $A(m, n) = \{(k, \ell) \in A : k \leq m, \ell \leq n\}$, and let $A \subset \mathbb{N} \times \mathbb{N}$ be a 2-dimensional set of $\mathbb{Z}^+$. Then, $\delta(A)$ represents the double density of A and can be expressed as follows:

$$\delta(A) = \lim_{m,n \to \infty} \frac{|A(m, n)|}{mn}.$$

Definition 1.4 (*cf.* [16])**.** If for any $\epsilon > 0$ the set $\{(j, k), j \leq n \;\&\; k \leq m : |x_{jk} - \ell| \geq \epsilon\}$ has double density zero, then a real double sequence $x = \{x_{jk}\}$ is statistically convergent to ℓ.

Definition 1.5 (*cf.* [10])**.** In a topological space (X, τ), a double sequence $\{x_{ij}\}$ is considered statistically convergent of order α, β $(0 < \alpha < 1, 0 < \beta < 1)$ if, for any neighborhood U of limit point ℓ, $\delta^{\alpha,\beta}(\{(m, n) \in \mathbb{N} \times \mathbb{N} : x_{mn} \notin U\}) = 0$.

$s^{\alpha,\beta}$-convergence is the representation for the statistical convergent of order α, β.

Definition 1.6 (*cf.* [6])**.** Consider a real sequence $x = \{x_k : k \in \mathbb{N}\}$, where $0 < \alpha < 1$. If there is a real number L such that $\lim_{n\to\infty} \frac{|\{k\leq n:|x_k-L|\geq\epsilon\}|}{n^\alpha} = 0$, then the sequence $\{x_k : k \in \mathbb{N}\}$ is statistically convergent of order α. And it is denoted by $s^\alpha\text{-}\lim_{k\to\infty} x_k = L$.

For a set $A \subset \mathbb{N}$, $\delta(\mathbb{N} \setminus A) = 1 - \delta(A)$. If $\delta^\alpha(A) = 0$, however, then $\delta^\alpha(\mathbb{N}/A) = \infty$, whose converse might not be true.

Definition 1.7. A subset $A \subset \mathbb{N} \times \mathbb{N}$ is said to be statistically α,β-dense (or $s^{\alpha,\beta}$-dense) if $\delta^{\alpha,\beta}(A) = \infty$.

Definition 1.8. For every neighbourhood U of x, the sequence $\{x_n\}$ in a topological space (X, τ) is statistically convergent of order α $(0 < \alpha < 1)$ to $x \in X$ when $\delta^\alpha(\{n \in \mathbb{N} : x_n \notin U\}) = 0$.

Definition 1.9. A sub-sequence $\{x_{n_k} : k \in \mathbb{N}\}$ of a sequence $\{x_n : n \in \mathbb{N}\}$ is called s^α-dense if

$$\{n \in \mathbb{N} : x_n \in \{x_{n_k} : k \in \mathbb{N}\}\}$$

has non-zero s^α-density.

2. $s^{\alpha,\beta,\gamma}$-convergence

Triple sequences, which offer a more sophisticated structure than single or double sequences, are concepts in topology that take the idea of sequences to three indices, much like how humans determine positions in three-dimensional space using coordinates for length, width, and height. An array of words indexed by three independent variables is represented by a triple sequence, which is commonly written as $\{x_{mnp}\}$, where $m, n, p \in \mathbb{N}$.

Definition 2.1. Let $A \subset \mathbb{N}^3$ be a 3-dimensional set of positive integers, and define

$$A = \{(i, j, k) \in A : (i, j, k) \leq (l, m, n)\}.$$

The 3-dimensional representation of natural density of order α,β,γ is then given by

$$\delta^{\alpha,\beta,\gamma}(A) = \lim_{l,m,n\to\infty} \frac{|A(l, m, n)|}{l^\alpha m^\beta n^\gamma},$$

which is referred to as the triple asymptotic density of order α,β,γ.

Definition 2.2. A triple sequence x_{ijk} in a topological space (X, τ) is said to be statistical convergent of order $\alpha,\beta,\gamma\,(0 < \alpha,\beta,\gamma < 1)$ to $p \in X$, if for every neighborhood U of p, the following condition holds:

$$\delta^{\alpha,\beta,\gamma}\left(\left\{(l, m, n) \in \mathbb{N}^3 : x_{lmn} \notin U\right\}\right) = 0.$$

This mode of convergence is referred to as statistical convergence of order α,β,γ and is denoted by $s^{\alpha,\beta,\gamma}$-convergence.

