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On multi q-Rung orthopair fuzzy Lie algebraic structures

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Abstract

In neutrosophic settings the membership, indeterminacy and non-membership degrees are determined by a single expert whereas in multi q-Rung orthopair fuzzy settings the same components are evaluated by hiring a group of independent expert which is more efficient, reliant and consistent than neutrosophic logic which motivates us to introduce for the first time the concept of Lie subalgebra and Lie ideal in the light of the concept of Smarandache's multi q-Rung orthopair fuzzy settings as a generalization of neutrosophic Lie subalgebra and Lie ideals. In this study, we proved that the intersection (resp. Cartesian product) of two multi q-Rung orthopair fuzzy Lie subalgebras is multi q-Rung orthopair fuzzy Lie subalgebra. Some more fundamental results and properties are also derived. In the light of analytical method we proved that every multi q-Rung orthopair fuzzy Lie ideal is multi q-Rung orthopair fuzzy Lie subalgebra but not reversible which is shown by utilizing counter example.

We also derive some preservation aspects of multi q-Rung orthopair fuzzy Lie subalgebra and Lie ideal, solvable and nilpotent multi q-Rung orthopair fuzzy Lie ideals under Lie homomorphism and its inverse. A significant finding from this study is homomorphic image of solvable (resp. nilpotent) multi q-Rung orthopair fuzzy Lie ideals is solvable (resp. nilpotent) multi q-Rung orthopair fuzzy Lie ideals. Finally, with the help of these newly defined concepts some more interesting results on solvable and nilpotent multi q-Rung orthopair fuzzy Lie ideals are investigated as a part of application. The findings offered here add to the larger framework of Lie algebraic theory, offering fresh perspectives for further study of the structural characteristics of BCK and BCI algebra using multi q-Rung orthopair fuzzy sets.

Keywords: Lie algebra, Lie ideal, MqROF set, MqROF Lie subalgebra, MqROF Lie ideal, solvable MqROF Lie ideal, nilpotent MqROF Lie ideal

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1. Introduction

Sophus Lie (1842-1899) first proposed the idea of the Lie algebra in an attempt to classify specific smooth subgroups of general linear groups (*cf.* [13]). Lie algebras, so named because they were invented by Sophus Lie, are a specific instance of general linear algebra. Following the introduction of this theory, Lie groups were used in mathematics and physics to categorise smooth subgroups. In more detail, Lie subalgebras and their characteristics were developed and examined in [1, 2, 3, 8, 9, 11, 12, 14, 15, 25, 26, 28, 30]. In the modern period, science and technology have been significantly tackling complex processes and phenomena for which there is little data. Mathematical models are created to deal with different kinds of systems with ambiguous and imprecise components in these circumstances. Numerous models are based on extensions of standard set theory, including fuzzy sets (*cf.* [29]), intuitionistic fuzzy

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sets (*cf.* [7]), neutrosophic sets (*cf.* [20]), Fermatean fuzzy sets (*cf.* [23]), q-Rung orthopair fuzzy sets (*cf.* [27]), and many more. The multi q-Rung orthopair fuzzy (in short, MqROF) model is a contemporary tool to handle uncertainty and vagueness in multi criteria decision-making problems. In this paper, we cover the core features of a multi q-Rung orthopair fuzzy Lie subalgebra of Lie algebra. Probably numerous domains along with decision making, genetic algorithms, signal processing, artificial intelligence, multiagent systems, computer networks, robotics, expert systems, neural networks, medical diagnosis and automata theory shall be benefited with the acquired outcomes.

The idea of a fuzzy set (shortly, FS) was introduced by Zadeh [29] for ambiguous, imprecise, and uncertain situations. In 1986, Atanassov [7] developed intuitionistic fuzzy set (shortly, IFS) as a generalization of fuzzy set. Since these developments, his theory gained widespread recognition as a crucial tool in the domains of science, technology, engineering, medicine, etc. Smarandache [19] introduced the neutrosophic set (shortly, NS) in 1995 as a generalization of intuitionistic fuzzy sets, as well as inconsistent intuitionistic fuzzy sets (shortly, IIFS), Pythagorean fuzzy sets (shortly, PFS), Fermatean fuzzy sets (shortly, FFS), spherical fuzzy sets, n-Hyperspherical fuzzy sets, and so forth. The Pythagorean fuzzy set has made significant contributions to our knowledge of decision-making issues. Peng and Yang [17] created division and subtraction using Pythagorean fuzzy aggregation operators and studied their characteristics, including monotonicity, idempotency, and boundedness. An example of evaluating the performance of an internet stock is given to illustrate the developed methodology. In 2018, Zeng [31] compared the Pythagorean fuzzy set with four different ranked techniques that were currently in use. In order to create model for vague and imprecise information, the model of q-Rung orthopair fuzzy (shortly, qROF) set was introduced by Yager [27] which is different from IFS model since it involves the condition $0 \leq T^q + F^q \leq 1, (q \geq 1)$, where T and F stand for membership and non-membership function respectively. Also these are broader variants of the intuitionistic fuzzy set and the Pythagorean fuzzy set. As q rises, a greater range of acceptable orthopairs becomes accessible, giving the user a more expansive mental framework to articulate their opinions regarding the belonging score. Naz et al. [22] create a multiple attribute group decision-making technique based on q-rung orthopair fuzzy 2-tuple linguistic set to enhance the capacity of 2-tuple linguistic words to describe complicated information.

In addition to creating original concepts of q-rung image fuzzy aggregate operators under the Einstein operators and outlining their use in multi-attribute decision-making, Akram et al. [4] expand the scope of Einstein operators' applications to the q-rung picture fuzzy environment. In decision making of material selection problems, q-Rung orthopair fuzzy set model has significant application proposed by Alrasheedi et al. [6]. The application range of solving real life problems such as decision making problems in qROF model is popularly increasing than that of the IFS model because number of pairs satisfying the condition $0 \leq T^q + F^q \leq 1$ $(q \geq 1)$ is higher than that of the condition $0 \leq T + F \leq 1$. Ali [5] redefined the q-ROFS utilizing a variety of crucial tools, covered the fundamental algebraic operation under action and qROFS fetchers, and introduced the orbit-based qROFS with the aid of examples and graphic explanations. By analyzing the function of supporting degree, non-supporting degree, and indeterminacy supporting degree among the rung orthopair fuzzy sets using cotangent and the standard cosine similarity measurements, Wang et al. [24] introduced the ten similarity measures.

