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# Complex fractal interpolation on 1-chains

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**Abstract**

In this paper, we introduce the novel concept of fractal interpolation on 1-chains in the complex plane. The approach presented here extends to a certain degree the results obtained by the author for curves in Banach spaces (*cf.* [18]). We work within the framework of iterated function systems and fractal functions and employ their fundamental properties which are mainly based on the Banach fixed point theorem. In particular, we establish the existence of continuous and $L^p$ fractal interpolation functions defined on 1-chains and present some examples. In addition, we derive an associated fractal operator for a space of continuous functions on 1-chains and discuss several of its properties.

*Keywords:* 1-chain, iterated function system, fractal interpolation, Read-Bajraktarević operator

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## 1. Introduction

Fractal interpolation functions, i.e., functions whose graphs are fractal sets and interpolate a given set of data points were first introduced via iterated functions systems (*cf.* [2, 11]). The main purpose of fractal interpolation theory is the description and modeling of intrinsically occurring complex geometric self-referential structures and to employ approximants that are well-suited to be adapted to these types of structures. This program has been successfully carried out over the last few decades and several different approaches and methodologies for fractal interpolation have been developed. The literature on the subject is vast and still expanding. The interested reader can find some of these constructions (*cf.* [13, 15]) or by searching for fractal functions using expressions such as "hidden variable," "$V$-variable," "coalescent," "super," "zipper," "radial," and "$\alpha$-fractal functions" to just name a few.

A fractal function generally satisfies a fixed point equation and the associated operator, called a Read-Bajraktarević operator, lies at the core of the theory. The domain of this operator is in many cases a space of functions and these functions are defined on certain subsets of Banach spaces, Hilbert spaces, or complete metric spaces.

Fractal interpolation over curves in Banach spaces was first introduced in [18] and it is therefore natural to consider the extension to 1-chains where the latter are defined as elements of a free abelian groups on a domain of the complex plane.

The organization of this paper is as follows. In Section 2, we give the formal definition of a 1-chain as an element of the free abelian group generated by paths lying in a domain of the complex plane. Then, in the next section, we present iterated function systems (IFSs) as they were used in the past to construct the graphs of fractal functions.

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Section 4 then gives a short introduction to the concept of fractal interpolation highlighting those themes that are necessary for the remainder of the paper. In particular, we emphasize the associated affine operator necessary to construct the type of fractal function considered here and mention the relation to the attractors of IFSs. The next two sections introduce the novel concept of continuous fractal interpolation over 1-chains and present a simple example. In a short Section 6, we also consider the setting of Lebesgue spaces. A fractal operator acting on functions defined on 1-chains is discussed in Section 7 and some of its properties are presented. A conclusion finishes the paper.

## 2. 1-chains in the complex plane

Here, we introduce some of the concepts needed in the remainder of this paper. The first such concept is that of a free abelian group. We follow the standard presentation of free abelian groups (*cf.* [22]) and we refer to the literature (*cf.* [11, 12]) for further details about 1-chains.

To this end, let $S$ be an arbitrary nonempty set. Let

$$F := \{f : S \to \mathbb{Z} : f(x) \neq 0, \text{ for only finitely many } s \in S\}.$$

Define an operation $+ : F \times F \to F$ by

$$\begin{aligned} (f, g) &\mapsto f + g, \\ (f + g)(s) &:= f(s) + g(s), \quad \forall\, s \in S. \end{aligned}$$

Hence, $(F, +)$ is an abelian group. Let $f_s \in F$ be defined by

$$f_s(t) := \begin{cases} 1, & \text{if } t = s; \\ 0, & \text{if } t \neq s. \end{cases}$$

Then, $\{f_s : s \in S\}$ constitutes a bases for $(F, +)$ as every $f \in F$ can be written in the form

$$f = \sum \alpha_\nu f_{s_\nu} \quad \text{with} \quad \alpha_\nu = f(s_\nu) \in \mathbb{Z}. \tag{2.1}$$

The mapping $s \mapsto f_s$ is bijective and, therefore, $\operatorname{card} S = \operatorname{card}\{f_s : s \in S\}$. For this reason, it is customary to write $s$ instead of $f_s$ and to express (2.1) as $\sum \alpha_\nu s_\nu$. The correspondence $\sum \alpha_\nu f_{s_\nu} \mapsto \sum \alpha_\nu s_\nu$ generates an abelian group containing $S$ which is isomorphic to $(F, +)$. This group is called the free abelian group generated by $S$ and is denoted by $F(S)$.

For our later purposes, we also need to following definitions (*cf.* [22, Theorem 22]).

**Definition 2.1.** Let $D \subseteq \mathbb{C}$ be a domain, i.e., a nonempty open connected subset of $\mathbb{C}$.

A Jordan arc in $D$ is the image of an injective continuous map of a compact interval $[a, b] \subset \mathbb{R}$ into $D$.

A Jordan curve or a simple closed curve in $D$ is the image of an injective continuous map of the unit circle $S^1 \subset \mathbb{R}^2$ into $D$.

A path is either a Jordan arc or a Jordan curve.

In the following, $D$ will always denote a domain in $\mathbb{C}$. Let $\Gamma := \{\gamma : [0, 1] \to D \subseteq \mathbb{C} : \gamma \text{ is a path}\}$. Here, we assume without loss of generality that all paths are defined on the interval $[0, 1]$.

The free abelian group $F(\Gamma)$ generated by the set $\Gamma$ is called the group of 1-chains and is denoted by $C_1 = C_1(D)$. Every element of $\gamma \in C_1$ has the form

$$\gamma = \sum \alpha_\nu \gamma_\nu,$$

where $\gamma_\nu \in \Gamma$ are distinct paths and $\alpha_\nu \in \mathbb{Z}$. We also write $\gamma_\nu \in \gamma$ to indicate that $\gamma_\nu \in \Gamma$ is a path which is a part of $\gamma \in C_1$.

The trace $\operatorname{tr}\gamma$ of a chain $\gamma = \sum_{\nu=1}^{m} \alpha_\nu \gamma_\nu$ is the compact subset of $D$ defined by

$$\operatorname{tr}\gamma := \bigcup_{\nu=1}^{m} \operatorname{tr}\gamma_\nu,$$

where $\operatorname{tr}\gamma_\nu := \gamma_\nu([0,1])$. Note that if $\alpha \in \mathbb{Z}$ and $\gamma \in C_1$ then

$$\operatorname{tr}(\alpha\,\gamma) = \begin{cases} \emptyset, & \alpha = 0; \\ \operatorname{tr}\gamma, & \alpha \neq 0. \end{cases}$$

Based on the definition of pushforward of chains (*cf.* [24]), a continuous function $f : D \to \mathbb{C}$ is applied to 1-chains as a group homomorphism from the free abelian group $C_1$ to the abelian group $\mathbb{C}$. By a slight abuse of notation, we denote this function again by $f$. More precisely, for $\gamma = \sum_{\nu=1}^{m} \alpha_\nu \gamma_\nu \in C_1$, we set

$$f(\gamma) = f\left(\sum_{\nu=1}^{m} \alpha_\nu \gamma_\nu\right) := \sum_{\nu=1}^{m} \alpha_\nu f(\gamma_\nu), \tag{2.2}$$

as the pushforward of a 1-chain should again be a 1-chain.

We define a valuation $v : C_1 \to \mathbb{R}$ by the linear map (*cf.* [12])

$$v(\gamma) := \sum \alpha_\nu v(\gamma_\nu). \tag{2.3}$$

For the purposes of this paper, we will use the following two valuations on the set of 1-chains:

$$v(\gamma_\nu) := |\gamma_\nu| := \max_{t\in[0,1]} |\gamma_\nu(t)| \tag{2.4}$$

and

$$v(\gamma_\nu) := |\gamma_\nu|_p := \left(\int_0^1 |\gamma_\nu(t)|^p dt\right)^{1/p}, \quad p \in [1,\infty), \tag{2.5}$$

and their extensions to $\gamma$ via (2.3).