Remark 2.3. Every statistically convergent sequence of order α,β,γ is a statistically convergent sequence where $\alpha = 1, \beta = 1, \gamma = 1$. But converse may not be true.

Example 2.4. Consider a topological space (X, τ) where $X = \{p, q\}$ and $\tau = \{\emptyset, \{p\}, \{q\}, \{p, q\}\}$. Also take a triple sequence $\{y_{lmn} : (l, m, n) \in \mathbb{N}^3\}$ in (X, τ) such that

$$y_{lmn} = \begin{cases} \{q\}, & \text{if } l = a^2, m = b^3 \text{ and } n = c^4 \text{ for some } (a, b, c) \in \mathbb{N}^3 \\ \{p\}, & \text{otherwise.} \end{cases}$$

Now the open neighborhoods of p are $U_1 = \{p\}$ and $U_2 = X$.

Here, $\delta(\{(l, m, n) \in \mathbb{N}^3 : x_{lmn} \notin U_1\}) = \lim_{l,m,n\to\infty} \frac{|(l,m,n)\in\mathbb{N}^3:x_{lmn}\notin U_1|}{lmn} = \lim_{l,m,n\to\infty} \frac{l^{\frac{1}{2}} m^{\frac{1}{3}} n^{\frac{1}{4}}}{lmn} = 0$. Also,

$$\delta(\{(l, m, n) \in \mathbb{N}^3 : x_{lmn} \notin U_2\}) = \delta(\{(l, m, n) \in \mathbb{N}^3 : x_{lmn} \notin X\}) = 0.$$

So for every open neighborhood U of p, $\delta(\{(l,m,n) \in \mathbb{N}^3 : x_{lmn} \notin U\}) = 0$. Therefore, x_{lmn} is statistically convergent to p. But,

$$\delta^{\frac{1}{3},\frac{1}{4},\frac{1}{5}}\left(\left\{(l,m,n) \in \mathbb{N}^3 : x_{lmn} \notin \{p\}\right\}\right) = \lim_{l,m,n\to\infty} \frac{l^{\frac{1}{2}} m^{\frac{1}{3}} n^{\frac{1}{4}}}{l^{\frac{1}{3}} m^{\frac{1}{4}} n^{\frac{1}{5}}} = \infty.$$

Further, $\delta^{\frac{1}{3},\frac{1}{4},\frac{1}{5}}(\{(l,m,n) \in \mathbb{N}^3 : x_{lmn} \notin \{q\}\}) = \infty$. Therefore, $\left\{x_{lmn} : (l,m,n) \in \mathbb{N}^3\right\}$ is not $s^{\alpha,\beta,\gamma}$-convergent for $\alpha = \frac{1}{3}$, $\beta = \frac{1}{4}, \gamma = \frac{1}{5}$.

Theorem 2.5. *If we consider a triple sequence $\{x_{ijk}\}$ which is $s^{\alpha,\beta,\gamma}$-converges to l then there exists $M \subset \mathbb{N}^3$ with $\delta^{\alpha,\beta,\gamma}(M) = \infty$ such that $\{x_{ijk} : (i,j,k) \in M\}$ converges to l.*

Proof. Let us consider a sequence $\{x_{ijk} : (i,j,k) \in \mathbb{N}^3\}$ which is $s^{\alpha,\beta,\gamma}$ converges to ℓ in a first countable space Y. We establish a countable decreasing local base $B_{1,\ell} \supseteq B_{2,\ell} \supseteq B_{3,\ell} \supseteq \ldots$ of ℓ. Let $A_q = \{(i,j,k) \in \mathbb{N}^3 : x_{ijk} \in B_{q,\ell}\}$ for each $q \in \mathbb{N}$. Thus $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$. But x_{ijk} $s^{\alpha,\beta,\gamma}$-converges to ℓ. Therefore, $\delta^{\alpha,\beta,\gamma}(\{(i,j,k) \in \mathbb{N}^3 : x_{ij} \notin B_{q,\ell}\}) = 0$, for all $q \in \mathbb{N}$ implies that $\delta^{\alpha,\beta,\gamma}({A_q}^c) = 0$ for all $q \in \mathbb{N}$, which shows that $\delta^{\alpha,\beta,\gamma}(A_q) = \infty$ for all $q \in \mathbb{N}$. Now consider a monotone increasing unbounded sequence $\{a_n\}$ so that for every $(f_1, g_1, h_1) \in A_1$, there exists a $(f_2, g_2, h_2) \in A_2$ such that $f_2 > f_1$, $g_2 > g_1$ and $h_2 \geq h_1$ or $f_2 > f_1$, $g_2 \geq g_1$ and $h_2 > h_1$ or $f_2 \geq f_1$, $g_2 > g_1$ and $h_2 > h_1$ such that