Additionally, q-rung orthopair fuzzy sets were utilized to handle multiple criteria decisions. By using the similarity measure under study to pattern classification, clustering, and medical issues, Peng and Liu [16] showed its accuracy. A few exemplary cases were addressed in order to validate the findings and demonstrate the availability and usefulness of similarity measures between qROFS. Razzaque and Razaq [18] produced some fundamental results and presented the innovative idea of the q-rung orthopair fuzzy subgroup. Additionally, the q-rung orthopair fuzzy left and right cosets, q-rung orthopair fuzzy level subgroups, and basic definitions and homomorphism under qROF set were explored. Smarandache [21] subsequently proposed the multi neutrosophic set in which multi q-Rung orthopair fuzzy set is one of the special model. Recently, Debnath and Smarandache [10] initiated the concept of multi intuitionistic fuzzy Lie subalgebras. These idea motivates us to introduce and study the notion of novel algebraic structure in multi q-Rung orthopair fuzzy (in short, MqROF) framework.

In the frame of literature review we notice that nearly all of the fuzzy Lie algebraic and neutrosophic Lie algebraic structures have been studied. Nevertheless, no research on algebraic structure in relation to MqROF sets has been done. There have been numerous efforts made on the notions of qROF sets. It was noticed that only a few papers were published in relation to qROF sets rather than MqROF sets. This motivated us to introduce for the first time the idea of novel algebraic structure in MqROF framework. To fill the gaps we clarify the concept of Lie subalgebras, Lie ideals, homomorphism, solvable and nilpotent Lie ideals using MqROF sets in Lie algebra.

The current research is formulated as follows: We focused on the introduction and review of literature in 1st

section. The focus of section 2 is on preliminary and common definitions. The notion of MqROF Lie ideals of Lie algebra and MqROF Lie subalgebras are explained in section 3. Some of their fundamental properties and operations like intersection and generalized Cartesian product are investigated. Moreover, the relationship between MqROF Lie subalgebras and MqROF Lie ideals are also established. In Section 4, we investigate the pre-images and images of MqROF Lie subalgebras and MqROF Lie ideals under Lie homomorphisms. Some interesting results on solvable and nilpotent MqROF Lie Ideals are established. In Section 5, we give the conclusion of the newly defined concept of MqROF Lie subalgebras and MqROF Lie ideals.

2. Preliminaries

Some common notations and definitions used throughout the text are included in this section. A Lie algebra is a vector space $\mathcal{L}$ over the field $\mathcal{F}$ ($\mathcal{F} = RorC$) on which the mapping $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$ defined by $(\gamma, \mu) = [\gamma, \mu]$ for all $\gamma, \mu \in \mathcal{L}$, where, $[\gamma, \mu]$ is called Lie bracket satisfying the following conditions:

(1) $[\gamma, \mu]$ is bilinear,

(2) $[\gamma, \gamma] = 0$ and

(3) $[[\gamma, \mu], \lambda] + [[\mu, \lambda], \gamma] + [[\lambda, \gamma], \mu] = 0$, (known as Jacobi identity) for all $\gamma, \mu, \lambda \in \mathcal{L}$.

We note that the operation Lie bracket [.,.] is neither associative nor commutative i.e,$[[\gamma, \mu], \lambda] \neq [[\mu, \lambda], \gamma]$ and $[\gamma, \mu] \neq [\mu, \gamma]$. But the operation Lie bracket [.,.] is anti-commutative i.e. $[\gamma, \mu] = -[\mu, \gamma]$. A subspace H of $\mathcal{L}$ is called a Lie subalgebra if it is closed under [.,.]. A subspace I of $\mathcal{L}$ is called a Lie ideal if $[I, L] \subset I$. It is always true that every Lie ideal is Lie subalgebra. Throughout the paper $\mathcal{L}$ will denote Lie algebra.

Definition 2.1 (*cf.* [29])**.** Let U represent a universal crisp set. Then the fuzzy set (briefly, FS) on U is described as $F = \{(x, \mu(x)) : x \in\}$ where, $\mu(x) \in [0, 1]$, denotes the degree of membership of $x \in X$.

Definition 2.2 (*cf.* [7])**.** Let U represent a universe of discourse. The Intuitionistic fuzzy set on U is then explained as $F = \{(x, \mu(x), \nu(x)) : x \in X\}$ where, $\mu(x), \nu(x) \in [0, 1]$, termed as the membership and non-membership degrees respectively satisfying

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Definition 2.3 (*cf.* [23])**.** Let U represent a universal crisp set. The Fermatean fuzzy set on U is then explained as $F = \{(x, \mu(x), \nu(x)) : x \in X\}$ where, $\mu(x), \nu(x) \in [0, 1]$, termed as membership and non-membership degrees respectively satisfying

$$0 \leq \mu_F^3(x) + \nu_F^3(x) \leq 1.$$

Definition 2.4 (*cf.* [19])**.** Suppose X denotes the crisp universal set. The neutrosophic set is then described by $N = \{(x, \tau(x), \lambda(x), \eta(x)), x \in X\}$ where, $\tau, \lambda, \eta \in [0, 1]$, termed as the truth, indeterminacy and falsehood degrees respectively which satisfy

$$0 \leq \inf(\tau) + \inf(\lambda) + \inf(\eta) \leq \sup(\tau) + \sup(\lambda) + \sup(\eta) \leq 3.$$