## 3. Iterated function systems

We will start by recalling some results from the theory of IFSs and their attractors. For general literature on fractals generated by IFSs and their properties, we refer the interested reader to the standard literature on these topics (*cf.* [2, 11, 15]).

**Definition 3.1.** A function $f : \mathsf{X} \to \mathsf{X}$ on a metric space $(\mathsf{X}, d_\mathsf{X})$ is a contraction if and only if $\operatorname{Lip}(f) < 1$, where

$$\operatorname{Lip}(f) := \sup_{x,y\in\mathsf{X}, x\neq y} \frac{d_\mathsf{X}(f(x), f(y))}{d_\mathsf{X}(x,y)}.$$

In the following, we use $\mathbb{N}_n$ to denote the subset $\{1,\ldots,n\}$ of $\mathbb{N}$ and set $\mathbb{N}_{0,n} := \{0\} \cup \mathbb{N}_n$.

**Definition 3.2** (*cf.* [2])**.** Let $(\mathsf{X}, d_\mathsf{X})$ be a complete metric space and let $\mathcal{F} := \{f_i : \mathsf{X} \to \mathsf{X},\ i \in \mathbb{N}_n\}$ be a set of contractions with contractivity constants $c_i \in [0,1)$. Then the pair $(\mathsf{X}, \mathcal{F})$ is called a hyperbolic IFS.

In this article, we exclusively deal with hyperbolic IFSs and therefore we drop the adjective "hyperbolic."

To construct attractors for IFSs, we need to consider a set-valued map associated with the collection $\mathcal{F}$ of point maps. For this purpose, denote by $\mathcal{H}(\mathsf{X})$ the set of all nonempty compact subsets of $\mathsf{X}$ endowed with the Hausdorff-Pompeiu metric $h : \mathcal{H}(\mathsf{X}) \times \mathcal{H}(\mathsf{X}) \to \mathbb{R}$ (*cf.* [10]):

$$h(A,B) := \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\}.$$

It is known (*cf.* [7, Theorem 17.6.5]) that if $(\mathsf{X}, d_\mathsf{X})$ is complete in the metric $d$ then $(\mathcal{H}(\mathsf{X}), h)$ is complete in the metric $h$.

The next result presents a way of constructing attractors of IFSs as fixed points of a certain set-valued map.

**Theorem 3.3** (*cf.* [2, 11, 15])**.** *Let* $(\mathsf{X}, \mathcal{F})$ *be an IFS on a complete metric space* $(\mathsf{X}, d_{\mathsf{X}})$*. Then, there exists a unique element* $\mathcal{A} \in \mathcal{H}(\mathsf{X})$ *such that* $\mathcal{F}(\mathcal{A}) = \mathcal{A}$*, where*

$$\begin{aligned} \mathcal{F} &: \mathcal{H}(\mathsf{X}) \to \mathcal{H}(\mathsf{X}), \\ \mathcal{F}(A) &:= \bigcup_{f_i \in \mathcal{F}} f_i(A), \end{aligned} \tag{3.1}$$

*for* $A \in \mathcal{H}(\mathsf{X})$*. Moreover,* $\mathcal{F}$ *is also a contraction on* $\mathcal{H}(\mathsf{X})$ *with contractivity constant* $c = \max_{i \in \mathbb{N}_n} c_i$*.*

The nonempty compact subset $\mathcal{A}$ is called the attractor of or the fractal (set) generated by the IFS $(\mathsf{X}, \mathcal{F})$. The reason for this terminology lies in the fact that $\mathcal{A}$ satisfies the self-referential (fixed point) equation

$$\mathcal{A} = \mathcal{F}(\mathcal{A}) = \bigcup_{f_i \in \mathcal{F}} f_i(\mathcal{A}) = \bigcup_{f_{i_1} \in \mathcal{F}} \cdots \bigcup_{f_{i_k} \in \mathcal{F}} f_{i_1} \circ \cdots \circ f_{i_k}(\mathcal{A}),$$

which expresses the attractor $\mathcal{A}$ at any given level as a finite copy of images of itself; one of the characteristics of a fractal.

We remark that the Banach fixed point theorem also provides a means of constructing the attractor or fractal: Choose any $A_0 \in \mathcal{H}(\mathsf{X})$ and, for $k \in \mathbb{N}$, define $A_k := \mathcal{F}^k(A_0)$, where $\mathcal{F}^k$ denotes the $k$-fold composition of $\mathcal{F}$. Then, $\mathcal{A}$ is given by $\mathcal{A} = \lim_{k \to \infty} \mathcal{F}^k(A_0)$ where the limit is taken with respect to the Hausdorff-Pompeiu metric $h$.

One particular class of attractors of iterated function systems consists of the graphs of functions, for instance interpolating a given set of data points or belonging to certain function spaces (*cf.* [1, 14]). Fractal interpolation arising in the context of IFSs is the topic of the next section. (See also, Remark 4.5 and Diagram 4.5.)

## 4. Fractal interpolation

The purpose of the current section is to introduce the rudimentaries of fractal interpolation and to relate the fractal interpolants to the solution of a system of functional equations.

Let $\mathsf{X} := (\mathsf{X}, d_{\mathsf{X}})$ be a compact metric space. Suppose $(\mathsf{Y}, d_{\mathsf{Y}})$ is an $F$-space (*cf.* [24]), i.e., a complete metric linear space endowed with a complete translation-invariant metric where addition and scalar multiplication are continuous operations. Recall that $d_{\mathsf{Y}}$ is a translation-invariant metric if

$$d_{\mathsf{Y}}(y_1 + a, y_2 + a) = d_{\mathsf{Y}}(y_1, y_2), \quad \text{for all} \quad a, y_1, y_2 \in \mathsf{Y}.$$

Suppose further that the metric $d_{\mathsf{Y}}$ is homogeneous, i.e.,

$$d_{\mathsf{Y}}(\alpha y_1, \alpha y_2) = |\alpha|\, d_{\mathsf{Y}}(y_1, y_2), \quad \forall\, \alpha \in \mathbb{K},\ \forall\, y_1, y_2 \in \mathsf{Y},$$

where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Let $\mathcal{L} := \{l_i : \mathsf{X} \to \mathsf{X} : i \in \mathbb{N}_n\}$ be a given finite family of contractions. Then, $(\mathsf{X}, \mathcal{L})$ is an IFS with (compact) attractor $\mathsf{I} \subseteq \mathsf{X}$.

Let $\{(x_j, y_j) \in \mathsf{I} \times \mathsf{Y},\ j \in \mathbb{N}_{0,n}\}$ be a set of interpolation points. Denote the restriction of the maps $l_i|_{\mathsf{I}}$ again by $l_i$ and assume that the maps $l_i$ are invertible over their ranges $l_i(\mathsf{I}) =: \mathsf{I}_i$. Furthermore, assume that

$$\forall\, i, j \in \mathbb{N}_n,\ |i - j| \notin \{0, 1\} : \mathsf{I}_i \cap \mathsf{I}_j = \emptyset; \tag{4.1}$$

$$\mathsf{I}_i \cap \mathsf{I}_{i+1} = \{x_i\}, \quad i \in \mathbb{N}_{n-1}. \tag{4.2}$$

We are looking for a global function $\psi : \mathsf{I} = \bigcup_{i=1}^{n} \mathsf{I}_i \to \mathsf{Y}$ satisfying a system of $n$ functional equations of the local form

$$\psi(l_i(x)) = q_i(x) + s_i(x)\psi(x), \quad \text{on } \mathsf{I}_i \text{ and for } i \in \mathbb{N}_n, \tag{4.3}$$

where $s_i$ is a given continuous function $\mathsf{I} \to \mathbb{R}$, called a scaling function, and $q_i$ a continuous function $\mathsf{I} \to \mathsf{Y}$. Note that as $\mathsf{Y}$ is an $F$-space, the expression on the right side of (4.3) is well-defined.