$$\frac{|\{(i,j,k) \in A_2 : i \leq m, j \leq n, k \leq p\}|}{m^\alpha n^\beta p^\gamma} > a_2, \text{ for all } m > f_2\ , n > g_2 \text{ and } \ p > h_2$$

since $\delta^{\alpha,\beta,\gamma}(A_2) = \infty$. Continuing in this manner results in $(f_q, g_q, h_q) \in A_q$ where $f_1 < f_2 < f_3 < \cdots < f_r$, $g_1 < g_2 < g_3 < \cdots < g_q$ and $h_1 < h_2 < h_3 < \cdots < h_q$ such that

$$\frac{|\{(i,j,k) \in A_q : i \leq m, j \leq n, k \leq p\}|}{m^\alpha n^\beta p^\gamma} > a_q, \text{ for all } m > f_q, n > g_q \text{ and } p > h_q.$$

Assume that A be the union of the sets $\{(i,j,k) \in \mathbb{N}^3 : i \in [1, f_1],\ j \in [1, g_1]\ \&\ k \in [1, h_1]\}$ and

$$\left\{(i,j,k) \in A_q : i \in [f_q, f_{q+1}],\ j \in [g_q, g_{q+1}]\ \&\ j \in [h_q, h_{q+1}], q = 1,2,3,\ldots\right\}.$$

So,

$$\frac{|\{(i,j,k) \in A : i \leq m, j \leq n, k \leq p\}|}{m^\alpha n^\beta p^\gamma} \geq \frac{|\{(i,j,k) \in A_q : i \leq m, j \leq n, k \leq p\}|}{m^\alpha n^\beta n^\gamma} > a_q,$$

for all $m,n,p \in \mathbb{N}$, where $f_q \leq m \leq f_{q+1}$, $g_q \leq n \leq g_{q+1}$ and $h_q \leq p \leq h_{q+1}$. Therefore $\delta^{\alpha,\beta,\gamma}(A) = \infty$, since $\{a_n\}$ be a monotonic increasing unbounded sequence.

Let W be a neighbourhood of ℓ and $B_{q,\ell} \subset W$. If $(i,j,k) \in A$ such that $i \geq f_q$, $j \geq g_q$ and $k \geq h_q$ then there is a $s > q$ with $f_s \leq i \leq f_{s+1}$, $g_s \leq j \leq g_{s+1}$ and $h_s \leq j \leq h_{s+1}$. Hence $(i,j,k) \in A_s$. Thus for every $(i,j,k) \in A$, $i \geq f_q$, $j \geq g_q$ and $k \geq h_q$, $x_{ijk} \in B_{s,\ell} \subset B_{q,\ell} \subset W$.

Therefore,

$$\lim_{m,n,p\to\infty,(i,j,k)\in A} x_{ijk} = \ell.$$

Hence $A \subset \mathbb{N}^3$ with $\delta^{\alpha,\beta,\gamma}(A) = \infty$ such that $\{x_{ijk} : (i,j,k) \in A\}$ converges to l. □

Example 2.6. Consider (X, τ) be a topological space where $X = \{a_1, a_2, a_3\}$ and $\tau = \{\emptyset, X, \{a_1, a_3\}, \{a_2\}\}$. Then take a triple sequence $\{x_{mnp} : (m,n,p) \in \mathbb{N}^3\}$ in (x, τ) such that

$$x_{mnp} = \begin{cases} \{a_2\}, & \text{if } m = i^3, n = j^3 \text{ and } p = k^4 \text{ for some } (i,j,k) \in \mathbb{N}^3 \\ \{a_1\}, & \text{otherwise.} \end{cases}$$