Definition 2.5 (*cf.* [27])**.** Suppose X denotes the crisp universal set. Then q-Rung orthopair fuzzy set (shortly, qROFS) on X is defined by $M = \{(x, x(T, F)) : x \in X\}$ and q-Rung orthopair fuzzy components truth and falsehood satisfy $T, F \subseteq [0, 1]$ along with $0 \leq T^q + F^q \leq 1$, where the degree of hesitancy (indeterminacy) I, can be found from the relation

$$T^q + I^q + F^q = 1.$$

Definition 2.6 (*cf.* [21])**.** Suppose X denotes the crisp universal set. The multi intuitionistic fuzzy set (shortly, multi IFS) on X is expressed by $M = \{(x, x(\tau_1, \tau_2, \ldots, \tau_p; \eta_1, \eta_2, \ldots, \eta_s)) : x \in X\}$ where p and s are integers ≥ 0, $p + s = n \geq 2$, and at least one of p and s is ≥ 2, in order to ensure the existence of multiplicity of at least one intuitionistic component: truth or falsehood; all subsets $\tau_1, \tau_2, \ldots, \tau_p, \eta_1, \eta_2, \ldots, \eta_s \subseteq [0, 1]$;

$$0 \leq \sum_{i=1}^{p} \inf(\tau_i) + \sum_{k=1}^{s} \inf(\eta_k) \leq \sum_{i=1}^{p} \sup(\tau_i) + \sum_{k=1}^{s} \sup(\eta_k) \leq n.$$

Definition 2.7 (*cf.* [21])**.** Suppose X denotes the crisp universal set. The multi q-rung orthopair fuzzy set (shortly, MqROFS) on X can be explained by $M = \left\{(x, x(T_1, T_2, \ldots, T_p; F_1, F_2, \ldots, F_s)) : x \in X\right\}$ where p and s are integers ≥ 3 with $p + s = n \geq 3$, and at least one of p and s is ≥ 2, in order to ensure the existence of multiplicity of at least one Fermatean component: truth or falsehood; all subsets $T_1, T_2, \ldots, T_p, F_1, F_2, \ldots, F_s \subseteq [0, 1]$ along with $0 \leq T_i^q + F_k^q \leq 1$, where the degree of hesitancy (Indeterminacy) I_j can be found by the relation

$$T_i^q + I_j^q + F_k^q = 1, \forall i = 1, 2, 3, \ldots, p; j = 1, 2, 3, \ldots, r; k = 1, 2, 3, \ldots, s.$$

3. Properties of multi q-Rung orthopair fuzzy Lie algebra

In this section, we first define q-Rung orthopair fuzzy (shortly, qROF) Lie subalgebras and q-Rung orthopair fuzzy Lie ideals and then the notion of multi q-Rung orthopair fuzzy (shortly, MqROF) Lie subalgebras and multi q-Rung orthopair fuzzy (shortly, MqROF) Lie ideals are initiated. Additionally, some characterizations, counterexamples, and fundamental characteristics are examined.

Definition 3.1. A q-Rung orthopair Fuzzy (briefly, qROF) set $W = (\tau, \eta)$ on $\mathcal{L}$ is said to be q-Rung orthopair fuzzy subalgebra if, $\forall \gamma, \mu \in \mathcal{L}; \forall c \in \mathcal{F}$, the following assumptions hold good:

(i) $\tau_W(\gamma + \mu) \geq \min\{\tau_W(\gamma), \tau_W(\mu)\}, \eta_W(\gamma + \mu) \leq \max\{\eta_W(\gamma), \eta_W(\mu)\}$.

(ii) $\tau_W(c\gamma) \geq \tau_W(\gamma), \eta_W(c\gamma) \leq \eta_W(\gamma)$.

(iii) $\tau_W[\gamma, \mu] \geq \min\{\tau_W(\gamma), \tau_W(\mu)\}, \eta_W[\gamma, \mu] \leq \max\{\eta_W(\gamma), \eta_W(\mu)\}$.

Definition 3.2. A q-Rung orthopair fuzzy (briefly, qROF) set $W = (\tau, \lambda, \eta)$ on $\mathcal{L}$ is said to be q-Rung orthopair fuzzy Lie ideal if, $\forall \gamma, \mu \in \mathcal{L}; \forall c \in \mathcal{F}$, the conditions (i) and (ii) of Definition 3.1 and the condition:

$$\tau_W([\gamma, \mu]) \geq \tau_W(\gamma), \eta_W([\gamma, \mu]) \leq \eta_W(\gamma)$$

holds good.

Definition 3.3. Let $N_l = \{1, 2, \ldots, l\}, N_n = \{1, 2, \ldots, n\}$ and $\mathcal{L}$ be a Lie algebra of vectors over the field $\mathcal{F}$. A multi q-Rung orthopair fuzzy set W = $(\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \eta_W^1, \eta_W^2, \ldots, \eta_W^n)$ on $\mathcal{L}$ is said to be multi q-Rung orthopair fuzzy (briefly, MqROF) Lie subalgebra over $\mathcal{L}$ if, $\forall\ \gamma,\ \eta \in \mathcal{L}$; $\forall\ i \in N_l, \forall\ k \in N_n$, $\forall c \in \mathcal{F}$, the following assumptions hold good:

(i) $\tau_W^i(\gamma + \mu) \geq \min\{\tau_W^i(\gamma), \tau_W^i(\mu)\}, \eta_W^k(\gamma + \mu) \leq \max\{\eta_W^k(\gamma), \eta_W^k(\mu)\}$.

(ii) $\tau_W^i(c\gamma) \geq \tau_W^i(\gamma), \eta_W^k(c\gamma) \leq \eta_W^k(\gamma)$.

(iii) $\tau_W^i[\gamma, \mu] \geq \min\{\tau_W^i(\gamma), \tau_W^i(\mu)\}, \eta_W^k[\gamma, \mu] \leq \max\{\eta_W^k(\gamma), \eta_W^k(\mu)\}$.