The principal idea is to consider (4.3) as the fixed point equation of an associated affine operator called a Read-Bajractarević (RB) operator acting on an appropriately defined space of functions (*cf.* [6, 8, 15]).

To this end, let

$$C_*(\mathsf{I},\mathsf{Y}) := \{f : \mathsf{I} \to \mathsf{Y} : f \text{ is continuous and } f(x_0) = y_0,\, f(x_n) = y_n\}$$

denote the complete metric space of continuous functions equipped with the metric

$$d(f,g) := \sup_{x\in\mathsf{I}} d_{\mathsf{Y}}(f(x), g(x)).$$

Define an RB operator $T : C_*(\mathsf{I},\mathsf{Y}) \to C_*(\mathsf{I},\mathsf{Y})$, by

$$(Tf)(l_i(x)) := q_i(x) + s_i(x)\cdot f(x), \quad x \in \mathsf{I}, \tag{4.4}$$

for $i \in \mathbb{N}_n$, or, equivalently, by

$$\begin{aligned} Tf(x) &= \sum_{i=1}^{n} (q_i \circ l_i^{-1})(x)\chi_{\mathsf{I}_i}(x) + \sum_{i=1}^{n} (s_i \circ l_i^{-1})(x)\cdot(f\circ l_i^{-1})(x)\chi_{\mathsf{I}_i}(x) \\ &= T(0) + \sum_{i=1}^{n} (s_i \circ l_i^{-1})(x)\cdot(f\circ l_i^{-1})(x)\chi_{\mathsf{I}_i}(x), \quad x \in \mathsf{I}_i, \end{aligned}$$

where $\chi_S$ denotes the characteristic function of a set $S$.

The following result is well-known (*cf.* [1, 4, 15, 25]).

**Theorem 4.1.** *The system of functional equations* (4.3) *has a unique continuous solution* $\psi \in C_*(\mathsf{I},\mathsf{Y})$ *provided that*

**(i)** *The functions* $l_i$*,* $q_i$*, and* $s_i$ *are continuous,*

**(ii)** $s := \max_{i\in\mathbb{N}_n} \sup_{x\in\mathsf{I}} |s_i(x)| < 1$,

**(iii)** *and* $\forall\, i, j \in \mathbb{N}_n$ *and* $\forall\, x_1, x_2 \in \mathsf{I}$*:*

$$\lim_{x\to x_1} l_j(x) = l_i(x_2) \quad \Longrightarrow \quad \lim_{x\to x_1} q_j(x) + s_j(x)\psi(x) = q_j(x_2) + s_j(x_2)\psi(x_2).$$

*The function* $\psi \in C_*(\mathsf{I},\mathsf{Y})$ *satisfies the fixed point or self-referential equation* $\psi = T\psi$*.*

*Proof.* We highlight the main ideas of the proof as it employs well-known and standard arguments (*cf.* [1, 4, 15, 25]). Conditions (i) and (iii) together with the equations (4.1) and (4.2) guarantee that $T$ maps $C_*(\mathsf{X},\mathsf{Y})$ into itself. To establish Condition (ii) and thus show that $T$ is contractive on $C_*(\mathsf{I},\mathsf{Y})$, we use the fact that $\mathsf{Y}$ has a translation-invariant and homogeneous metric. Indeed, for $x \in \mathsf{I}_i$,

$$\begin{aligned} d_{\mathsf{Y}}(Tf(x), Tg(x)) &= d_{\mathsf{Y}}(T(0) + (s_i \circ l_i^{-1})(x)\cdot(f\circ l_i^{-1})(x), T(0) + (s_i \circ l_i^{-1})(x)\cdot(g\circ l_i^{-1})(x)) \\ &= d_{\mathsf{Y}}((s_i \circ l_i^{-1})(x)\cdot(f\circ l_i^{-1})(x), (s_i \circ l_i^{-1})(x)\cdot(g\circ l_i^{-1})(x)) \\ &= \left|(s_i \circ l_i^{-1})(x)\right| d_{\mathsf{Y}}((f\circ l_i^{-1})(x), (g\circ l_i^{-1})(x)) \\ &\le s\, d_{\mathsf{Y}}(f,g), \end{aligned}$$

where we used the translation-invariance of $d_{\mathsf{Y}}$ to obtain the second equality and its homogeneity to deduce the third equality.

Thus, by the completeness of the metric and the Banach Fixed Point theorem, $T$ has a unique fixed point $\psi \in C_*(\mathsf{I},\mathsf{Y})$. □

*Remark* 4.2. The fixed point $\psi \in C_*(\mathsf{I}, \mathsf{Y})$ of the RB operator $T$ is referred to as a *continuous fractal function*. The self-referential equation $T\psi = \psi$ expresses the fractal nature of the graph of $\psi$: It is made up of a finite number of copies of itself with each copy being supported on the local sets $\mathsf{I}_i$. The fixed point $\psi$ depends on $n$, the sets $\{\mathsf{I}_i : i \in \mathbb{N}_n\}$ determined by the functions $\boldsymbol{l} := (l_1, \ldots, l_n)$, the $n$-tuple of functions $\boldsymbol{s} := (s_1, \ldots, s_n)$ and $\boldsymbol{q} := (q_1, \ldots, q_n)$ with different choices yielding different continuous fractal functions. Therefore, the fixed point $\psi$ is actually a three-parameter family $\psi = \psi(\boldsymbol{l}, \boldsymbol{s}, \boldsymbol{q})$ of functions. Emphasizing the dependence of $\psi$ on the functions $s_i$, the expression *$\boldsymbol{s}$-fractal function* can be found in the literature (*cf.* [19, 20]). In this context, one considers a fractal function as the image of a given (non-fractal) function $f$ under a so-called fractal operator $\mathfrak{F}^s$. See also Section 7 for properties of the fractal operator in the current setting.

*Remark* 4.3. The constructive proof of Banach's Fixed Point Theorem provides an algorithm for the generation of $\psi$: Choose *any* function $\psi_0 \in C_*(\mathsf{I}, \mathsf{Y})$ and iteratively define the following sequence of functions:

$$\psi_k := T\psi_{k-1}, \quad k \in \mathbb{N}.$$

Then, $\psi = \lim_{k\to\infty} \psi_k$ where the limit is taking with respect to the metric $d_{\mathsf{Y}}$.

*Remark* 4.4. Functional equations such as (4.3) are also related to *fractels* (*cf.* [3, 15]) and the approximation theory of so-called rough functions (*cf.* [5]).

*Remark* 4.5. There is a relation (*cf.* [16]) between the graph of a (continuous) fractal function and its interpretation as the attractor of an underlying IFS. This relationship can be summarized in the following commuting diagram:

$$\begin{array}{ccc} \mathcal{H}(\mathsf{X}) & \xrightarrow{\mathcal{F}} & \mathcal{H}(\mathsf{X}) \\ \uparrow G & & \uparrow G \\ \mathbb{C}_*(\mathsf{I}, \mathsf{Y}) & \xrightarrow{T} & \mathbb{C}_*(\mathsf{I}, \mathsf{Y}) \end{array} \tag{4.5}$$

where $G$ is the mapping

$$C_*(\mathsf{I}, \mathsf{Y}) \ni g \mapsto G(g) = \{(x, g(x)) : x \in [a, b]\} \in \mathcal{H}(\mathsf{X}).$$

In other words, fractal functions may be defined using IFSs and the set-valued map $\mathcal{F}$ or an appropriate function space and the RB operator $T$. Diagram (4.5) expresses again the fact that the fixed point $\psi$, respectively its graph, has a fractal structure.

## 5. Continuous fractal interpolation on 1-chains

In this section, we introduce the concept of continuous fractal interpolation on 1-chains using contractive RB operators defined on an appropriate function space.

