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Characterization of linear canonical ambiguity function on some function spaces

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Abstract

An important time-frequency analytic tool, the classical ambiguity function (AF), was developed for the examination of the time-frequency properties of non-stationary signals. Radar waveform characteristics such as time-delay resolution, Doppler resolution and Doppler-tolerance characteristic may all be accurately and efficiently analyzed using it. This paper is concerned with the continuity properties of the linear canonical ambiguity function, which is a generalization of the classical ambiguity function using the linear canonical transform. It is demonstrated that the linear canonical ambiguity function yields continuity results on Hardy, BMO, weighted Hardy and weighted BMO spaces. The Hardy, BMO, weighted Hardy and weighted BMO distances of two linear canonical ambiguity functions connected to different functionals and signals are also investigated in this work.

Keywords: Linear canonical ambiguity function, Hardy space, weighted Hardy space, BMO space, weighted BMO space

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1. Introduction

One of the most widely utilized techniques in signal processing and many other scientific domains is Fourier analysis. The fractional Fourier transform (FrFT), a generalization of the classical Fourier transform (FT), was introduced to the mathematical literature in the 1980s. First introduced by Wiener [14] in 1929, the FrFT was designed to solve the quantum mechanical differential equation. Optical problems can also be interpreted with the FrFT. The FrFT has been used in many scientific disciplines such as signal and image processing, communications, wave propagation. The subject came into the limelight with the work of Namias [6] in 1980. Many ideas related to the FT have been extended to the linear canonical transform (LCT) domain as a further expansion of the FRFT, which is demonstrated to have a significant impact on optics and signal processing. First-order optical systems, which are theoretically similar to the linear canonical transformations (LCTs), include transmission through narrow lenses, propagation across quadratic graded-index media, free space propagation under the Fresnel approximation, and their arbitrary combinations. LCT is more adaptable due to the additional degrees of freedom, and it has been demonstrated to be an effective tool for filter construction, time-frequency analysis, phase retrieval, pattern recognition, etc.

The ambiguity function (AF) is a significant instrument employed in the field of signal processing for the purpose of delineating methodologies that facilitate the recovery of amplitude and phase information from intensity measurements. In accordance with the traditional concept of ambiguity function (AF), Pei and Ding [7] first examine the

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linear canonical ambiguity function (LCAF) and derive several significant characteristics. In the field of optical signal processing, Zhao et al. [15] have recently looked at the characteristics and physical significance of the LCAF. Tian-Wen et al. [12] presented a new definition for LCAF that differs from the definitions of AF found in [7, 15]. The key characteristics and use of the novel AF in the linear frequency-modulated signal processing were also covered.

The continuity problem of LCAF, as defined in [12], is examined in this study in Hardy, BMO, weighted Hardy, and weighted BMO spaces. It is also worth noting that our current investigation into the continuity of LCAF on Hardy and BMO spaces draws upon the analytical framework recently established in [11]. In [11], the author demonstrated the boundedness of the short-time fractional Fourier transform (STFrFT) on unweighted and weighted Hardy and BMO spaces. Since the LCAF can be interpreted as a kernel-based time-frequency distribution that shares structural similarities with the STFrFT, specifically regarding modulation and shifting properties within the linear canonical transform domain, our results maintain a direct continuity with those findings. By extending this analytical approach to the LCAF, we establish that the boundedness properties, which were previously proven for the STFrFT, hold consistently across these related time-frequency representations, thereby strengthening the theoretical framework of the field.

1.1. Preliminaries

This section introduces the fundamental mathematical definitions and properties required for the subsequent developments in this study.

The Linear Canonical Transform (LCT): Let

$$\Theta = (a, b; c, d) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

be a matrix parameter satisfying $det\Theta = ad - bc = 1$. The LCT of a signal $h \in L^1(\mathbb{R})$ is defined by

$$L_\Theta(h(t))(u) = \begin{cases} \int_{\mathbb{R}} h(t)\, K_\Theta(t,u)\, dt, & b \neq 0 \\ \sqrt{d}\, e^{i\frac{c}{d}u^2}\, h(du), & b = 0 \end{cases}$$

where $K_\Theta(t,u)$ is so-called the kernel of the LCT given by

$$K_\Theta(t,u) = \frac{1}{\sqrt{2\pi b i}}\, e^{\frac{i}{2b}(at^2 - 2tu + du^2)}.$$

It is clear from the LCT definition that a signal's LCT is basically a chirp multiplication when $b = 0$. Consequently, in this study, the assumption is made that $b \neq 0$ (*cf.* [7, 12, 15]).

The cross-ambiguity function (AF) of functionals h and f, which are elements of the $L^2(\mathbb{R})$ space, is defined as follows

$$A(h, f)(u,\mu) = \int_{\mathbb{R}} e^{-i\mu\nu}\, h(\nu + \tfrac{u}{2})\, \overline{f(\nu - \tfrac{u}{2})}\, d\nu,$$

if h is equal to f, then $Ah(u,\mu)$ is called the auto-ambiguity function of h, which is given by

$$Ah(u,\mu) = \int_{\mathbb{R}} e^{-i\mu\nu}\, h(\nu + \tfrac{u}{2})\, \overline{h(\nu - \tfrac{u}{2})}\, d\nu.$$

It is important to note that the term "ambiguity functions" is used to refer to both auto-ambiguity and cross-ambiguity functions (*cf.* [3]).