Definition 3.4. Let $N_l = \{1, 2, \ldots, l\}, N_n = \{1, 2, \ldots, n\}$ and $\mathcal{L}$ be a Lie algebra of vectors over the field $\mathcal{F}$. A multi q-Rung orthopair fuzzy set $W = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \eta_W^1, \eta_W^2, \ldots, \eta_W^n)$ on $\mathcal{L}$ is said to be MqROF Lie ideal if,$\forall \gamma, \eta \in \mathcal{L}; \forall i \in N_l, \forall k \in N_n, \forall c \in \mathcal{F}$, the conditions (i)-(ii) of Definition 3.3 along with the following condition is satisfied: $\forall \gamma, \eta \in \mathcal{L}$,

$$\tau_W^i[\gamma, \mu] \geq \tau_W^i(\gamma), \eta_W^k[\gamma, \mu] \leq \eta_W^k(\gamma).$$

It follows from condition (ii) of Definition 3.3 that

$$\tau_W^i(0) \geq \tau_W^i(\gamma), \eta_W^k(0) \leq \eta_W^k(\gamma)$$

and

$$\tau_W^i(-\gamma) \geq \tau_W^i(\gamma), \eta_W^k(-\gamma) \leq \eta_W^k(\gamma).$$

Theorem 3.5. *Let $M = (\mathcal{L}, \tau_M^1, \tau_M^2, \ldots, \tau_M^l; \eta_M^1, \eta_M^2, \ldots, \eta_M^n)$ be a MqROF Lie ideal over $\mathcal{L}$ and let $N_l = \{1, 2, \ldots, l\}$, $N_n = \{1, 2, \ldots, n\}$, then $\tau_M^i(0) = \sup\{\tau_M^i(\gamma) : \gamma \in \mathcal{L}\}$ and $\eta_M^k(0) = \inf\{\eta_M^k(\gamma) : \gamma \in \mathcal{L}\}, \forall i \in N_l, \forall k \in N_n$*

Proof. Since M is MqROF Lie ideal over $\mathcal{L}$, so by Definition 3.4, we must have the two conditions: $\tau^i_M(0) \geq \tau^i_M(\gamma)$ and $\eta^k_M(0) \leq \eta^k_M(\gamma)$. As γ runs over $\mathcal{L}$, taking supremum on both sides of first inequality and infimum of second inequality we get,

$$\sup\{\tau^i_M(0) : \gamma \in \mathcal{L}\} = \sup\{\tau^i_M(\gamma) : \gamma \in \mathcal{L}\}$$

and

$$\inf\{\eta^k_M(0) : \gamma \in \mathcal{L}\} = \inf\{\eta^k_M(\gamma) : \gamma \in \mathcal{L}\}.$$

This implies that $\tau^i_M(0) = \sup\{\tau^i_M(\gamma) : \gamma \in \mathcal{L}\}$ and $\eta^k_M(0) = \inf\{\eta^k_M(\gamma) : \gamma \in \mathcal{L}\}$. □

Proposition 3.6. *Let $M = (\mathcal{L}, \{\tau^i_M\}^l_{i=1}, \{\eta^k_M\}^n_{k=1})$ be a MqROF Lie ideal over $\mathcal{L}$. Then for each $\psi, \sigma \in [0, 1]$ satisfying $\tau^i_M(0) \geq \psi$, $\eta^k_M(0) \leq \sigma$ and $0 \leq \psi + \sigma \leq 1$, the (ψ, σ)-level subset $\mathcal{L}^{(\psi,\sigma)}_M$ is a MqROF Lie ideal of Lie algebra $\mathcal{L}$.*

Proof. Let $N_l = \{1, 2, \ldots, l\}, N_n = \{1, 2, \ldots, n\}, \gamma, \mu \in \mathcal{L}^{(\psi,\sigma)}_M$ and $\mathcal{F}$ be a field such that $t \in \mathcal{F}$. We know that (ψ, σ)-level subset $\mathcal{L}^{(\psi,\sigma)}_M$ is defined by $\mathcal{L}^{(\psi,\sigma)}_M = \{\gamma \in \mathcal{L} : \tau^i_W(\gamma) \geq \psi, \eta^k_W(\gamma) \leq \sigma\}$. Since $M = (\mathcal{L}, \{\tau^i_M\}^l_{i=1}, \{\eta^k_M\}^n_{k=1})$ be a multi q-Rung orthopair fuzzy Lie ideal over $\mathcal{L}$, then $\forall i \in N_l, \forall k \in N_n$, we have

$$\tau^i_W(\gamma + \mu) \geq \min\{\tau^i_W(\gamma), \tau^i_W(\mu)\} \geq \psi,$$

$$\eta^k_W(\gamma + \mu) \leq \max\{\eta^k_W(\gamma), \eta^k_W(\mu)\} \leq \sigma$$

$$\Rightarrow \gamma + \mu \in \mathcal{L}^{(\psi,\sigma)}_M.$$

Also we obtain

$$\tau^i_W(c\gamma) \geq \tau^i_W(\gamma) \geq \psi$$

and

$$\eta^k_W(c\gamma) \leq \eta^k_W(\gamma) \leq \sigma$$

$$\Rightarrow c\gamma \in \mathcal{L}^{(\psi,\sigma)}_M.$$

Thus, $\mathcal{L}^{(\psi,\sigma)}_M$ is subspace of $\mathcal{L}$. Again suppose that $\gamma \in \mathcal{L}, \mu \in \mathcal{L}^{(\psi,\sigma)}_M$, then

$$\tau^i_W[\gamma, \mu] \geq \min\{\tau^i_W(\gamma), \tau^i_W(\mu)\} \geq \psi$$

and

$$\eta^k_W[\gamma, \mu] \leq \max\{\eta^k_W(\gamma), \eta^k_W(\mu)\} \leq \sigma,$$

from this it follows that $[\gamma, \eta] \in \mathcal{L}^{(\psi,\sigma)}_M$. Hence, $\mathcal{L}^{(\psi,\sigma)}_M$ is a MqROF Lie ideal of $\mathcal{L}$. □

Theorem 3.7. *If δ is a fixed element of $\mathcal{L}$ and $W = (\mathcal{L}, \{\tau^i_W\}^l_{i=1}, \{\eta^k_W\}^n_{k=1})$ is a MqROF Lie ideal of $\mathcal{L}$. Then the set defined by $W^\delta = \{\gamma \in \mathcal{L} : \tau^i_W(\gamma) \geq \tau^i_W(\delta), \eta^k_W(\gamma) \leq \eta^k_W(\delta)\}$ is a MqROF Lie ideal of $\mathcal{L}$.*