In this case, a_1 has two neighbourhoods U_1 and U_2 i.e., $U_1 = \{a_1, a_3\}$ and $U_2 = X$. Now,

$$\delta^{\frac{1}{2},\frac{1}{2},\frac{1}{3}}\left(\left\{(m,n,p) \in \mathbb{N}^3 : x_{mnp} \notin U_1\right\}\right) = 0.$$

Also,

$$\delta^{\frac{1}{2},\frac{1}{2},\frac{1}{3}}\left(\left\{(m,n,p)\in\mathbb{N}^3 : x_{mnp}\notin U_2\right\}\right)=\lim_{m,n,p\to\infty}\frac{m^{\frac{1}{3}}n^{\frac{1}{3}}p^{\frac{1}{4}}}{m^{\frac{1}{2}}n^{\frac{1}{2}}p^{\frac{1}{3}}}=0.$$

Therefore, for every open neighborhood U of a_1, $\delta^{\frac{1}{2},\frac{1}{2},\frac{1}{3}}\left(\left\{(m,n,p)\in\mathbb{N}^3 : x_{mnp}\notin U\right\}\right) = 0$. So $\{x_{m,n,p}\}$ is $s^{\frac{1}{2},\frac{1}{2},\frac{1}{3}}$-convergent. Thus, the limit of a $s^{\alpha,\beta,\gamma}$-convergent sequence may not be unique.

Theorem 2.7. *In a T_2 space, the limit of an $s^{\alpha,\beta,\gamma}$-convergent sequence is unique.*

Proof. Let us consider a sequence $\{x_{ijk} : (i,j,k)\in\mathbb{N}^3\}$ in a topological space (X,τ) such that $x_{ijk}\xrightarrow{s^{\alpha,\beta,\gamma}lim} K$ and $x_{ijk}\xrightarrow{s^{\alpha,\beta,\gamma}lim} L$ with $K\neq L$. Since X is a T_2 space, there exist $U,V\in\tau$ such that $K\in U, L\in V$ and $U\cap V=\phi$. Now $\delta^{\alpha,\beta,\gamma}(\{(i,j,k)\in\mathbb{N}^3 : x_{ijk}\notin U\})=0$. i.e., $\delta^{\alpha,\beta,\gamma}(\{(i,j,k)\in\mathbb{N}^3 : x_{ijk}\in U^c\})=0$. But $V\subset U^c$, since $U\cap V=\phi$. So, $\delta^{\alpha,\beta,\gamma}(\{(i,j,k)\in\mathbb{N}^3 : x_{ijk}\in V\})\leq\delta^{\alpha,\beta,\gamma}(\{(i,j,k)\in\mathbb{N}^3 : x_{ijk}\in U^c\})=0$. i.e., $\delta^{\alpha,\beta,\gamma}(\{(i,j,k)\in\mathbb{N}^3 : x_{ijk}\in V\})=0$. i.e., $\delta^{\alpha,\beta,\gamma}(\{(i,j,k)\in\mathbb{N}^3 : x_{ijk}\notin V^c\})=\infty$, which contradicts the fact that $x_{ijk}\xrightarrow{s^{\alpha,\beta,\gamma}lim} L$, since V is a neighbourhood of L.

Therefore the limit must be unique. □

In a topological space (X,τ), the symbols listed below are used in this way:

M_0 : The collection of all sequences that are statistically convergent in a space (X,τ).

M_0^{α} : The collection of all sequences that are statistically convergent sequences of order α in a space (X,τ).

$M_0^{\alpha,\beta}$: The collection of all sequences that are statistically convergent double sequences of order α,β in a space (X,τ).

$M_0^{\alpha,\beta,\gamma}$: The collection of all sequences that are statistically convergent triple sequences of order α,β,γ in a space (X,τ).