By substituting the kernel of the Fourier transform (FT) with the kernel of the LCT in the AF definition, we may derive a definition of the linear canonical ambiguity function (LCAF) from the definition of the AF associated with the FT (*cf.* [7, 12, 15]):

$$A^\Theta(h, f)(u,\mu) = \int_{\mathbb{R}} K_\Theta(\nu,\mu)\, h(\nu + \tfrac{u}{2})\, \overline{f(\nu - \tfrac{u}{2})}\, d\nu. \tag{1.1}$$

Let ω be a function on $\mathbb{R}$ that is positive. It is called a "temperate weight function" if $C > 0$ and $N > 0$ exist such that

$$\omega(u + \mu) \leq (1 + C|u|)^N \omega(\mu), \qquad u, \mu \in \mathbb{R}.$$

In this context, the symbol $\mathcal{W}$ will represent the collection of all such functions.

For $p \in [1, \infty]$, the space $L^p_\omega(\mathbb{R})$ consists of all complex-valued measurable functions on $\mathbb{R}$ that satisfy

$$\int_{\mathbb{R}} |h(y)|^p \omega(y)\, dy < \infty.$$

We define the $L^p_\omega(\mathbb{R})$ norm of h by

$$\|h\|_{L^p_\omega} = \|h\|_{p,\omega} = \left(\int_{\mathbb{R}} |h(y)|^p \omega(y)\, dy \right)^{\frac{1}{p}}.$$

Then $L^p_\omega(\mathbb{R})$ is complete with the norm $\|\cdot\|_{p,\omega}$. If $\omega = 1$, then the space $L^p_\omega(\mathbb{R})$ is the Lebesgue space $L^p(\mathbb{R})$.

It is known that a $\mathbb{C}$-valued function h defined on $\mathbb{R}$ is called a locally integrable function if $\int_K |h(x)|\, dx < \infty$, where $K \subset \mathbb{R}$ is compact. $L^1_{loc}(\mathbb{R})$ denotes the spaces of locally integrable functions.

There are significant connections between the theory of Hardy Space (HS) and many areas of mathematical study, such as Fourier analysis, harmonic analysis, operator theory and singular integrals, signal and image processing, and control theory. The literature has demonstrated that the HS is more appropriate than the Lebesgue space for certain harmonic analysis concerns. The maximal functions can be defined as follows, and this will allow us to give an equivalent definition of HS $H^1(\mathbb{R})$: We are going to take a function that is both integrable and smooth. This function will be denoted by φ and its domain will be the real line, with its support lying in the unit ball. In addition, it should be $\int_{\mathbb{R}} \varphi = 1$. Let us set $\varphi_t(y) = t^{-1}\varphi(yt^{-1})$, $t > 0$. For an integrable function h, the maximal operator, represented by the symbol $\mathcal{M}_\varphi$, is

$$\mathcal{M}_\varphi h(y) = \sup_{t>0} |h * \varphi_t (y)|.$$

The space HS is defined as the collection of functions $h \in L^1(\mathbb{R})$ satisfying the condition that, given any test function $\varphi \in \mathcal{S}(\mathbb{R})$ (where the Schwartz space is denoted by $\mathcal{S}(\mathbb{R})$ and the integral of φ over $\mathbb{R}$ is normalized to 1), the resulting maximal operator $\mathcal{M}_\varphi h$ remains within $L^1(\mathbb{R})$. Furthermore, for any $h \in H^1(\mathbb{R})$, the translation operator $L_u h(y) = h(y - u)$ for $y, u \in \mathbb{R}$ maps HS back into itself, preserving the Hardy space norm such that $\|L_u h\|_{H^1} = \|h\|_{H^1}$.

The "weighted Hardy space" (WHS) $H^1_\omega(\mathbb{R})$ is the linear space of all $h \in L^1_\omega(\mathbb{R})$ if, for some $\varphi \in \mathcal{S}(\mathbb{R})$ with $\int_{\mathbb{R}} \varphi = 1$, $\mathcal{M}_\varphi h$ is in $L^1_\omega(\mathbb{R})$.

Functions with bounded mean oscillations (BMO) are the class of functions whose variation from their means over cubes is bounded. In [4], John-Nirenberg developed the space $BMO(\mathbb{R})$ of functions with bounded mean oscillation. $BMO(\mathbb{R})$ represents the space of all functions $h \in L^1_{loc}(\mathbb{R})$ such that

$$\|h\|_{BMO} = \sup_{I \subset \mathbb{R}} I\left(|h - I(h)|\right) < \infty.$$

In this expression, the supremum is computed across the collection of all axes-aligned cubes I. Here, the notation $|I|$ denotes the Lebesgue measure of the cube I, while $I(h)$ signifies the average value of the function h within that specific cube, that is

$$I(h) = |I|^{-1} \int_I h(y)\, dy \ \leq\ |I|^{-1} \int_I |h(y)|\, dy \ \leq M\ < \infty. \tag{1.2}$$

The weighted bounded mean oscillation space (WBMO) $BMO_\omega(\mathbb{R})$ is defined as the set of all functions $h \in L^{1,\omega}_{loc}(\mathbb{R})$ such that

$$\|h\|_{BMO_\omega} = \sup_{I \subset \mathbb{R}} |I|^{-1}_\omega \int_I |h(y) - I(h)|\, \omega(y) dy < \infty,$$

where $|I|_\omega = \int_I \omega(y)\, dy$ and the supremum is taken over balls I in $\mathbb{R}$, (*cf.* [1, 2, 4, 5], [8]-[11], [13]).