Proof. Suppose that $\gamma, \mu \in W^\delta, i \in N_l, k \in N_n$ be arbitrary. Then

$$\tau^i_W(\gamma + \mu) \geq \min\{\tau^i_W(\gamma), \tau^i_W(\mu)\} \geq \tau^i_W(\delta); \eta^k_W(\gamma + \mu) \leq \max\{\eta^k_W(\gamma), \eta^k_W(\mu)\} \leq \eta^i_W(\delta).$$

This implies that $\gamma + \mu \in W^\delta$. Now, $\forall \gamma \in W^\delta, \forall i \in N_l, \forall k \in N_n, \forall c \in \mathcal{F}$, we have,

$$\tau^i_W(c\gamma) \geq \tau^i_W(\gamma) \geq \tau^i_W(\delta); \eta^k_W(c\gamma) \leq \eta^k_W(\gamma) \leq \eta^i_W(\delta).$$

This follows that $c\gamma \in W^\delta$. Also for every $\gamma \in W^\delta$ and for every $\mu \in W^\delta, \forall i \in N_l, \forall j \in N_m, \forall k \in N_n$

$$\tau^i_W[\gamma, \mu] \geq \min\{\tau^i_W(\gamma), \tau^i_W(\mu)\} \geq \tau^i_W(\delta); \eta^k_W[\gamma, \mu] \leq \max\{\eta^k_W(\gamma), \eta^k_W(\mu)\} \leq \eta^i_W(\delta),$$

which shows that $[\gamma, \eta] \in W^\delta$. Hence, W^δ is a MqROF Lie ideal of $\mathcal{L}$. □

Corollary 3.8. *If $W = (\mathcal{L}, \{\tau^i_W\}^l_{i=1}, \{\eta^k_W\}^n_{k=1})$ is a MqROF Lie ideal of $\mathcal{L}$. Then the set defined by*

$$W^0 = \left\{\gamma \in \mathcal{L} : \tau^i_W(\gamma) \geq \tau^i_W(0), \eta^k_W(\gamma) \leq \eta^k_W(0), \forall i \in N_l, \forall k \in N_n\right\}$$

is a MqROF Lie ideal of $\mathcal{L}$.

Proof. Let $\gamma, \mu \in W^0, i \in N_l, k \in N_n$. Then $\forall \gamma, \mu \in W^0, \forall i \in N_l, \forall k \in N_n$, we have,

$$\tau^i_W(\gamma + \mu) \geq \min\{\tau^i_W(\gamma), \tau^i_W(\mu)\} \geq \tau^i_W(0)$$

and

$$\eta^k_W(\gamma + \mu) \leq \max\{\eta^k_W(\gamma), \eta^k_W(\mu)\} \leq \eta^i_W(0)$$
$$\Rightarrow \gamma + \mu \in W^0.$$

Now, $\forall \gamma \in W^0, \forall i \in N_l, \forall k \in N_n, \forall c \in \mathcal{F}$, we find that

$$\tau^i_W(c\gamma) \geq \tau^i_W(\gamma) \geq \tau^i_W(0)$$

and

$$\eta^k_W(c\gamma) \leq \eta^k_W(\gamma) \leq \eta^i_W(0)$$
$$\Rightarrow c\gamma \in W^0.$$

Also for each $\gamma \in W^0$ and for each $\mu \in W^0, \forall i \in N_l, \forall j \in N_m, \forall k \in N_n$, we get

$$\tau^i_W[\gamma, \mu] \geq \min\{\tau^i_W(\gamma), \tau^i_W(\mu)\} \geq \tau^i_W(0)$$

and

$$\eta^k_W[\gamma, \mu] \leq \max\{\eta^k_W(\gamma), \eta^k_W(\mu)\} \leq \eta^i_W(0),$$

from which it follows that $[\gamma, \eta] \in W^0$. Hence, W^0 is a MqROF Lie ideal of $\mathcal{L}$. □

Theorem 3.9. *Let $W = (\{\tau^i_W\}^l_{i=1}, \{\eta^k_W\}^n_{k=1})$ be a MqROF Lie subalgebra of a Lie algebra $\mathcal{L}$ and $\mathcal{R} \subseteq \mathcal{L} \times \mathcal{L}$ be a binary relation on $\mathcal{L}$ defined by $\mathcal{R} = \{(\gamma, \mu) \in \mathcal{L} \times \mathcal{L} \mid \tau^i_W(\gamma - \mu) = \tau^i_W(0), \eta^k_W(\gamma - \mu) = \eta^k_W(0), \gamma, \mu \in \mathcal{L}, i \in N_l, k \in N_n\}$, then $\mathcal{R}$ is a congruence relation on $\mathcal{L}$.*

Proof. First of all we need to prove that the relation $\mathcal{R}$ is equivalence relation on $\mathcal{L}$. Now

(i) Reflexive: Since for each $\gamma \in \mathcal{L}$,

$$\tau^i_W(\gamma - \gamma) = \tau^i_W(0)$$

and

$$\eta^k_W(\gamma - \gamma) = \eta^k_W(0).$$

Thus, $(\gamma, \gamma) \in \mathcal{R}, \forall \gamma \in \mathcal{L}, \forall i \in N_l, \forall k \in N_n$, and consequently $\mathcal{R}$ is reflexive relation on $\mathcal{L}$.

(ii) Symmetric: Let, $(\gamma, \mu) \in \mathcal{R}$. Then $\tau^i_W(\gamma - \mu) = \tau^i_W(0)$. By second assumption of Definition 3.3, we have

$$\tau^i_W(-(\mu - \gamma)) \geq \tau^i_W((\mu - \gamma)) = \tau^i_W(0).$$

By similar argument we have,

$$\eta^k_W(\gamma - \mu) = \eta^k_W(0),$$
$$\Rightarrow \eta^k_W(-(\mu - \gamma)) \leq \eta^j_W((\mu - \gamma)) = \eta^j_W(0).$$

Thus, $(\mu, \gamma) \in \mathcal{R}, \forall \gamma, \mu \in \mathcal{L}, \forall i \in N_l, \forall k \in N_n$, so that $\mathcal{R}$ is symmetric relation on $\mathcal{L}$.