Suppose that $\gamma \in C_1$ and $\gamma_\nu \in \gamma$, $\nu \in \mathbb{N}_m$. Set $I := [0, 1]$, let $1 < n_\nu \in \mathbb{N}$ and for $i \in \mathbb{N}_{n_\nu}$ define contractive maps $l_{\nu,i} : I \to I$ which are invertible on their ranges $I_{\nu,i} := l_{\nu,i}(I)$, satisfy (4.1) and (4.2), as well as

$$I = \bigcup_{i\in\mathbb{N}_{n_\nu}} l_{\nu,i}(I), \tag{5.1}$$

i.e., $I$ is the attractor of the IFS $\mathcal{I}_\nu := (I, \{l_{\nu,i} : i \in \mathbb{N}_{n_\nu}\})$. Denote the unique element in the intersection $I_{\nu,i} \cap I_{\nu,i+1}$ by $\{t_{\nu,i}\}$, $i \in \mathbb{N}_{n_\nu-1}$, and set $t_{\nu,0} := 0$ and $t_{\nu,n} := 1$. Then, $\{t_{\nu,0} < t_{\nu,1} < \cdots < t_{\nu,n_\nu}\}$ is a partition of $I$ induced by the maps $l_{\nu,i}$.

The parametrization $\gamma_\nu$ applied to $I$ induces then a partition $\Delta_\nu$ on $\operatorname{tr}\gamma_\nu$ via $z_{\nu,j} := \gamma_\nu(t_{\nu,j})$ for a Jordan arc and $z_{\nu,j} := \gamma_\nu \circ \exp(2\pi\mathrm{i}\, t_{\nu,j})$ for a Jordan curve, $j \in \mathbb{N}_{0,n_\nu}$. In the latter expression, we identified $\mathbb{R}^2$ with $\mathbb{C}$ via $\mathbb{R}^2 \ni (x, y) \mapsto x + \mathrm{i}\, y \in \mathbb{C}$. Note that $(\operatorname{tr}\gamma_\nu, \{\gamma_\nu \circ l_{\nu,i} : i \in \mathbb{N}_{n_\nu}\})$, respectively, $(\operatorname{tr}\gamma_\nu, \{\gamma_\nu \circ \exp(2\pi\mathrm{i}\, l_{\nu,i}) : i \in \mathbb{N}_{n_\nu}\})$ defines an IFS on $\operatorname{tr}\gamma_\nu$ with attractor $\operatorname{tr}\gamma_\nu$.

Now, suppose $D \subset \mathbb{C}$ is a domain, $\gamma \in C_1$, and $f \in C(D, \mathbb{C})$. The restriction of $f$ to $\operatorname{tr}\gamma$, denoted again by $f$, yields a function $f \in C(\operatorname{tr}\gamma, \mathbb{C})$. Then, $(f \circ \gamma)(t) = \sum \alpha_\nu (f \circ \gamma_\nu)(t)$, $t \in I$, or more succinctly, by denoting the local coordinate on $\gamma_\nu$ by $z_\nu$, $f(z) = \sum \alpha_\nu f(z_\nu)$.

For $\nu \in \mathbb{N}_m$, consider the complete metric space of complex-valued functions

$$C_*(\operatorname{tr}\gamma_\nu) := \{f : \operatorname{tr}\gamma_\nu \to \mathbb{C} : f(z_{\nu,0}) =: w_{\nu,0},\ f(z_{\nu,n_\nu}) =: w_{\nu,n_\nu}\},$$

for given $w_{\nu,0}, w_{\nu,n_\nu} \in \mathbb{C}$.

Using the convention in (2.2), we define an RB operator $T : C_*(\operatorname{tr}\gamma) \to C_*(\operatorname{tr}\gamma)$ by

$$Tf(\gamma) := \sum \alpha_\nu T_\nu f(\gamma_\nu), \tag{5.2}$$

where $T_\nu : C_*(\operatorname{tr}\gamma_\nu) \to C_*(\operatorname{tr}\gamma_\nu)$ is the RB operator given by

$$T_\nu f(\gamma_\nu \circ l_{\nu,i})(t) := q_{\nu,i} \circ \gamma_\nu(t) + s_{\nu,i} \circ \gamma_\nu(t) \cdot f \circ \gamma_\nu(t), \qquad t \in I, \tag{5.3}$$

or, equivalently,

$$T_\nu(f \circ \gamma_\nu)(t) := q_{\nu,i} \circ \gamma_\nu \circ l_{\nu,i}^{-1}(t) + (s_{\nu,i} \circ \gamma_\nu \circ l_{\nu,i}^{-1}(t)) \cdot (f \circ \gamma_\nu \circ l_{\nu,i}^{-1}(t)), \qquad t \in I_{\nu,i}. \tag{5.4}$$

Here, $q_{\nu,i}$ and $s_{\nu,i}$ denotes continuous functions $\operatorname{tr}\gamma_\nu \to \mathbb{C}$. We further require that

$$T_\nu(f \circ \gamma_\nu)(t_{\nu,i}-) = T_\nu(f \circ \gamma_\nu)(t_{\nu,i}+), \tag{5.5}$$

for all $i \in \mathbb{N}_{n_\nu}$. This will guarantee continuity across the interval $I = [0, 1]$. Eqn. (5.5) can also be expressed by a system of equations of the form

$$q_{\nu,i}(z_{\nu,n_\nu}) + s_{\nu,i}(z_{\nu,n_\nu}) \cdot w_{\nu,n_\nu} = q_{\nu,i+1}(z_{\nu,0}) + s_{\nu,i+1}(z_{\nu,0}) \cdot w_{\nu,0}, \quad i \in \mathbb{N}_{n_\nu}. \tag{5.6}$$

Imposing condition (5.5) or equivalently (5.6) implies that $T_\nu$ maps $C_*(\operatorname{tr}\gamma_\nu)$ into itself, and then a posteriori that $T$ maps $C_*(\operatorname{tr}\gamma)$ into itself.

Alternatively, instead of imposing (5.5), suppose that for each $\nu \in \mathbb{N}_m$ and a fixed compact subset $K \subset \mathbb{C}$,

$$P_\nu := \{(z_{\nu,j}, w_{\nu,j}) \in \operatorname{tr}\gamma_\nu \times K : j \in \mathbb{N}_{0,n_\nu}\}, \tag{5.7}$$

is a given set of interpolation points with $z_{\nu,j} := \gamma_\nu(t_{\nu,j})$. One can guarantee the validity of Eqs. (5.5) or (5.6) by requiring that the auxiliary functions

$$\begin{gathered} k_{\nu,i} : \operatorname{tr}\gamma_\nu \times K \to K, \\ k_{\nu,i}(z_\nu, w) := q_{\nu,i}(z_\nu) + s_{\nu,i}(z_\nu) \cdot w, \quad z_\nu = \gamma_\nu(t), \end{gathered}$$

satisfy the following conditions:

$$k_{\nu,i}(z_{\nu,0}, w_{\nu,0}) = w_{\nu,i-1} \quad \text{and} \quad k_{\nu,i}(z_{\nu,n_\nu}, w_{\nu,n_\nu}) = w_{\nu,i}, \quad i \in \mathbb{N}_{n_\nu}. \tag{5.8}$$

For then,

$$\begin{aligned} T_\nu(f \circ \gamma_\nu)(t_{\nu,i}-) &= k_{\nu,i}(\gamma_\nu(l_{\nu,i}^{-1}(t_{\nu,i})), f(\gamma_\nu)(l_{\nu,i}^{-1}(t_{\nu,i}))) \\ &= k_{\nu,i}(z_{\nu,n_\nu}, f(z_{\nu,n_\nu})) = k_{\nu,i}(z_{\nu,n_\nu}, w_{\nu,n_\nu}) \\ &= k_{\nu,i+1}(z_{\nu,0}, w_{\nu,0}) = k_{\nu,i+1}(z_{\nu,0}, f(\gamma_\nu)(t_{\nu,0})) \\ &= k_{\nu,i+1}(\gamma_\nu(l_{\nu,i+1}^{-1}(t_{\nu,i})), f(\gamma_\nu)(l_{\nu,i+1}^{-1}(t_{\nu,i}))) = T_\nu(f \circ \gamma_\nu)(t_{\nu,i}+). \end{aligned}$$