2. Continuity of LCAF

This main part of the paper investigates the continuity problem of LCAF in Hardy, BMO, weighted Hardy, and weighted BMO spaces. However, we shall first obtain the following novel formula for the LCAF given in (1.1), which is more practical in operations.

Theorem 2.1. *Let $h, f \in L^2(\mathbb{R})$. Then the LCAF can be expressed as follows:*

$$A^{\Theta}(h,f)(u,\mu) = \frac{1}{2\pi b}\, e^{\frac{i}{2b}u\,(au-\mu)} \int_{\mathbb{R}} h(m+u)\,\overline{f(m)}\, e^{\frac{i}{b}m\,(au-\mu)}\, dm. \tag{2.1}$$

Proof. In consideration of AF, a novel expression for LCAF is obtained as demonstrated below:

$$\begin{aligned} A(h,f)(u,\mu) &= \int_{\mathbb{R}} h(v+\tfrac{u}{2})\,\overline{f(v-\tfrac{u}{2})}\, e^{-i\mu v}\, dv \\ &= \int_{\mathbb{R}} h(v+\tfrac{u}{2})\, e^{-i\frac{\mu}{2}(v+\frac{u}{2})}\, \overline{f(v-\tfrac{u}{2})}\, e^{-i\frac{\mu}{2}(v-\frac{u}{2})}\, dv \\ &= \int_{\mathbb{R}} h_{\mu}(v+\tfrac{u}{2})\,\overline{f_{\mu}(v-\tfrac{u}{2})}\, dv, \end{aligned} \tag{2.2}$$

where

$$h_{\mu}(v) = h(v)\, e^{-i\frac{\mu}{2}v}, \qquad f_{\mu}(v) = f(v)\, e^{i\frac{\mu}{2}v}. \tag{2.3}$$

The replacement of $e^{-i\frac{\mu}{2}v}$ and $e^{i\frac{\mu}{2}v}$ in (2.3) with $K_{\Theta}(v, \frac{\mu}{2})$ and $K_{\Theta}(v, -\frac{\mu}{2})$, resp., results in the following equations:

$$h^{\Theta}_{\mu}(v) = h(v)\, K_{\Theta}(v, \tfrac{\mu}{2}), \qquad f^{\Theta}_{\mu}(v) = f(v)\, K_{\Theta}(v, -\tfrac{\mu}{2}). \tag{2.4}$$

Therefore, a novel kind of LCAF is obtained by substituting h_{μ} with h^{Θ}_{μ} and f_{μ} with f^{Θ}_{μ} in (2.2):

$$\begin{aligned} A^{\Theta}(h,f)(u,\mu) &= \int_{\mathbb{R}} h^{\Theta}_{\mu}(v+\tfrac{u}{2})\,\overline{f^{\Theta}_{\mu}(v-\tfrac{u}{2})}\, dv \\ &= \int_{\mathbb{R}} h(v+\tfrac{u}{2})\, K_{\Theta}(v+\tfrac{u}{2}, \tfrac{\mu}{2})\, \overline{f(v-\tfrac{u}{2})\, K_{\Theta}(v-\tfrac{u}{2}, -\tfrac{\mu}{2})}\, dv \\ &= \frac{1}{2\pi b}\int_{\mathbb{R}} h(v+\tfrac{u}{2})\,\overline{f(v-\tfrac{u}{2})}\, e^{\frac{i}{b}v\,(au-\mu)} dv. \end{aligned} \tag{2.5}$$

Substitution of the equation (2.5) with $v - \frac{u}{2} = m$ gives rise to the following revised equation:

$$A^{\Theta}(h,f)(u,\mu) = \frac{1}{2\pi b}\, e^{\frac{i}{2b}u\,(au-\mu)} \int_{\mathbb{R}} h(m+u)\,\overline{f(m)}\, e^{\frac{i}{b}m\,(au-\mu)}\, dm.$$

□

2.1. Continuity on HS and BMO

In this section, we investigate the continuity properties of the linear canonical ambiguity function, a generalization of the classical ambiguity function, within the framework of HS. Throughout this section, unless otherwise stated, the parameter μ is assumed to be fixed for the operator $A^{\Theta}(h,f)(\cdot,\mu)$.