(iii) Transitive: Let, $(\gamma,\mu),(\mu,\sigma)\in\mathcal{R}$. Then

$$\tau^i_W(\gamma-\mu)=\tau^i_W(0),\tau^i_W(\mu-\sigma)=\tau^i_W(0)$$

and

$$\eta^k_W(\gamma-\mu)=\eta^k_W(0),\eta^k_W(\mu-\sigma)=\eta^k_W(0).$$

From which it follows that,

$$\begin{aligned}\tau^i_W(\gamma-\sigma) &= \tau^i_W\{(\gamma-\mu)+(\mu-\sigma)\}\geq\min\{\tau^i_W(\gamma-\mu),\tau^i_W(\mu-\sigma)\}\\ &= \tau^i_W(0),\eta^k_W(\gamma-\sigma)=\eta^k_W\{(\gamma-\mu)+(\mu-\sigma)\}\\ &\leq \max\{\eta^k_W(\gamma-\mu),\eta^k_W(\mu-\sigma)\}=\eta^k_W(0).\end{aligned}$$

Hence, $(\gamma,\sigma)\in\mathcal{L},\forall i\in N_l,\forall k\in N_n$ and consequently, $\mathcal{R}$ is transitive relation on $\mathcal{L}$. Hence, $\mathcal{R}$ is an equivalence relation on $\mathcal{L}$.

We now want to verify that $\mathcal{R}$ is an congruence relation on $\mathcal{L}$. Suppose that $(\gamma,\mu),(\mu,\sigma)\in\mathcal{R}$. Then

$$\tau^i_W(\gamma-\mu)=\tau^i_W(0),\tau^i_W(\mu-\sigma)=\tau^i_W(0)$$

and

$$\eta^k_W(\gamma-\mu)=\eta^k_W(0),\eta^k_W(\mu-\sigma)=\eta^k_W(0)$$

Now if $\gamma_1,\gamma_2,\mu_1,\mu_2\in\mathcal{L}$, then assumption (i) of Definition 3.3 produces,

$$\tau^i_W\{(\gamma_1+\gamma_2)-(\mu_1+\mu_2)\}=\tau^i_W\{(\gamma_1-\mu_1)+(\gamma_2-\mu_2)\}\geq\min\{\tau^i_W(\gamma_1-\mu_1),\tau^i_W(\gamma_2-\mu_2)\}=\tau^i_W(0);$$

$$\eta^k_W\{(\gamma_1+\gamma_2)-(\mu_1+\mu_2)\}=\eta^k_W\{(\gamma_1-\mu_1)+(\gamma_2-\mu_2)\}\leq\max\{\eta^k_W(\gamma_1-\mu_1),\eta^k_W(\gamma_2-\mu_2)\}=\eta^k_W(0)$$

By second assumption of Definition 3.3, we obtain,

$$\tau^i_W(c\gamma_1-c\mu_1)=\tau^i_W\{c(\gamma_1-\mu_1)\}\geq\tau^i_W(\gamma_1-\mu_1)=\tau^i_W(0)$$

and

$$\eta^k_W(c\gamma_1-c\mu_1)=\eta^k_W\{c(\gamma_1-\mu_1)\}\geq\eta^k_W(\gamma_1-\mu_1)=\eta^k_W(0),$$

By third assumption of Definition 3.3, we get,

$$\tau^i_W\{[\gamma_1,\gamma_2]-[\mu_1,\mu_2]\}=\tau^i_W[(\gamma_1-\mu_1),(\gamma_2-\mu_2)]\geq\min\{\tau^i_W(\gamma_1-\mu_1),\tau^i_W(\gamma_2-\mu_2)\}=\tau^i_W(0)$$

and

$$\eta^k_W\{[\gamma_1,\gamma_2]-[\mu_1,\mu_2]\}=\eta^k_W[(\gamma_1-\mu_1),(\gamma_2-\mu_2)]\leq\max\{\eta^k_W(\gamma_1-\mu_1),\eta^k_W(\gamma_2-\mu_2)\}=\eta^k_W(0),$$

Thus, $(\gamma_1+\gamma_2)\mathcal{R}(\mu_1+\mu_2)$, $c\gamma_1\mathcal{R}c\mu_1$ and $[\gamma_1,\gamma_2]\,\mathcal{R}\,[\mu_1,\mu_2]$. Hence, $\mathcal{R}$ is a congruence relation on $\mathcal{L}$. □

Theorem 3.10. *Every multi q-Rung orthopair fuzzy Lie ideal is multi q-Rung orthopair fuzzy Lie subalgebra.*

Proof. Let $W=\left(\tau^1_W,\tau^2_W,\ldots,\tau^l_W;\eta^1_W,\eta^2_W,\ldots,\eta^n_W\right)$ be multi q-rung orthopair fuzzy Lie ideal. Then $\forall\gamma,\eta\in\mathcal{L};\forall i\in N_l,\forall k\in N_n,\forall c\in\mathcal{F}$, the following three assumptions of Definition 3.3 easily hold good:

(i) $\tau^i_W(\gamma+\mu)\geq\min\{\tau^i_W(\gamma),\tau^i_W(\mu)\},\eta^k_W(\gamma+\mu)\leq\max\{\eta^k_W(\gamma),\eta^k_W(\mu)\}$.

(ii) $\tau^i_W(c\gamma)\geq\tau^i_W(\gamma),\eta^k_W(c\gamma)\leq\eta^k_W(\gamma)$.

(iii) $\tau^i_W[\gamma,\mu]\geq\min\{\tau^i_W(\gamma),\tau^i_W(\mu)\},\eta^k_W[\gamma,\mu]\leq\max\{\eta^k_W(\gamma),\eta^k_W(\mu)\}$.