Moreover, in this case, the following equations hold at the endpoints:

$$\begin{aligned} T_\nu(f \circ \gamma_\nu)(t_{\nu,0}) &= k_{\nu,1}(\gamma_\nu(l_{\nu,0}^{-1}(t_{\nu,0})), f(\gamma_\nu)(l_{\nu,0}^{-1}(t_{\nu,0}))) \\ &= k_{\nu,1}(z_{\nu,0}, f(\gamma_\nu)(t_{\nu,0})) = k_{\nu,1}(z_{\nu,0}, w_{\nu,0}) = w_{\nu,0}, \\ T_\nu(f \circ \gamma_\nu)(t_{\nu,n_\nu}) &= k_{\nu,n_\nu}(\gamma_\nu(l_{\nu,n_\nu}^{-1}(t_{\nu,n_\nu})), f(\gamma_\nu)(l_{\nu,n_\nu}^{-1}(t_{\nu,n_\nu}))) \\ &= k_{\nu,n_\nu}(z_{\nu,n_\nu}, f(\gamma_\nu)(t_{\nu,n_\nu})) = k_{\nu,n_\nu}(z_{\nu,n_\nu}, w_{\nu,n_\nu}) = w_{\nu,n_\nu}. \end{aligned}$$

**Theorem 5.1.** *Suppose $D \subseteq \mathbb{C}$ is a domain and $\gamma \in C_1(D)$ with $\gamma = \sum \alpha_\nu \gamma_\nu$. For each $\nu \in \mathbb{N}_m$, let there be given a set of interpolation points $P_\nu := \{(z_{\nu,j}, w_{\nu,j}) \in \operatorname{tr}\gamma_\nu \times K : j \in \mathbb{N}_{0,n_\nu}\}$. On $C_*(\operatorname{tr}\gamma)$ define an RB operator $T$ of the form* (5.2) *where each $T_\nu$ is given by* (5.3)*. Further suppose that continuous functions $q_{\nu,i}, s_{\nu,i} : \operatorname{tr}\gamma_\nu \to \mathbb{C}$ are chosen such that the join-up conditions* (5.5) *or equivalently* (5.6) *are satisfied for all $i \in \mathbb{N}_{n_\nu}$. If*

$$\sigma := \max_{\nu\in\mathbb{N}_m} \max_{i\in\mathbb{N}_{n_\nu}} \max_{t\in[0,1]} \left|s_{\nu,i} \circ \gamma_\nu(t)\right| < 1, \tag{5.9}$$

*then the RB operator $T$ has a unique fixed point $\psi \in C_*(\operatorname{tr}\gamma)$. This fixed point has the form*

$$\psi(\gamma) = \sum \alpha_\nu \psi(\gamma_\nu), \tag{5.10}$$

*where each $\psi(\gamma_\nu)$ satisfies the self-referential equation*

$$\psi(\gamma_\nu)(l_{\nu,i})(t) := q_{\nu,i} \circ \gamma_\nu(t) + s_{\nu,i} \circ \gamma_\nu(t) \cdot \psi \circ \gamma_\nu(t), \quad i \in \mathbb{N}_{n_\nu}, \quad t \in I. \tag{5.11}$$

*$\psi$ is referred to as a continuous fractal interpolation function on the 1-chain $\gamma$.*

*Proof.* Only the contractivity of $T$ under condition (5.9) remains to be shown. For this purpose, we employ the specific from of an valuation on $C_1(D)$ mentioned in (2.4). Let $f, g \in C_*(\operatorname{tr}\gamma)$. Then, for $t \in I_{\nu,i}$:

$$\begin{aligned} |T_\nu f \circ \gamma_\nu(t) - T_\nu g \circ \gamma_\nu(t)| &= \left|s_{\nu,i} \circ \gamma_\nu \circ l_{\nu,i}^{-1}(t) \cdot (f-g) \circ \gamma_\nu \circ l_{\nu,i}^{-1}(t)\right| \\ &\le \sigma \cdot |(f-g) \circ \gamma_\nu|. \end{aligned}$$

By definition of the valuation, this implies that $|Tf(\gamma) - Tg(\gamma)| \le \sigma\, |f(\gamma) - g(\gamma)|$.

Eq. (5.11) is a direct consequence of the definitions. □

Next, we present an example of a continuous fractal interpolation function constructed on a single path $\gamma : [0,1] \to \mathbb{C}$.

**Example 5.2.** Let $\gamma(t) := e^{\mathrm{i}\pi t}$, $t \in [0,1]$, and let the interpolation set $P$ be given by

$$P := \left\{(0, -1-\mathrm{i}), \left(\tfrac{1}{4}, -1+\mathrm{i}\right), \left(\tfrac{1}{2}, 1+\mathrm{i}\right), \left(\tfrac{3}{4}, 1-\mathrm{i}\right), (1, -1-\mathrm{i})\right\}.$$

Note that giving $P$ is equivalent to providing a partition of $I$ by maps $l_i$ of the form $l_i(t) = \frac{1}{4}(t+i-1)$, $i \in \mathbb{N}_4$. The partition points on $\gamma$ are then given by $z_j = e^{\mathrm{i}\pi j/4}$, $j \in \mathbb{N}_{0,4}$. As the scaling functions we select constants $s_i \in \left(\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\right)$.

Choosing for the $q_i$ the affine functions $q_i(z) := a_i z + b_i$, the join-up conditions (5.6) become

$$a_i z_4 + b_i + s_i w_4 = a_{i+1} z_0 + b_{i+1} + s_{i+1} w_0, \quad i \in \mathbb{N}_3,$$

which are automatically satisfied by applying the eight interpolation conditions

$$\begin{aligned} k_i(z_0, w_0) &= w_{i-1} = a_i z_0 + b_i + s_i w_0, \\ k_i(z_4, w_4) &= w_i = a_i w_4 + b_i + s_i w_4. \end{aligned}$$

These uniquely determine the eight unknowns $a_i$ and $b_i$.

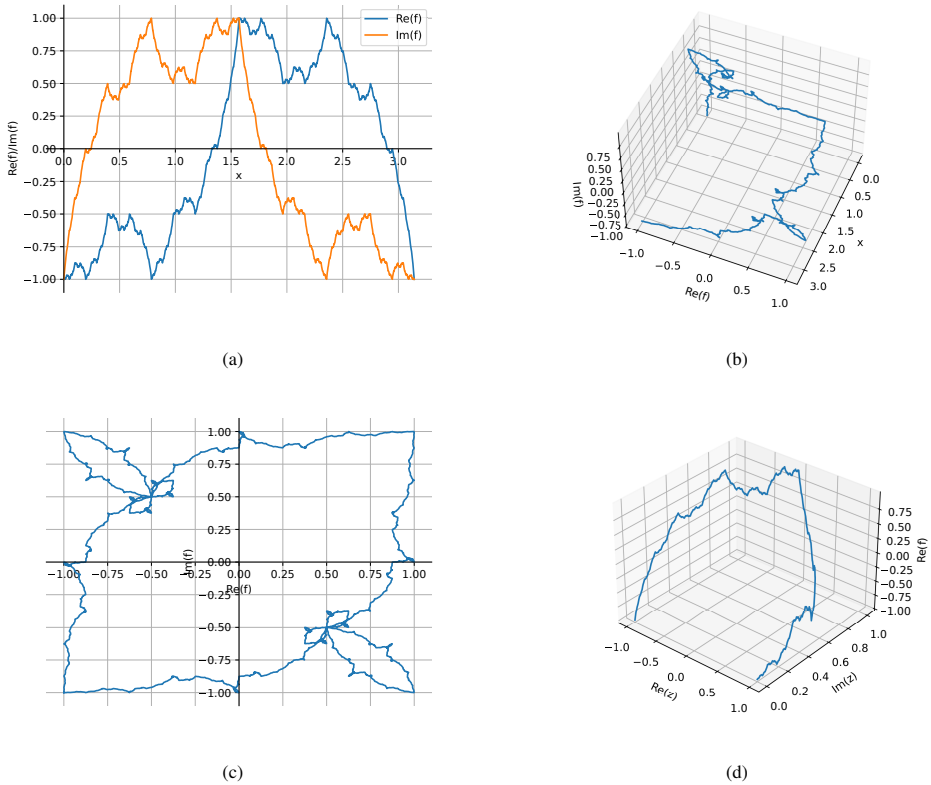
The interpolation conditions at the boundary points of $\operatorname{tr}\gamma$ read

$$a_1 z_0 + b_1 + s_1 w_0 = w_0 \quad \text{and} \quad a_4 z_4 + b_4 + s_4 w_4 = w_4.$$