Theorem 2.2. *Let f be in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. The function $h \to A^{\Theta}(h,f)(\cdot,\mu)$ defines a continuous operator from $H^1(\mathbb{R})$ to itself. Moreover,*

$$\left\|A^{\Theta}(h,f)(\cdot,\mu)\right\|_{H^1} \leq \frac{1}{|2\pi b|}\, \|f\|_1\, \|h\|_{H^1}.$$

Proof. Firstly, the demonstration of $A^{\Theta}(h,f)(\cdot,\mu)$ being contained within the set $L^1(\mathbb{R})$ is required. For any $h \in H^1(\mathbb{R})$, we have $h \in L^1(\mathbb{R})$. Utilizing the equation (2.1) and the Fubini theorem, we find

$$\begin{aligned}
\|A^{\Theta}(h,f)(\cdot,\mu)\|_1 &= \int_{\mathbb{R}} \left| \frac{1}{2\pi b} e^{\frac{i}{2b}u(au-\mu)} \int_{\mathbb{R}} h(m+u)\,\overline{f(m)}\, e^{\frac{i}{b}m(au-\mu)}\, dm \right| du \\
&\leq \frac{1}{|2\pi b|} \int_{\mathbb{R}} |\overline{f(m)}| \int_{\mathbb{R}} |h(m+u)|\, du\, dm \\
&= \frac{1}{|2\pi b|} \|h\|_1 \int_{\mathbb{R}} |f(m)|\, dm \\
&= \frac{1}{|2\pi b|} \|h\|_1 \|f\|_1 < \infty.
\end{aligned}$$

Again, we get

$$\begin{aligned}
\left| \left(A^{\Theta}(h,f)(\cdot,\mu) * \varphi_t \right)(u) \right| &= \left| \int_{\mathbb{R}} A^{\Theta}(h,f)(u-\eta,\mu)\, \varphi_t(\eta)\, d\eta \right| \\
&= \left| \int_{\mathbb{R}} \left(\frac{1}{2\pi b} e^{\frac{i}{2b}(u-\eta)(a(u-\eta)-\mu)} \int_{\mathbb{R}} h\,(m+u-\eta)\, \overline{f(m)} e^{\frac{i}{b}m(a(u-\eta)-\mu)}\, dm \right) \varphi_t(\eta)\, d\eta \right| \\
&\leq \frac{1}{|2\pi b|} \int_{\mathbb{R}} |f(m)| \int_{\mathbb{R}} |h(m+u-\eta)|\, |\varphi_t(\eta)|\, d\eta\, dm \\
&= \frac{1}{|2\pi b|} \int_{\mathbb{R}} |f(m)| \int_{\mathbb{R}} |L_{-m}h(u-\eta)|\, |\varphi_t(\eta)|\, d\eta\, dm,
\end{aligned}$$

where $L_{-m}h(u) = h(u+m)$. Applying the Minkowski's inequality for integrals and by using the translation invariant property of Hardy space, we have

$$\begin{aligned}
\left\| A^{\Theta}(h,f)(\cdot,\mu) \right\|_{H^1} &= \int_{\mathbb{R}} \sup_{t>0} \left| \left(A^{\Theta}(h,f)(\cdot,\mu) * \varphi_t \right)(u) \right| du \\
&\leq \frac{1}{|2\pi b|} \int_{\mathbb{R}} |f(m)| \left(\int_{\mathbb{R}} \sup_{t>0} \left(\int_{\mathbb{R}} |L_{-m}h(u-\eta)|\, |\varphi_t(\eta)|\, d\eta \right) du \right) dm \\
&\leq \frac{1}{|2\pi b|} \int_{\mathbb{R}} |f(m)| \left(\int_{\mathbb{R}} \sup_{t>0} (|L_{-m}h| * |\varphi_t|)(u)\, du \right) dm \\
&= \frac{1}{|2\pi b|} \int_{\mathbb{R}} |f(m)| \|L_{-m}h\|_{H^1}\, dm \\
&= \frac{1}{|2\pi b|} \|f\|_1 \, \|h\|_{H^1}.
\end{aligned}$$

□

Applying the previous theorem allows us to establish the following relationship.

Corollary 2.3. *If $h \in H^1(\mathbb{R})$ and $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then*

$$\left\| A^{\Theta}(h,f)(\cdot,\mu) \right\|_{H^1} = O\left(\frac{1}{|2\pi b|} \right).$$

Our next step is to evaluate the $H^1(\mathbb{R})$ metric for a pair of LCAFs.

Theorem 2.4. *Let $\chi, \rho \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. If $h_1, h_2 \in H^1(\mathbb{R})$, the following conclusion is reached:*

$$\left\| A^{\Theta}(h_1,\chi)(\cdot,\mu) - A^{\Theta}(h_2,\rho)(\cdot,\mu) \right\|_{H^1} \leq \frac{1}{|2\pi b|} \left(\|\chi-\rho\|_1 \, \|h_1\|_{H^1} + \|\rho\|_1 \, \|h_1 - h_2\|_{H^1} \right).$$

Proof. We write

$$\begin{aligned}\left\|A^{\Theta}(h_1,\chi)(\cdot,\mu)-A^{\Theta}(h_2,\rho)(\cdot,\mu)\right\|_{H^1} \quad \leq \quad & \left\|A^{\Theta}(h_1,\chi)(\cdot,\mu)-A^{\Theta}(h_1,\rho)(\cdot,\mu)\right\|_{H^1} \\ & +\left\|A^{\Theta}(h_1,\rho)(\cdot,\mu)-A^{\Theta}(h_2,\rho)(\cdot,\mu)\right\|_{H^1}. \end{aligned} \tag{2.6}$$