Thus W is multi q-Rung orthopair fuzzy Lie subalgebra. □

The converse of the above Theorem 3.10 is not true which can be seen from the following example:

Example 3.11. Suppose that $F = \mathbb{R}$, the set of real numbers and $\mathcal{L} = \{(\alpha, \beta, \gamma) : \alpha, \beta, \gamma \in \mathbb{R}\}$ be the Lie algebra. Let us define the mapping $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$ by $[u, v] = u \times v + v$, where $\times$ denotes the vector product (or cross product). Consider the multi q-Rung orthopair fuzzy set $W = (\tau_W^1, \tau_W^2, \ldots, \tau_W^4; \eta_W^1, \eta_W^2, \ldots, \eta_W^5) : \mathcal{L} \to [0,1] \times [0,1] \times [0,1] \times [0,1] \times [0,1] \times [0,1] \times [0,1] \times [0,1] \times [0,1]$ described by

$$\tau_W^1(\alpha,\beta,\gamma) = \begin{cases} 0.9 & \text{if } \alpha = \beta = \gamma = 0 \\ 0.5 & \text{if } \alpha \neq 0, \beta = \gamma = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\tau_W^2(\alpha,\beta,\gamma) = \begin{cases} \frac{o.9}{2} & \text{if } \alpha = \beta = \gamma = 0 \\ \frac{o.5}{2} & \text{if } \alpha \neq 0, \beta = \gamma = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\tau_W^3(\alpha,\beta,\gamma) = \begin{cases} \frac{o.9}{3} & \text{if } \alpha = \beta = \gamma = 0 \\ \frac{o.5}{3} & \text{if } \alpha \neq 0, \beta = \gamma = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\tau_W^4(\alpha,\beta,\gamma) = \begin{cases} \frac{o.9}{4} & \text{if } \alpha = \beta = \gamma = 0 \\ \frac{o.5}{4} & \text{if } \alpha \neq 0, \beta = \gamma = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\eta_W^1(\alpha,\beta,\gamma) = \begin{cases} 0 & \text{if } \alpha = \beta = \gamma = 0 \\ 0.7 & \text{if } \alpha \neq 0, \beta = \gamma = 0 \\ 1 & \text{otherwise} \end{cases}$$

$$\eta_W^2(\alpha,\beta,\gamma) = \begin{cases} 0 & \text{if } \alpha = \beta = \gamma = 0 \\ \frac{o.7}{2} & \text{if } \alpha \neq 0, \beta = \gamma = 0 \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

$$\eta_W^3(\alpha,\beta,\gamma) = \begin{cases} 0 & \text{if } \alpha = \beta = \gamma = 0 \\ \frac{o.7}{3} & \text{if } \alpha \neq 0, \beta = \gamma = 0 \\ \frac{1}{3} & \text{otherwise} \end{cases}$$

$$\eta_W^4(\alpha,\beta,\gamma) = \begin{cases} 0 & \text{if } \alpha = \beta = \gamma = 0 \\ \frac{o.7}{4} & \text{if } \alpha \neq 0, \beta = \gamma = 0 \\ \frac{1}{4} & \text{otherwise} \end{cases}$$

$$\eta_W^5(\alpha,\beta,\gamma) = \begin{cases} 0 & \text{if } \alpha = \beta = \gamma = 0 \\ \frac{o.7}{5} & \text{if } \alpha \neq 0, \beta = \gamma = 0 \\ \frac{1}{5} & \text{otherwise.} \end{cases}$$

Then it is easy to verify that W is a MqROF Lie subalgebra of $\mathcal{L}$ but it is not MqROF Lie ideal of $\mathcal{L}$ because for each $i = 1, 2, 3, 4$, we find that

$$\tau_W^i[(2,0,0),(i,-i,i)] = \tau_W^i(i,-3i,-i) = 0 \not\geq \frac{o.5}{i} = \tau_W^i(2,0,0).$$

Theorem 3.12. *Let $N_l = \{1, 2, \ldots, l\}, N_n = \{1, 2, \ldots, n\}$ and $\mathcal{L}$ be a Lie Algebra of vectors over the field $\mathcal{F}$. The necessary and sufficient condition for a multi q-Rung orthopair fuzzy set $W = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \eta_W^1, \eta_W^2, \ldots, \eta_W^n)$ to be a MqROF Lie subalgebra over $\mathcal{L}$ is that, $\forall r \in [0,1]$, non-empty upper r-level cut $U_r(\tau_W^i) = \{\gamma \in \mathcal{L} : \tau_W^i(\gamma) \geq r, \forall i \in N_l\}$ and non-empty lower r-level cut $V_r(\eta_W^k) = \{\gamma \in \mathcal{L} : \eta_W^k(\gamma) \leq r, \forall k \in N_n\}$ are Lie subalgebra over $\mathcal{L}$.*

Proof. Suppose that $W = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \eta_W^1, \eta_W^2, \ldots, \eta_W^n)$ is a MqROF Lie subalgebra over $\mathcal{L}$ and $r \in [0,1]$ is such that $U_r(\tau_W^i) \neq \phi$. Let $\gamma, \eta \in U_r(\tau_W^i)$. Then $\forall \gamma, \eta \in \mathcal{L}; \forall i \in N_l, \forall k \in N_n, \forall c \in \mathcal{F}$,

$$\tau_W^i(\gamma+\mu) \geq \min\{\tau_W^i(\gamma), \tau_W^i(\mu)\} \geq r, \eta_W^k(\gamma+\mu) \leq \max\{\eta_W^k(\gamma), \eta_W^k(\mu)\} \leq r,$$

$$\tau_W^i(c\gamma) \geq \tau_W^i(\gamma) \geq r, \eta_W^k(c\gamma) \leq \eta_W^k(\gamma) \leq r$$

and

$$\tau_W^i[\gamma,\mu] \geq \min\{\tau_W^i(\gamma), \tau_W^i(\mu)\} \geq r, \eta_W^k[\gamma,\mu] \leq \max\{\eta_W^k(\gamma), \eta_W^k(\mu)\} \leq r.$$

From which it follows that,

$$\gamma+\mu, c\gamma, [\gamma,\mu] \in U_r(\tau_W^i)$$

and

$$\gamma+\mu, c\gamma, [\gamma,\mu] \in V_r(\eta_W^k).$$

Hence, $U_r(\tau_W^i)$ and $V_r(\eta_W^k)$ constitute Lie sub algebra over $\mathcal{L}$.