Theorem 2.8. *Let $M_0^{\alpha,\beta,\gamma}$ be the collection of triple sequences which are all statistically convergent of order α,β,γ in a space (X,τ). Then for $0<\alpha<\delta<1$, $0<\beta<\mu<1$ and $0<\gamma<\eta<1$, $M_0^{\alpha,\beta,\gamma}\subseteq M_0^{\delta,\mu,\eta}$.*

Proof. Let us consider a triple sequence $\left\{x_{ijk} : (i,j,k)\in\mathbb{N}^3\right\}\in M^{\alpha,\beta,\gamma}$ in a topological space (X,τ) where $0<\alpha<\delta<1$, $0<\beta<\mu<1$ and $0<\gamma<\eta<1$. Therefore, for any neighborhood U of l, there exists a $l\in X$ such that

$$\lim_{m,n,p\to\infty}\frac{|\{(p,q,r)\in\mathbb{N}^3 : x_{pqr}\notin U\}|}{m^{\alpha}n^{\beta}p^{\gamma}}=0.$$

But,

$$\lim_{m,n,p\to\infty}\frac{|\{(i,j,k)\in\mathbb{N}^3 : x_{ijk}\notin U\}|}{m^{\delta}n^{\mu}p^{\eta}}\leq\lim_{m,n,p\to\infty}\frac{|\{(p,q,r)\in\mathbb{N}^3 : x_{pqr}\notin U\}|}{m^{\alpha}n^{\beta}p^{\gamma}}=0.$$

So, for every neighbourhood U of l,

$$\lim_{m,n,p\to\infty}\frac{|\{(i,j,k)\in\mathbb{N}^3 : x_{ijk}\notin U\}|}{m^{\delta}n^{\mu}p^{\eta}}=0,$$

i.e., $\left\{x_{ijk} : (i,j,k)\in\mathbb{N}^3\right\}\in M^{\delta,\mu,\eta}$. Therefore $M_0^{\alpha,\beta,\gamma}\subseteq M_0^{\delta,\mu,\eta}$. □

Definition 2.9. A subset $M\subset\mathbb{N}^3$ is said to be statistically α,β,γ-dense (or $s^{\alpha,\beta,\gamma}$-dense) if $\delta^{\alpha,\beta,\gamma}(M)=\infty$.

Definition 2.10. A sub-sequence $\{x_{m_i,n_j,l_k} : (i,j,k)\in\mathbb{N}^3\}$ of a sequence $\{x_{mnl} : (m,n,l)\in\mathbb{N}^3\}$ is called $s^{\alpha,\beta,\gamma}$-dense if $\{(m,n,l)\in\mathbb{N}^3 : x_{mnl}\in\{x_{m_i,n_j,l_k} : (i,j,k)\in\mathbb{N}^3\}\}$ is α,β,γ-dense (or $s^{\alpha,\beta,\gamma}$-dense).

Theorem 2.11. *A sequence is $s^{\alpha,\beta,\gamma}$-convergent in a topological space (X,τ) if and only if every $s^{\alpha,\beta,\gamma}$-dense sub-sequence is $s^{\alpha,\beta,\gamma}$-convergent.*

Proof. Let us consider a arbitrary sequence $\left\{x_{mnp} : (m,n,p) \in \mathbb{N}^3\right\}$ in topological space (X,τ) and every $s^{\alpha,\beta,\gamma}$-dense sub-sequence of $\{x_{mnp} : (m,n,p) \in \mathbb{N}^3\}$ is $s^{\alpha,\beta,\gamma}$-convergent. Now $\{x_{mnp} : (m,n,p) \in \mathbb{N}^3\}$ is a sub-sequence to itself. Now,

$$\begin{aligned}\lim_{m,n,p\to\infty} \frac{|\{(i,j,k) \in \mathbb{N}^3 : x_{ijk} \in \{x_{mnp} : (m,n,p) \in \mathbb{N}^3\}\}|}{m^\alpha n^\beta p^\gamma} &= \lim_{m,n,p\to\infty} \frac{mnp}{m^\alpha n^\beta p^\gamma} \\ &= \lim_{m,n,p\to\infty} m^{1-\alpha} n^{1-\beta} p^{1-\gamma} = \infty.\end{aligned}$$

Therefore, $\{x_{mnp} : (m,n,p) \in \mathbb{N}^3\}$ is $s^{\alpha,\beta,\gamma}$-dense in itself. So, by the hypothesis, $\{x_{mnp} : (m,n,p) \in \mathbb{N}^3\}$ is a $s^{\alpha,\beta,\gamma}$-convergent.