It is evident that

$$\begin{aligned} A^{\Theta}(h_1,\chi)(u,\mu)-A^{\Theta}(h_1,\rho)(u,\mu) &= A^{\Theta}(h_1,\chi-\rho)(u,\mu) \\ &= \frac{1}{2\pi b}\, e^{\frac{i}{2b}u\,(au-\mu)}\int_{\mathbb{R}} h_1(m+u)\,\overline{(\chi-\rho)(m)}\, e^{\frac{i}{b}m\,(au-\mu)}\,dm \end{aligned}$$

and

$$\begin{aligned}&\left(\left(A^{\Theta}(h_1,\chi)-A^{\Theta}(h_1,\rho)\right)(\cdot,\mu)*\varphi_t(\cdot)\right)(u) \\ &= \frac{1}{2\pi b}\int_{\mathbb{R}}\int_{\mathbb{R}} h_1(m+u-\eta)\,\overline{(\chi-\rho)(m)}\, e^{\frac{i}{2b}(u-\eta)\,(a(u-\eta)-\mu)}\, e^{\frac{i}{b}m\,(a(u-\eta)-\mu)}\,\varphi_t(\eta)\,d\eta\,dm.\end{aligned}$$

According to Theorem 2.2, the leading term of the right-hand side of (2.6) becomes

$$\left\|A^{\Theta}(h_1,\chi)(\cdot,\mu)-A^{\Theta}(h_1,\rho)(\cdot,\mu)\right\|_{H^1} \leq \frac{1}{|2\pi b|}\,\|\chi-\rho\|_1\,\|h_1\|_{H^1}. \tag{2.7}$$

Again, with $A^{\Theta}(h_1,\rho)(u,\mu)-A^{\Theta}(h_2,\rho)(u,\mu)=A^{\Theta}(h_1-h_2,\rho)(u,\mu)$, we obtain

$$\begin{aligned}&\left(\left(A^{\Theta}(h_1,\rho)-A^{\Theta}(h_2,\rho)\right)(\cdot,\mu)*\varphi_t(\cdot)\right)(u) \\ &= \frac{1}{2\pi b}\int_{\mathbb{R}}\int_{\mathbb{R}} (h_1-h_2)(m+u-\eta)\,\overline{\rho(m)}\, e^{\frac{i}{2b}(u-\eta)\,(a(u-\eta)-\mu)}\, e^{\frac{i}{b}m\,(a(u-\eta)-\mu)}\,\varphi_t(\eta)\,d\eta\,dm.\end{aligned}$$

And so, using Theorem 2.2

$$\left\|A^{\Theta}(h_1,\rho)(\cdot,\mu)-A^{\Theta}(h_2,\rho)(\cdot,\mu)\right\|_{H^1} \leq \frac{1}{|2\pi b|}\,\|\rho\|_1\,\|h_1-h_2\|_{H^1}. \tag{2.8}$$

Then from the inequalities by (2.6), (2.7) and (2.8), we obtain

$$\left\|A^{\Theta}(h_1,\chi)(\cdot,\mu)-A^{\Theta}(h_2,\rho)(\cdot,\mu)\right\|_{H^1} \leq \frac{1}{|2\pi b|}(\|\chi-\rho\|_1\,\|h_1\|_{H^1}+\|\rho\|_1\,\|h_1-h_2\|_{H^1}).$$

□

We will now present the BMO continuity of the LCAF.

Theorem 2.5. *Let $f\in L^1(\mathbb{R})$ be compactly supported. The function $h\to A^{\Theta}(h,f)(\cdot,\mu)$ defines a continuous operator from $BMO(\mathbb{R})$ to itself and holds*

$$\left\|A^{\Theta}(h,f)(\cdot,\mu)\right\|_{BMO} \leq \frac{1}{|2\pi b|}\,\|f\|_1\,(\|h\|_{BMO}+2M),$$

where M is the boundedness constant of h in space BMO, as defined in inequality (1.2).

Proof. Taking any $h\in BMO(\mathbb{R})$, we have $h\in L^1_{loc}(\mathbb{R})$. For any compact set $I\subset\mathbb{R}$, we can write from (2.1)

$$\begin{aligned}\int_I\left|A^{\Theta}(h,f)(u,\mu)\right|du &\leq \frac{1}{|2\pi b|}\int_{\mathbb{R}}|f(m)|\left(\int_I|h\,(m+u)|\;du\right)dm \\ &= \frac{1}{|2\pi b|}\int_{\mathbb{R}}|f\,(m)|\left(\int_{I'}|h\,(\eta)|\;d\eta\right)dm, \qquad I'=I+m.\end{aligned} \tag{2.9}$$

As $I' \subset \sup pf + I$ is a closed and bounded set in $\mathbb{R}$ and $h \in L^1_{loc}(\mathbb{R})$, then

$$\int_I \left| A^{\Theta}(h,f)(u,\mu) \right| du \leq \frac{1}{|2\pi b|} K \|f\|_1 < \infty.$$