The RB operator is then given by

$$Tf(e^{\mathrm{i}\pi t}) = a_i e^{\mathrm{i}\pi(4t-i+1)} + b_i + s_i f(e^{\mathrm{i}\pi(4t-i+1)}), \quad t \in \left[\tfrac{i-1}{4}, \tfrac{i}{4}\right], \quad i \in \mathbb{N}_4.$$

In Figure 1, several views of the graph of the function $f(\gamma(t))$ are shown.




(a) (b) (c) (d)

Figure 1: Sub-figure 1a shows the graphs of the functions $\mathrm{Re}(f(\gamma(t)))$ and $\mathrm{Im}(f(\gamma(t)))$. In sub-figure 1b, the values of $t$, $\mathrm{Re}(f(\gamma(t)))$, and $\mathrm{Im}(f(\gamma(t)))$ are laid out along the $x$, $y$ and $z$ axis, respectively. Sub-figure 1c shows the $\mathrm{Re}(f)$ versus the $\mathrm{Im}(f)$, and 1d is a plot of the $\mathrm{Re}(f)$ values along the curve. (Images courtesy of Aliaksei Kuzmenka, Student Project "Complex Fractal Interpolation on a Curve" supervised by Peter Massopust, Department of Mathematics, Technical University of Munich, Germany, March 2025)

## 6. $L^p$ fractal interpolation on 1-chains

In this short section, we provide a construction for fractal interpolation on a 1-chain in the setting of the Lebesgue spaces $L^p$, $p \in [1, \infty)$.

To this end, we consider the same initial set-up as in Section 5 but instead of considering the function space $C_*$, we choose $L^p$ instead. The RB operators $T$ and $T_\nu$ are selected as in Eqs. (5.2) and (5.3), respectively, but with complex-valued functions $q_{\nu,i} \in L^p(\operatorname{tr}\gamma_\nu)$ and $s_{\nu,i} \in L^\infty(\operatorname{tr}\gamma_\nu)$ instead. Then, the following theorem holds.

**Theorem 6.1.** *Suppose $D \subseteq \mathbb{C}$ is a domain and $\gamma \in C_1(D)$ with $\gamma = \sum \alpha_\nu \gamma_\nu$. On $L^p(\operatorname{tr}\gamma)$, $p \in [1, \infty)$, define an RB operator $T$ of the form* (5.2) *acting on $L^p(\operatorname{tr}\gamma_\nu)$ where each $T_\nu$ is given by* (5.3) *with functions $q_{\nu,i} \in L^p(\operatorname{tr}\gamma_\nu)$ and $s_{\nu,i} \in L^\infty(\operatorname{tr}\gamma_\nu)$. If*

$$\sigma_p := \max_{\nu \in \mathbb{N}_m} \max_{i \in \mathbb{N}_{n_\nu}} \|s_{\nu,i} \circ \gamma_\nu\|_\infty < 1, \tag{6.1}$$

*then the RB operator $T$ has a unique fixed point $\psi \in L^p(\operatorname{tr}\gamma)$. Here, $\|\cdot\|_\infty$ denotes the $L^\infty$-norm. This fixed point is given by*

$$\psi(\gamma) = \sum \alpha_\nu \psi(\gamma_\nu), \tag{6.2}$$

*where each $\psi_\nu$ satisfies a self-referential equation of the form* (5.11)*. Such a fixed point is termed a fractal interpolation function of class $L^p$ on the 1-chain $\gamma$.*

*Proof.* It is clear from the set-up that the RB operator $T$ maps $L^p(\operatorname{tr}\gamma)$ into itself as $L^\infty$ is a multiplier space for $L^p$. It suffices to establish contractivity under assumption (6.1). For this purpose, let $f, g \in L^p(\operatorname{tr}\gamma, \mathbb{C})$. Then, using the

validation given in Eqn. (2.5), we obtain

$$\begin{aligned}|T_\nu f \circ \gamma_\nu(t) - T_\nu g \circ \gamma_\nu(t)|_p^p &= \int_{I_{\nu,i}} \left|s_{\nu,i} \circ \gamma_\nu \circ l_{\nu,i}^{-1}(t) \cdot (f-g) \circ \gamma_\nu \circ l_{\nu,i}^{-1}(t)\right|^p dt \\ &\leq \sigma_p^p \cdot |(f-g) \circ \gamma_\nu|_p,\end{aligned}$$

using a change of variables. This establishes the result. □

## 7. A fractal operator

In this section, we introduce a fractal operator associated with a more general RB operator defined on $C_*(\operatorname{tr}\gamma)$ and discuss some of its properties.

We use the set-up at the beginning of Section 5. Suppose that $\psi \in C(\operatorname{tr}\gamma)$ is a fixed continuous complex-valued function. Let its restriction to $\gamma_\nu \in \gamma$ be denoted by $\psi_\nu$. For $\nu \in \mathbb{N}_m$, let

$$w_{\nu,0} := \psi_\nu(z_{\nu,0}) = \psi_\nu(\gamma_\nu(t_{\nu,0})) \quad \text{and} \quad w_{\nu,n_\nu} := \psi_\nu(z_{\nu,n_\nu}) = \psi_\nu(\gamma_\nu(t_{\nu,n_\nu})).$$

Now, let $b_\nu \in C_*(\operatorname{tr}\gamma_\nu)$ be a fixed function called the base function (*cf.* [1, 15]) and let

$$C_0(\operatorname{tr}\gamma_\nu) := \{f \in C(\operatorname{tr}\gamma_\nu) : f(z_{\nu,0}) = 0 = f(z_{\nu,n_\nu})\}.$$

Let $L_\nu : C_0(\operatorname{tr}\gamma_\nu) \to C_0(\operatorname{tr}\gamma_\nu)$ be a given bounded linear operator satisfying $L_\nu g(z_{\nu,0}) = 0 = L_\nu g(z_{\nu,n_\nu})$. Denote the class of these operators by $\mathcal{L}_\nu$. Define an RB operator $T_\Delta^L : C_*(\operatorname{tr}\gamma) \to C_*(\operatorname{tr}\gamma)$ by

$$T_\Delta^L g(\gamma) := \sum \alpha_\nu T_{\Delta_\nu}^{L_\nu} g(\gamma_\nu), \tag{7.1}$$

where $T_{\Delta_\nu}^{L_\nu}$ in an RB operator on $C_*(\operatorname{tr}\gamma_\nu)$ given by

$$T_{\Delta_\nu}^{L_\nu}(g \circ \gamma_\nu) := \psi_\nu \circ \gamma_\nu + L_\nu(g - b_\nu)(\gamma_\nu \circ l_{\nu,i}^{-1}), \quad \text{on } I_{\nu,i}. \tag{7.2}$$

We remark that choosing for $b_\nu$ the affine function connecting the points $(z_{\nu,0}, w_{\nu,0})$ and $(z_{\nu,n_\nu}, w_{\nu,n_\nu})$, the application of $L_\nu$ to the difference $g - b_\nu$, sometimes called the height function, scales along the vertical direction. In the literature, one finds for $L_\nu$ examples such as $L_\nu(g - b_\nu) := s_\nu \cdot (g - b_\nu)$ where either $s_\nu \in \mathbb{C}$ (*cf.* [1]) or an appropriately chosen function (*cf.* [15, 19, 20]).