Conversely, let $\{x_{mnp} : (m,n,p) \in \mathbb{N}^3\}$ is a $s^{\alpha,\beta,\gamma}$-convergent sequence and $\{x_{m_i n_j p_k} : (i,j,k) \in \mathbb{N}^3\}$ be a $s^{\alpha,\beta,\gamma}$-dense sub-sequence of $\{x_{mnp} : (m,n,p) \in \mathbb{N}^3\}$. If possible, let $\{x_{m_i n_j p_k} : (i,j,k) \in \mathbb{N}^3\}$ is not $s^{\alpha,\beta,\gamma}$-convergent. So, for every $b \in X$ there is a neighbourhood U of b with

$$\delta^{\alpha,\beta,\gamma}\left(\left\{(m_i,n_j,p_k) \in \mathbb{N}^3 : x_{m_i n_j p_k} \notin U\right\}\right) \neq 0.$$

Now, $\{(m,n,p) \in \mathbb{N}^3 : x_{mnp} \notin U\} \supseteq \{(m_i,n_j,p_k) \in \mathbb{N}^3 : x_{m_i n_j p_k} \notin U\}$. Thus,

$$\delta^{\alpha,\beta,\gamma}\left(\left\{(m,n,p) \in \mathbb{N}^3 : x_{mnp} \notin U\right\}\right) \geq \delta^{\alpha,\beta,\gamma}\left(\left\{(m_i,n_j,p_k) \in \mathbb{N}^3 : x_{m_i n_j p_k} \notin U\right\}\right) \neq 0,$$

which is a contradiction to the fact that $\{x_{mnp} : (m,n,p) \in \mathbb{N}^3\}$ is a $s^{\alpha,\beta,\gamma}$-convergent. Therefore every $s^{\alpha,\beta,\gamma}$-dense sub-sequence of a $s^{\alpha,\beta,\gamma}$- convergent triple sequence must be $s^{\alpha,\beta,\gamma}$-convergent. □

3. $s^{\alpha,\beta,\gamma}$-convergence in product space

Triple density is the foundation for defining the statistical convergence of triple sequences, and it is defined for the cartesian product of $\mathbb{N}$ to $\mathbb{N}$ to $\mathbb{N}$(i.e., $\mathbb{N}^3$). Therefore, it appears that it is crucial to look at the relationships between the statistical convergence of triple sequences of ordered pairs in a product space and the statistical convergence of sequences in topological spaces.

Theorem 3.1. *Consider three sequences $\{x_l : l \in \mathbb{N}\}$, $\{y_m : m \in \mathbb{N}\}$ and $\{z_n : n \in \mathbb{N}\}$ in three topological space (X,τ), (Y,σ) and (Z,η) respectively such that $x_l \xrightarrow{s-\lim} x_0$, $y_m \xrightarrow{s-\lim} y_0$ and $z_n \xrightarrow{s-\lim} z_0$. Then the triple sequence $\{k_{l,m,n} = (x_l, y_m, z_n) : (l,m,n) \in \mathbb{N}^3\}$ is s-triple convergent in the product space $X \times Y \times Z$.*

Proof. Let us consider $\mathcal{B} = \{(U,V,W) : U \in \tau,\ V \in \sigma\ \&\ W \in \eta\}$ be the base of the product space of $X \times Y \times Z$. Assume that R is any arbitrary open neighborhood of (x_0, y_0, z_0). Then there exist $U \times V \times W \in \mathcal{B}$ such that $(x_0, y_0, z_0) \in U \times V \times W \subset R$ where $x_0 \in U \in \tau$, $y_0 \in V \in \sigma$ and $z_0 \in W \in \eta$. But, $x_m \xrightarrow{s-\lim} x_0$, $y_n \xrightarrow{s-\lim} y_0$ and $z_p \xrightarrow{s-\lim} z_0$. So, $\delta(\{m \in \mathbb{N} : x_m \notin U\}) = 0$, $\delta(\{n \in \mathbb{N} : y_n \notin V\}) = 0$ and $\delta(\{p \in \mathbb{N} : z_p \notin W\}) = 0$.