So $A^{\Theta}(h,f)(\cdot,\mu) \in L^1_{loc}(\mathbb{R})$. Furthermore, since

$$\begin{aligned} I\left(A^{\Theta}(h,f)\right) &= |I|^{-1} \int_I A^{\Theta}(h,f)(\eta,\mu)\, d\eta \\ &= |I|^{-1} \int_I \left(\frac{1}{2\pi b} \int_{\mathbb{R}} h(m+\eta)\, e^{\frac{i}{2b}\eta\left(ma+\frac{a\eta-\mu}{2}\right)} \overline{f(m)\, e^{\frac{i}{b}m\mu}}\, dm \right) d\eta \\ &= \frac{1}{2\pi b} \int_{\mathbb{R}} \overline{f(m)\, e^{\frac{i}{b}m\mu}} \left(|I|^{-1} \int_I h(m+\eta)\, e^{\frac{i}{2b}\eta\left(ma+\frac{a\eta-\mu}{2}\right)} d\eta \right) dm, \end{aligned} \tag{2.10}$$

we obtain

$$\begin{aligned} \left\| A^{\Theta}(h,f)(\cdot,\mu) \right\|_{BMO} &= \sup_{I\subset\mathbb{R}} |I|^{-1} \int_I \left| A^{\Theta}(h,f)(u,\mu) - I\left(A^{\Theta}(h,f)\right) \right| du \\ &= \sup_{I\subset\mathbb{R}} |I|^{-1} \int_I \left| \frac{1}{2\pi b} \int_{\mathbb{R}} \overline{f(m)\, e^{\frac{i}{b}m\mu}}\, h(m+u)\, e^{\frac{i}{b}u\left(ma+\frac{au-\mu}{2}\right)} dm \right. \\ &\quad \left. -\frac{1}{2\pi b} \int_{\mathbb{R}} \overline{f(m)\, e^{\frac{i}{b}m\mu}} \left(|I|^{-1} \int_I h(m+\eta)\, e^{\frac{i}{b}\eta\left(ma+\frac{a\eta-\mu}{2}\right)} d\eta \right) dm \right| du \\ &\leq \frac{1}{|2\pi b|} \int_{\mathbb{R}} |f(m)| \left(\sup_{I\subset\mathbb{R}} |I|^{-1} \int_I \left| h(m+u)\, e^{\frac{i}{b}u\left(ma+\frac{au-\mu}{2}\right)} - |I|^{-1} e^{\frac{i}{b}u\left(ma+\frac{au-\mu}{2}\right)} \int_I h(m+\eta)\, d\eta \right| du \right. \\ &\quad + \sup_{I\subset\mathbb{R}} |I|^{-1} \int_I \left| |I|^{-1} e^{\frac{i}{b}u\left(ma+\frac{au-\mu}{2}\right)} \int_I h(m+\eta)\, d\eta \right| du \\ &\quad \left. + \sup_{I\subset\mathbb{R}} |I|^{-1} \int_I \left| |I|^{-1} \int_I h(m+\eta)\, e^{\frac{i}{b}\eta\left(ma+\frac{a\eta-\mu}{2}\right)} d\eta \right| du \right) dm \\ &\leq \frac{1}{|2\pi b|} \int_{\mathbb{R}} |f(m)| \left(\sup_{I\subset\mathbb{R}} |I|^{-1} \int_I \left| h(m+u) - |I|^{-1} \int_I h(m+\eta)\, d\eta \right| du \right. \\ &\quad \left. + 2\sup_{I\subset\mathbb{R}} |I|^{-1} \int_I |I|^{-1} \int_I |h(m+\eta)|\, d\eta\, du \right) dm. \end{aligned}$$

Using the set I', we get

$$\begin{aligned} \left\| A^{\Theta}(h,f)(\cdot,\mu) \right\|_{BMO} &\leq \frac{1}{|2\pi b|} \int_{\mathbb{R}} |f(m)| \left(\sup_{I'\subset\mathbb{R}} \left|I'\right|^{-1} \int_{I'} \left| h(z) - |I'|^{-1} \int_{I'} h(z')\, dz' \right| dz \right. \\ &\quad \left. + 2\sup_{I'\subset\mathbb{R}} |I'|^{-1} \int_{I'} |I'|^{-1} \int_{I'} |h(z')|\, dz'\, dz \right) dm \\ &\leq \frac{1}{|2\pi b|} \int_{\mathbb{R}} |f(m)| \left(\|h\|_{BMO} + 2\, |I'|^{-1} M\, |I'| \right) dm \\ &= \frac{1}{|2\pi b|} \left(\|h\|_{BMO} + 2M \right) \|f\|_1 . \end{aligned}$$

□

Corollary 2.6. *If h belong to $BMO(\mathbb{R})$ and $f \in L^1(\mathbb{R})$ be a function with compact support, then we have*

$$\left\| A^{\Theta}(h,f)(\cdot,\mu) \right\|_{BMO} = O\left(\frac{1}{|2\pi b|} \right).$$

The following result's proof is comparable to that of Theorem 2.4.

Corollary 2.7. *Let χ, ρ be compactly supported functions in $L^1(\mathbb{R})$. If $h_1, h_2 \in BMO(\mathbb{R})$, then*

$$\left\|A^{\Theta}(h_1,\chi)(\cdot,\mu) - A^{\Theta}(h_2,\rho)(\cdot,\mu)\right\|_{BMO} \leq \frac{1}{|2\pi b|}\Big(\|\chi-\rho\|_1 \left(\|h_1\|_{BMO} + 2M\right) + \|\rho\|_1 \left(\|h_1 - h_2\|_{BMO} + 2M\right) \Big).$$

2.2. Continuity on WHS and WBMO spaces

In this section, we demonstrate the continuity of LCAF in both the weighted Hardy space and the weighted BMO space. For this section, the following assumption is made: the functions, denoted by f, χ and ρ are assumed to be compactly supported, with their support belonging to a ball of radius r centred on the origin.