Note that $T_{\Delta_\nu}^{L_\nu}$ depends on the choice of the operator $L_\nu$, the fixed base function $b_\nu$ and, through the $l_{\nu,i}$, also on the choice of the partition $\Delta_\nu$. In this section, it is important that this dependence is expressed explicitly in the notations with the exception of $b_\nu$ as these functions do not affect the contractivity of the RB operator (see proof of Theorem 7.1).

The continuity of $T_{\Delta_\nu}^{L_\nu} g$ follows from

$$\begin{aligned}(T_{\Delta_\nu}^{L_\nu} g)(z_{\nu,i}-) &= \psi_\nu(z_{\nu,i}) + L_\nu(g - b_\nu)(\gamma_\nu \circ l_{\nu,i}^{-1}(t_{\nu,i}-)) = \psi_\nu(z_{\nu,i}) + L_\nu(g - b_\nu)(z_{\nu,n_\nu}) = \psi_\nu(z_{\nu,i}), \\ (T_{\Delta_\nu}^{L_\nu} g)(z_{\nu,i}+) &= \psi_\nu(z_{\nu,i}) + L_\nu(g - b_\nu)(\gamma_\nu \circ l_{\nu,i+1}^{-1}(t_{\nu,i}+)) = \psi_\nu(z_{\nu,i}) + L_\nu(g - b_\nu)(z_{\nu,0}) = \psi_\nu(z_{\nu,i}),\end{aligned}$$

for all $i \in \mathbb{N}_{n_\nu}$.

Now, assume that $g, h \in C_*(\operatorname{tr}\gamma_\nu)$. Then,

$$\begin{aligned}\left|T_{\Delta_\nu}^{L_\nu} g - T_{\Delta_\nu}^{L_\nu} h\right| &= \left|L_\nu(g - b_\nu)(\gamma_\nu \circ l_{\nu,i}^{-1}) - L_\nu(h - b_\nu)(\gamma_\nu \circ l_{\nu,i}^{-1})\right| \\ &\leq |L_\nu(g - b_\nu) - L_\nu(h - b_\nu)| \leq \|L_\nu\|\, |g - h|,\end{aligned}$$

where $\|\cdot\|$ denotes the (strong) operator norm with respect to the valuation (2.4).

Therefore, if $\max\{\|L_\nu\| : \nu \in \mathbb{N}_m\} < 1$, the RB operator $T_{\Delta_\nu}^{L_\nu}$ is contractive on $C_*(\operatorname{tr}\gamma_\nu)$ and, as $C_*(\operatorname{tr}\gamma_\nu)$ is a compact metric space, it has a unique fixed point $\psi_{\Delta_\nu}^{L_\nu} \in C_*(\operatorname{tr}\gamma_\nu)$.

These results are summarized in the following theorem.

**Theorem 7.1.** *Suppose $D \subseteq \mathbb{C}$ is a domain and $\gamma \in C_1(D)$ with $\gamma = \sum \alpha_\nu \gamma_\nu$. On $C_*(\operatorname{tr}\gamma)$ define an RB operator $T^L_\Delta$ of the form* (7.1) *where each $T^{L_\nu}_{\Delta_\nu}$ is given by* (7.2)*. Further suppose that $L_\nu \in \mathcal{L}_\nu$, $\nu \in \mathbb{N}_m$. If*

$$\lambda := \max\{\|L_\nu\| : \nu \in \mathbb{N}_m\} < 1, \tag{7.3}$$

*then the RB operator $T^{L_\nu}_{\Delta_\nu}$ has a unique fixed point $\psi^L_\Delta \in C_*(\operatorname{tr}\gamma)$. This fixed point is given by*

$$\psi^L_\Delta(\gamma) = \sum \alpha_\nu \psi^{L_\nu}_{\Delta_\nu}(\gamma_\nu), \tag{7.4}$$

*where each $\psi^{L_\nu}_{\Delta_\nu}$ satisfies a self-referential equation of the form*

$$\psi^{L_\nu}_{\Delta_\nu} \circ \gamma_\nu = \psi_\nu \circ \gamma_\nu + L_\nu(\psi^{L_\nu}_{\Delta_\nu} - b_\nu)(\gamma_\nu \circ l^{-1}_{\nu,i}), \quad \text{on } I_{\nu,i}, \quad i \in \mathbb{N}_{n_\nu}.$$

We denote the correspondence $C(\operatorname{tr}\gamma) \ni \psi \mapsto \psi^{L_\nu}_{\Delta_\nu} \in C(\operatorname{tr}\gamma))$ by $\mathfrak{F}^L_\Delta$. More precisely,

$$\mathfrak{F}^L_\Delta \psi := \sum \alpha_\nu \mathfrak{F}^{L_\nu}_{\Delta_\nu} \psi_\nu := \sum \alpha_\nu \psi^{L_\nu}_{\Delta_\nu}. \tag{7.5}$$

$\mathfrak{F}^L_\Delta$ and $\mathfrak{F}^{L_\nu}_{\Delta_\nu}$ are referred to as fractal operators on $C(\operatorname{tr}\gamma)$ and $C(\operatorname{tr}\gamma_\nu)$, respectively. Fractal operators associated with fixed points of an RB operator were first considered by Navascués (*cf.* [19, 20]).

The fractal operator $\mathfrak{F}^L_\Delta$ enjoys the properties listed below.

**Proposition 7.2.** *The mapping $\mathfrak{F}^L_\Delta : C(\operatorname{tr}\gamma) \to C(\operatorname{tr}\gamma)$, $\psi \mapsto \mathfrak{F}^L_\Delta(\psi) = \psi^L_\Delta$, is a bounded linear operator satisfying*

- **(i)** *If $L_\nu \in \mathcal{L}_\nu$ is the zero operator $O$ then $\mathfrak{F}^O_\Delta = \mathrm{id}_{C(\operatorname{tr}\gamma)}$,*
- **(ii)** *$1 \in \sigma_p(\mathfrak{F}^L_\Delta)$, where $\sigma_p$ denotes the point spectrum of the operator,*
- **(iii)** *$1 \le \|\mathfrak{F}^L_\Delta\| \le (1-\lambda)^{-1}$.*

*Proof.* That $\mathfrak{F}^L_\Delta$ is a linear operator follows directly from the fact that the fixed point of $T^L_\Delta$ is unique and (7.5). To this end, let $\phi, \psi \in C_*(\operatorname{tr}\gamma)$ and $\lambda \in \mathbb{C}$. Then,

$$\begin{aligned} \phi^{L_\nu}_{\Delta_\nu} \circ \gamma_\nu &= \phi_\nu + L_\nu(\phi^{L_\nu}_{\Delta_\nu} - b_\nu)(\gamma_\nu \circ l^{-1}_{\nu,i}), \\ \psi^{L_\nu}_{\Delta_\nu} \circ \gamma_\nu &= \psi_\nu + L_\nu(\psi^{L_\nu}_{\Delta_\nu} - b_\nu)(\gamma_\nu \circ l^{-1}_{\nu,i}), \end{aligned}$$

on $I_{\nu,i}$. The uniqueness of the fixed point together with (7.5) implies

$$\mathfrak{F}^L_\Delta(\phi + \lambda\psi) = \mathfrak{F}^L_\Delta(\phi) + \lambda \mathfrak{F}^L_\Delta(\psi).$$

Property (i) follows immediately.