Now,

$$\begin{aligned}&\lim_{m,n,p\to\infty} \frac{|\{(m,n,p) \in \mathbb{N}^3 : q_{m,n,p} = (x_m, y_n, z_p) \notin U \times V \times W\}|}{mnp} \\ &\leq \lim_{m,n,p\to\infty} \frac{|\{m \in \mathbb{N} : x_m \notin U\}|np + |\{n \in \mathbb{N} : y_n \notin V\}|mp + |\{p \in \mathbb{N} : z_p \notin W\}|mn}{mnp} \\ &= \lim_{m\to\infty} \frac{|\{m \in \mathbb{N} : x_m \notin U\}|}{m} + \lim_{n\to\infty} \frac{|\{n \in \mathbb{N} : y_n \notin V\}|}{n} + \lim_{p\to\infty} \frac{|\{p \in \mathbb{N} : z_p \notin W\}|}{p} = 0.\end{aligned}$$

Thus,

$$\delta(\{m \in \mathbb{N} : x_m \notin U\}) + \delta(\{n \in \mathbb{N} : y_n \notin V\}) + \delta(\{p \in \mathbb{N} : z_p \notin W\}) = 0.$$

Hence the triple sequence $\{k_{l,m,n} = (x_l, y_m, z_n) : (l,m,n) \in \mathbb{N}^3\}$ is s-triple convergent in the product space $X \times Y \times Z$. □

Example 3.2. The s-triple convergence of the triple sequence $\{k_{l,m,n} = (x_l, y_m, z_n) : (l, m, n) \in \mathbb{N}^3\}$ in the product space of $X \times Y \times Z$, does not guarantee that the sequences $\{x_l : l \in \mathbb{N}\}$, $\{y_m : n \in \mathbb{N}\}$ and $\{z_n : n \in \mathbb{N}\}$ are s-convergent in the respective topological spaces.

Consider a topological space (X, τ), where $X = \{a, b\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, X\}$. In this space take three sequences $\{x_l : l \in \mathbb{N}\}$, $\{y_m : n \in \mathbb{N}\}$ and $\{z_n : n \in \mathbb{N}\}$ where

$$x_l = \begin{cases} a, & \text{if } l = p^2 \text{ for some } p \in \mathbb{N} \\ b, & \text{otherwise} \end{cases}$$

and

$$y_m = \begin{cases} a, & \text{if } m = 2q \text{ for some } q \in \mathbb{N} \\ b, & \text{otherwise} \end{cases}$$

and

$$z_n = \begin{cases} a, & \text{if } n = r^3 \text{ for some } r \in \mathbb{N} \\ b, & \text{otherwise.} \end{cases}$$

Now, $\{(x_l, y_m, z_n) : (l, m, n) \in \mathbb{N}^3\}$ is a triple sequence in the product space $X \times X \times X$. In $X \times X \times X$, the smallest open set containing (b, b, b) is $\{b\} \times \{b\} \times \{b\}(= W$ say). So for every open set V containing (b, b, b),

$$\begin{aligned} \delta(\{(i, j, k) \in \mathbb{N}^3 : (x_i, y_j, z_k) \notin V\}) &\leq \delta(\{(i, j, k) \in \mathbb{N}^3 : (x_i, y_j, z_k) \notin W\}) \\ &= \lim_{l,m,n\to\infty} \frac{|(i, j, k) : i \leq l, j \leq m \,\&\, k \leq n \text{ and } (x_i, y_j, z_k) \notin W|}{lmn} \\ &= \lim_{l,m,n\to\infty} \frac{\sqrt{l}\frac{m}{2}\sqrt[3]{n}}{lmn} = 0. \end{aligned}$$

So, the triple sequence $\{(x_l, y_m, z_n) : (l, m, n) \in \mathbb{N}^3\}$ statistically triple converges to (b, b, b). On the other side, although $\{x_l : l \in \mathbb{N}\}$ and $\{z_n : n \in \mathbb{N}\}$ statistically converges to b, the sequence $\{y_m : m \in \mathbb{N}\}$ does not statistically converges to b.

Open Problem: If $\{x_m : m \in \mathbb{N}\}$,$\{y_n : n \in \mathbb{N}\}$ and $\{z_p : p \in \mathbb{N}\}$ be three sequence in topological spaces (X, τ) ,(Y, σ) and (Z, η) respectively such that $x_m \xrightarrow{s^\alpha - lim} x_0$, $y_n \xrightarrow{s^\beta - lim} y_0$ and $z_p \xrightarrow{s^\gamma - lim} z_0$, does the triple sequence $\{q_{m,n,p} = (x_m, y_n, z_p) : (m, n, p) \in \mathbb{N}^3\}$ is $s^{\alpha,\beta,\gamma}$- convergent to (x_0, y_0, z_0) in the product space of $X \times Y \times Z$.