Theorem 2.8. *Let $\omega \in \mathcal{W}$. If f is in $L^1(\mathbb{R})$, then the operator $A^{\Theta}(\cdot,f) : H^1_{\omega}(\mathbb{R}) \to H^1_{\omega}(\mathbb{R})$, $h \to A^{\Theta}(h,f)(\cdot,\mu)$ is continuous. Moreover,*

$$\left\|A^{\Theta}(h,f)(\cdot,\mu)\right\|_{H^1_{\omega}} \leq \frac{1}{|2\pi b|}(1+Cr)^N \|h\|_{H^1_{\omega}} \|f\|_1 .$$

Proof. Let us take any function $h \in H^1_{\omega}(\mathbb{R})$. Then $h \in L^1_{\omega}(\mathbb{R})$ and so we get $A^{\Theta}(h,f)(\cdot,\mu) \in L^1_{\omega}(\mathbb{R})$ as follows:

$$\begin{aligned}
\left\|A^{\Theta}(h,f)(\cdot,\mu)\right\|_{1,\omega} &= \int_{\mathbb{R}} \left| \frac{1}{2\pi b}\, e^{\frac{i}{2b}u\,(au-\mu)} \int_{\mathbb{R}} h(m+u)\,\overline{f(m)}\, e^{\frac{i}{b}m\,(au-\mu)}\, dm \right| \omega(u)du \\
&\leq \frac{1}{|2\pi b|} \int_{\mathbb{R}} |f(m)| \left(\int_{\mathbb{R}} |h(\vartheta)|\, \omega(\vartheta - m)\, d\vartheta \right) dm \\
&\leq \frac{1}{|2\pi b|} \left(\int_{|m|\leq r} |f(m)|\, (1 + C\,|m|)^N dm \right) \left(\int_{\mathbb{R}} |h(\vartheta)|\, \omega(\vartheta)\, d\vartheta \right) \\
&\leq \frac{1}{|2\pi b|}(1+Cr)^N \|h\|_{1,\omega} \|f\|_1 < \infty.
\end{aligned}$$

Furthermore, utilizing the equality

$$\begin{aligned}
\left|\left(A^{\Theta}(h,f)(\cdot,\mu) * \varphi_t\right)(u)\right| &= \left| \int_{\mathbb{R}} \left(\frac{1}{2\pi b} \int_{\mathbb{R}} \overline{f(m)\, e^{\frac{i}{b}m\mu}}\, h\,(m+u-\eta)\;\, e^{\frac{i}{b}u(ma+\frac{a(u-\eta)-\mu)}{2}}\, dm \right) \varphi_t(\eta)\, d\eta \right| \\
&\leq \frac{1}{|2\pi b|} \int_{\mathbb{R}} |f(m)| \int_{\mathbb{R}} |h(m+u-\eta)|\, |\varphi_t(\eta)|\, d\eta\, dm \\
&= \frac{1}{|2\pi b|} \int_{\mathbb{R}} |f(m)|\, (|h| * |\varphi_t|)(m+u)\, dm.
\end{aligned}$$

Applying the Minkowski's inequality for integrals, we have

$$\begin{aligned}
\left\|A^{\Theta}(h,f)(\cdot,\mu)\right\|_{H^1_{\omega}} &= \int_{\mathbb{R}} \sup_{t>0} \left|\left(A^{\Theta}(h,f)(\cdot,\mu) * \varphi_t\right)(u)\right| \omega(u)\, du \\
&\leq \frac{1}{|2\pi b|} \int_{\mathbb{R}} |f(m)| \left(\int_{\mathbb{R}} \sup_{t>0} (|h| * |\varphi_t|)(m+u)\, \omega(u)\, du \right) dm \\
&= \frac{1}{|2\pi b|} \int_{\mathbb{R}} |f(m)| \left(\int_{\mathbb{R}} \sup_{t>0} (|h| * |\varphi_t|)(\eta)\, \omega(\eta - m)\, d\eta \right) dm \\
&\leq \frac{1}{|2\pi b|} \left(\int_{|m|\leq r} |f(m)|\, (1 + C|m|)^N \right) \left(\int_{\mathbb{R}} \sup_{t>0} (|h| * |\varphi_t|)(\eta)\, \omega(\eta)\, d\eta \right) \\
&\leq \frac{1}{|2\pi b|}(1+Cr)^N \|f\|_1 \|h\|_{H^1_{\omega}} .
\end{aligned}$$

□

The continuity of LCAF on BMO_{ω} is expressed by the following theorem.

Theorem 2.9. *Let f be in $L^1(\mathbb{R})$ and $\omega \in \mathcal{W}$. Then $A^{\Theta}(\cdot, f) : BMO_\omega(\mathbb{R}) \to BMO_\omega(\mathbb{R})$, $h \to A^{\Theta}(h, f)(\cdot, \mu)$ is continuous and holds*

$$\left\|A^{\Theta}(h, f)(\cdot, \mu)\right\|_{BMO_\omega} \leq \frac{1}{|2\pi b|}\,(1 + Cr)^{2N}\,(\|h\|_{BMO_\omega} + 2M)\,\|f\|_1\,,$$

where M denotes the boundedness constant from the BMO definition.