To show (ii), consider the constant function $\psi_c$, $\psi_c(\gamma) = \sum \alpha_\nu \psi_c(\gamma_\nu) = \sum \alpha_\nu \cdot c$, $c \in \mathbb{C}$. Then, $(\psi_c)^{L_\nu}_{\Delta_\nu} \circ \gamma_\nu = \psi_c + L_\nu((\psi_c)^{L_\nu}_{\Delta_\nu} - b_\nu) \circ (\gamma_\nu \circ l^{-1}_i)$. As $L_\nu(\psi_c)^{L_\nu}_{\Delta_\nu}(z_{\nu,0}) = 0 = L_\nu(\psi_c)^{L_\nu}_{\Delta_\nu}(z_{\nu,n_\nu})$, $\mathfrak{F}^{L_\nu}_{\Delta_\nu}(\psi_c) = (\psi_c)^{L_\nu}_{\Delta_\nu} = \psi_c$, and therefore $1 \in \sigma_p(\mathfrak{F}^{L_\nu}_{\Delta_\nu})$ and the statement follows.

To establish the estimate in (iii), note that if $L_\nu, L'_\nu \in \mathcal{L}_\nu$,

$$\left|\psi^{L_\nu}_{\Delta_\nu}\right| - \left|\psi^{L'_\nu}_{\Delta_\nu}\right| \le \left|\psi^{L_\nu}_{\Delta_\nu} - \psi^{L'_\nu}_{\Delta_\nu}\right| \le \|L_\nu\| \left|\psi^{L_\nu}_{\Delta_\nu}\right| + \|L'_\nu\| \left|\psi^{L'_\nu}_{\Delta_\nu}\right|,$$

and that setting $L'_\nu := O$ implies

$$\left|\psi^{L_\nu}_{\Delta_\nu}\right| - |\psi_\nu| \le \|L_\nu\| \left|\psi^{L_\nu}_{\Delta_\nu}\right|,$$

i.e.,

$$\left|\psi^{L_\nu}_{\Delta_\nu}\right| \le \frac{|\psi_\nu|}{1 - \|L_\nu\|} \le \frac{|\psi_\nu|}{1-\lambda},$$

which together with item (ii) and (2.4) verifies the statement. □

It is also possible to derive estimates between $\psi_\Delta^L$ and the original function $\psi$ as well as $\psi_\Delta^L - \psi_\Delta^{L'}$. To this end, let $\nu \in \mathbb{N}_m$ and suppose $L_\nu, L'_\nu \in \mathcal{L}_\nu$. Then, for any $g \in C_*(\operatorname{tr}\gamma_\nu)$,

$$\left|T_{\Delta_\nu}^{L_\nu} g - T_{\Delta_\nu}^{L'_\nu} g\right| = \left|(L_\nu(g - b_\nu) - L'_\nu(g - b_\nu))(\gamma_\nu \circ l_i^{-1})\right| \leq \|L_\nu - L'_\nu\| \cdot |g - b_\nu|.$$

Let $\psi_{\Delta_\nu}^{L_\nu}$ and $\psi_{\Delta_\nu}^{L'_\nu}$ be the fixed points of the RB operators $T_{\Delta_\nu}^{L_\nu}$ and $T_{\Delta_\nu}^{L'_\nu}$, respectively. Then,

$$\begin{aligned}\left|\psi_{\Delta_\nu}^{L_\nu} - \psi_{\Delta_\nu}^{L'_\nu}\right| &= \left|T_{\Delta_\nu}^{L_\nu}\psi_{\Delta_\nu}^{L_\nu} - T_{\Delta_\nu}^{L_\nu}\psi_{\Delta_\nu}^{L'_\nu} + T_{\Delta_\nu}^{L_\nu}\psi_{\Delta_\nu}^{L'_\nu} - T_\Delta^{L'}\psi_\Delta^{L'}\right| \\ &\leq \left|T_\Delta^L(\psi_\Delta^L - \psi_\Delta^{L'})\right| + \|L_\nu - L'_\nu\| \cdot \left|\psi_\Delta^{L'} - b_\nu\right| \\ &\leq \|L_\nu\| \cdot \left|\psi_{\Delta_\nu}^{L_\nu} - \psi_{\Delta_\nu}^{L'_\nu}\right| + \|L_\nu - L'_\nu\| \cdot \left|\psi_{\Delta_\nu}^{L'_\nu} - b_\nu\right|.\end{aligned}$$

Hence,

$$\left|\psi_{\Delta_\nu}^{L_\nu} - \psi_{\Delta_\nu}^{L'_\nu}\right| \leq \frac{\|L_\nu - L'_\nu\|}{1 - \|L_\nu\|}\left|\psi_{\Delta_\nu}^{L'_\nu} - b_\nu\right|. \tag{7.6}$$

Interchanging $L$ and $L'$ in (7.6) yields

$$\begin{aligned}\left|\psi_{\Delta_\nu}^{L_\nu} - \psi_{\Delta_\nu}^{L'_\nu}\right| &\leq \frac{\|L_\nu - L'_\nu\|}{1 - \|L_\nu\|}\max\left\{\left|\psi_{\Delta_\nu}^{L_\nu} - b_\nu\right|, \left|\psi_{\Delta_\nu}^{L'_\nu} - b_\nu\right|\right\} \\ &\leq \frac{\lambda + \lambda'}{1 - \lambda}\max\left\{\left|\psi_{\Delta_\nu}^{L_\nu} - b_\nu\right|, \left|\psi_{\Delta_\nu}^{L'_\nu} - b_\nu\right|\right\},\end{aligned} \tag{7.7}$$

where $\lambda' := \max\{\|L'_\nu\| : \nu \in \mathbb{N}_m\} < 1$. Note that the left-hand side of (7.7) can also be written as $\left|(\psi_{\Delta_\nu}^{L_\nu} - b_\nu) - (\psi_{\Delta_\nu}^{L'_\nu} - b_\nu)\right|$ expressing more clearly the relation to the right-hand side.

Setting $L' := O$ in (7.6) gives an error estimate between $\psi_{\Delta_\nu}^{L_\nu}$ and $\psi_\nu$ and consequently between $\psi_\Delta^L$ and $\psi$ using (2.4).

**Proposition 7.3.** *Let $\psi \in C(\operatorname{tr}\gamma)$ and let $\psi_\Delta^L$ be the fractal function generated by the RB operator $T_\Delta^L$. Then,*

$$\left|(\psi_\Delta^L - b) - (\psi - b)\right| \leq \frac{\lambda}{1 - \lambda}|\psi - b|,$$

*where $b \in C_*(\operatorname{tr}\gamma)$ is the function given by $b|_{\operatorname{tr}\gamma_\nu} = b_\nu$, $\nu \in \mathbb{N}_m$.*

## 8. Conclusion

In this paper, we introduced the novel concept of fractal interpolation on 1-chains in the complex plane generalizing to a certain extend fractal interpolation on curves in Banach spaces (*cf.* [17]). We presented a construction in the setting of continuous and $L^p$-functions, $p \in [1, \infty)$. Moreover, we introduced a fractal operator on $C(\operatorname{tr}\gamma)$ associated with such fractal functions and discussed some of its properties.

The paper also lays the foundations for further studies of this topic. In particular, one may extend the given constructions to function spaces such as Sobolev, Hölder, Besov and Triebel-Lizorkin spaces (*cf.* [14]) as they play important roles in interpolation and approximation theory. This would be a natural extension of the $L^p$-case considered in this paper.

Instead of just considering 1-chains, the higher-dimensional analogs, namely $n$-chains could be investigated. Here, one would work within the settings of fractal interpolation for surfaces and hypersurfaces (*cf.* [15, 17]) and exploit the rich theory of simplicial chains (*cf.* [11]).

The fractal methodologies presented in this paper are based on Banach contractions. Recently, more general types of contraction mappings such as Ratkotch or $F$-contractions (*cf.* [21]) have been considered in the literature. It would be worthwhile to explore these more general contractions in the setting of this paper.

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