Example 3.3. The $s^{\alpha,\beta,\gamma}$- convergence of the triple sequence $\{q_{m,n,p} = (x_m, y_n, z_p) : (m, n, p) \in \mathbb{N}^3\}$ in the product space of $X \times Y \times Z$, does not guarantee that the sequences $\{x_m : m \in \mathbb{N}\}$,$\{y_n : n \in \mathbb{N}\}$ and $\{z_p : p \in \mathbb{N}\}$ are s^α ,s^β and s^γ convergent respectively in the respective topological spaces.

Consider the topological space of Example 3.2 and three sequences $\{x_m : m \in \mathbb{N}\}$,$\{y_n : n \in \mathbb{N}\}$ and $\{z_p : p \in \mathbb{N}\}$, where

$$x_m = \begin{cases} a, & \text{if } m = r^2 \text{ for some } r \in \mathbb{N} \\ b, & \text{otherwise} \end{cases}$$

and

$$y_n = \begin{cases} a, & \text{if } n = s^3 \text{ for some } s \in \mathbb{N} \\ b, & \text{otherwise} \end{cases}$$

and

$$z_p = \begin{cases} a, & \text{if } p = t^4 \text{ for some } t \in \mathbb{N} \\ b, & \text{otherwise.} \end{cases}$$

Now, $\{(x_m, y_n, z_p) : (m,n,p) \in \mathbb{N}^3\}$ is a triple sequence in the product space $X \times X \times X$. In $X \times X \times X$, the smallest open set containing (b,b,b) is $\{b\} \times \{b\} \times \{b\}(= W$ say). So for every open set V containing (b,b,b),

$$\begin{aligned}
\delta^{\frac{1}{2},\frac{1}{2},\frac{1}{2}}(\{(i,j,k) \in \mathbb{N}^3 : (x_i, y_j, z_k) \notin V\}) &\leq \delta^{\frac{1}{2},\frac{1}{2},\frac{1}{2}}(\{(i,j,k) \in \mathbb{N}^3 : (x_i, y_j, z_k) \notin W\}) \\
&= \lim_{m,n,p\to\infty} \frac{|(i,j,k) : i \leq m, j \leq n, k \leq p \text{ and } (x_i, y_j, z_k) \notin W|}{m^{\frac{1}{2}} n^{\frac{1}{2}} p^{\frac{1}{2}}} \\
&= \lim_{m,n,p\to\infty} \frac{m^{\frac{1}{2}} n^{\frac{1}{3}} p^{\frac{1}{4}}}{m^{\frac{1}{2}} n^{\frac{1}{2}} p^{\frac{1}{2}}} = 0.
\end{aligned}$$

So, the triple sequence $\{(x_m, y_n, z_p) : (m,n,p) \in \mathbb{N}^3\}$ $s^{\frac{1}{2},\frac{1}{2},\frac{1}{2}}$-converges to (b,b,b). On the other side, $\{y_n : n \in \mathbb{N}\}$ and $\{z_p : p \in \mathbb{N}\}$ $s^{\frac{1}{2}}$-converges to b. But $\delta^{\frac{1}{2}}(\{m \in \mathbb{N} : m \notin \{b\}\}) = 1$, i.e., the sequence $\{x_m : m \in \mathbb{N}\}$ does not $s^{\frac{1}{2}}$-converges to b.

4. Conclusion

The notion of $s^{\alpha,\beta,\gamma}$-convergence refines statistical convergence for triple sequences by requiring that the set of indices where the terms leave any chosen neighborhood has triple asymptotic density of order α, β, γ equal to zero. This provides a controlled statistical approach toward an arbitrary point and guarantees the existence of a unique limit, particularly in a T_2 space. In this work, we also introduced the notion of $s^{\alpha,\beta,\gamma}$-dense sub-sequences for triple sequences showed that a triple sequence is $s^{\alpha,\beta,\gamma}$-convergent precisely when all of its $s^{\alpha,\beta,\gamma}$-dense sub-sequences converge in the same manner. Furthermore, several applications of $s^{\alpha,\beta,\gamma}$-convergence for triple sequences in product spaces have been established. This work also provides a basis for further continuation of statistical convergence studies in more general ideals, fuzzy structures, and higher-dimensional product spaces.

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