Proof. Let I be a compact ball in $\mathbb{R}$ and $h \in BMO_\omega(\mathbb{R})$. It is simple to see that $A^{\Theta}(h, f)(\cdot, \mu) \in L^{1,\omega}_{loc}(\mathbb{R})$ using a proof method analogous to that in (2.9). Furthermore, by using (2.10), we find

$$\begin{aligned}
\left\|A^{\Theta}(h, f)(\cdot, \mu)\right\|_{BMO_\omega} &= \sup_{I\subset\mathbb{R}} |I|_\omega^{-1} \int_I \left|A^{\Theta}(h, f)(u, \mu) - I\left(A^{\Theta}(h, f)\right)\right| \omega(u)\,du \\
&\leq \frac{1}{|2\pi b|} \int_{\mathbb{R}} |f(m)| \left(\sup_{I\subset\mathbb{R}} |I|_\omega^{-1} \int_I \left|h(m+u) - |I|_\omega^{-1} \int_I h(m+\eta)\,d\eta\right| \omega(u)\,du\right. \\
&\quad \left. + 2\sup_{I\subset\mathbb{R}} |I|_\omega^{-1} \int_I |I|_\omega^{-1} \int_I |h(m+\eta)|\,d\eta\,\omega(u)\,du\right) dm.
\end{aligned}$$

Let us say $I' = I + m$, $m \in \mathbb{R}$. Hence, we get

$$|I'|_\omega = \int_{I'} \omega(\vartheta)\,d\vartheta = \int_I \omega(m+u)\,du \leq (1 + C|m|)^N\,|I|_\omega$$

and so

$$\begin{aligned}
\left\|A^{\Theta}(h, f)(\cdot, \mu)\right\|_{BMO_\omega} &\leq \frac{1}{|2\pi b|} \int_{\mathbb{R}} |f(m)| \left((1 + C|m|)^{2N} \sup_{I'\subset\mathbb{R}} \left|I'\right|_\omega^{-1} \int_{I'} \left|h(\vartheta) - |I'|^{-1} \int_{I'} h(z)\,dz\right| \omega(\vartheta)\,d\vartheta\right. \\
&\quad \left. + 2(1 + C|m|)^{2N} \sup_{I'\subset\mathbb{R}} \left|I'\right|_\omega^{-1} \int_{I'} |I'|^{-1} \int_{I'} |h(z)|\,dz\,\omega(\vartheta)\,d\vartheta\right) dm \\
&\leq \frac{1}{|2\pi b|} \int_{|m|\leq r} |f(m)|\,(1 + C|m|)^{2N} \left(\|h\|_{BMO_\omega} + 2\,|I'|_\omega^{-1} M\,|I'|_\omega\right) dm \\
&\leq \frac{1}{|2\pi b|}(1 + Cr)^{2N}\,(\|h\|_{BMO_\omega} + 2M)\,\|f\|_1\,.
\end{aligned}$$

□

In what follows, the H^1_ω-distance and BMO_ω-distance of two LCAFs are presented. Using the principles of Theorems 2.8 and 2.9, the proof of the following result can be carried out similarly to that of Theorem 2.4.

Corollary 2.10. *Let $\omega \in \mathcal{W}$ and $\chi, \rho \in L^1(\mathbb{R})$.*

i) *If $h_1, h_2 \in H^1_\omega(\mathbb{R})$, then*

$$\left\|A^{\Theta}(h_1,\chi)(\cdot, \mu) - A^{\Theta}(h_2,\rho)(\cdot, \mu)\right\|_{H^1_\omega} \leq \frac{1}{|2\pi b|}(1 + Cr)^N \left(\|\chi - \rho\|_1\,\|h_1\|_{H^1_\omega} + \|\rho\|_1\,\|h_1 - h_2\|_{H^1_\omega}\right).$$

ii) *If h_1 and h_2 are in $BMO_\omega(\mathbb{R})$, then*

$$\begin{aligned}
&\left\|A^{\Theta}(h_1,\chi)(\cdot, \mu) - A^{\Theta}(h_2,\rho)(\cdot, \mu)\right\|_{BMO_\omega} \\
&\qquad \leq \frac{1}{|2\pi b|}(1 + Cr)^{2N}\Big(\|\chi - \rho\|_1\,(\|h_1\|_{BMO_\omega} + 2M) + \|\rho\|_1\left(\|h_1 - h_2\|_{BMO_\omega} + 2M\right)\Big).
\end{aligned}$$

3. Conclusion

Pei and Ding [7] analyzed the LCAF in accordance with the conventional notion of AF, and they deduce some important characteristics. Subsequent research has addressed this in a number of papers [12, 15]. This study explores the continuity properties of LCAF on Hardy, WHS, BMO and WBMO spaces. These function spaces are central to theoretical mathematics, particularly within the realms of operator theory and singular integrals. By focusing on these challenging analytical frameworks, this study not only bridges significant theoretical foundations with practical inquiry but also establishes a basis for potential novel applications in engineering.

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