Conversely, suppose that $\forall i \in i \in N_l$ and $\forall r \in [0,1]$, $U_r(\tau_W^i) \neq \phi$ is a Lie subalgebra over $\mathcal{L}$ and if possible suppose that for some $\gamma, \mu \in \mathcal{L}$.

$$\tau_W^i(\gamma+\mu) \ngeq \min\{\tau_W^i(\gamma), \tau_W^i(\mu)\}.$$

If we choose

$$r_0 = \frac{1}{2}(\tau_W^i(\gamma+\mu) + \min\{\tau_W^i(\gamma), \tau_W^i(\mu)\}),$$

then by properties of inequality we must have,

$$\tau_W^i(\gamma+\mu) < r_0 < \min\{\tau_W^i(\gamma), \tau_W^i(\mu)\}.$$

This implies that, $\gamma+\mu \notin U_r(\tau_W^i), \gamma, \mu \in U_r(\tau_W^i)$, which is a contradiction. Hence

$$\tau_W^i(\gamma+\mu) \geq \min\{\tau_W^i(\gamma), \tau_W^i(\mu)\}, \forall \gamma, \mu \in \mathcal{L}.$$

In a similar manner we can prove that

$$\tau_W^i(c\gamma) \geq \tau_W^i(\gamma); \tau_W^i[\gamma,\mu] \geq \min\{\tau_W^i(\gamma), \tau_W^i(\mu)\}, \forall c \in \mathcal{F}, \forall i \in N_l.$$

Similarly we can prov for the case $V_r(\eta_W^k)$. This completes the proof. □

Theorem 3.13. *Let $N_l = \{1,2,\ldots,l\}, N_n = \{1,2,\ldots,n\}$ and $\mathcal{L}$ be a Lie Algebra of vectors over the field $\mathcal{F}$. If $V = (\tau_V^1, \tau_V^2, \ldots, \tau_V^l; \eta_V^1, \eta_V^2, \ldots, \eta_V^n)$ and $W = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \eta_W^1, \eta_W^2, \ldots, \eta_W^n)$ are two MqROF Lie subalgebra over $\mathcal{L}$, then their intersection $V \cap W = H = (\tau_H^1, \tau_H^2, \ldots, \tau_H^l; \eta_H^1, \eta_H^2, \ldots, \eta_H^n)$ is also MqROF Lie subalgebra over $\mathcal{L}$.*

Proof. Suppose that $\gamma, \mu \in \mathcal{L}$ be arbitrary and $\forall i \in N_l, \forall k \in N_n, \forall c \in \mathcal{F}$. So by definition of intersection of two MqROF sets we will have,

$$\begin{aligned}
\tau_H^i(\gamma+\mu) &= \min\{\tau_V^i(\gamma+\mu), \tau_W^i(\gamma+\mu)\} \geq \min\{\min\{\tau_V^i(\gamma), \tau_V^i(\mu)\}, \min\{\tau_W^i(\gamma), \tau_W^i(\mu)\}\} \\
&= \min\{\min\{\tau_V^i(\gamma), \tau_W^i(\gamma)\}, \min\{\tau_V^i(\mu), \tau_W^i(\mu)\}\} \\
&= \min\{\tau_H^i(\gamma), \tau_H^i(\mu)\}
\end{aligned}$$

and

$$\begin{aligned}
\eta_H^k(\gamma+\mu) &= \max\{\eta_V^k(\gamma+\mu), \eta_W^k(\gamma+\mu)\} \leq \max\{\max\{\eta_V^k(\gamma), \eta_V^k(\mu)\}, \max\{\eta_W^k(\gamma), \eta_W^k(\mu)\}\} \\
&= \max\{\max\{\eta_V^k(\gamma), \eta_W^k(\gamma)\}, \max\{\eta_V^k(\mu), \eta_W^k(\mu)\}\} \\
&= \max\{\eta_H^k(\gamma), \eta_H^k(\mu)\}.
\end{aligned}$$

Also, we see that

$$\tau_H^i(c\gamma) = \min\{\tau_V^i(c\gamma), \tau_W^i(c\gamma)\} \geq \{\min\{\tau_V^i(\gamma), \tau_W^i(\gamma)\} = \tau_H^i(\gamma)$$

and

$$\eta_H^k(c\gamma) = \max\{\eta_V^k(c\gamma), \eta_W^k(c\gamma)\} \leq \{\max\{\eta_V^k(\gamma), \eta_W^k(\gamma)\} = \eta_H^k(\gamma).$$

Now,

$$\begin{aligned}\tau_H^i[\gamma,\mu] &= \min\{\tau_V^i[\gamma,\mu], \tau_W^i[\gamma,\mu]\} \geq \min\{\min\{\tau_V^i(\gamma), \tau_V^i(\mu)\}, \min\{\tau_W^i(\gamma), \tau_W^i(\mu)\}\} \\ &= \min\{\min\{\tau_V^i(\gamma), \tau_W^i(\gamma)\}, \min\{\tau_V^i(\mu), \tau_W^i(\mu)\}\} = \min\{\tau_H^i(\gamma), \tau_H^i(\mu)\}\end{aligned}$$

and

$$\begin{aligned}\eta_H^k[\gamma,\mu] &= \max\{\eta_V^k[\gamma,\mu], \eta_W^k[\gamma,\mu]\} \leq \max\{\max\{\eta_V^k(\gamma), \eta_V^k(\mu)\}, \max\{\eta_W^k(\gamma), \eta_W^k(\mu)\}\} \\ &= \max\{\max\{\eta_V^k(\gamma), \eta_W^k(\gamma)\}, \max\{\eta_V^k(\mu), \eta_W^k(\mu)\}\} \\ &= \max\{\eta_H^k(\gamma), \eta_H^k(\mu)\}.\end{aligned}$$

Hence, $V \cap W = H$ is MqROF Lie subalgebra over $\mathcal{L}$. □

Definition 3.14. Let $N_l = \{1, 2, \ldots, l\}, N_n = \{1, 2, \ldots, n\}$ and $\mathcal{L}$ be a Lie Algebra of vectors over the field $\mathcal{F}$. If $V = (\tau_V^1, \tau_V^2, \ldots, \tau_V^l; \eta_V^1, \eta_V^2, \ldots, \eta_V^n)$ and $W = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \eta_W^1, \eta_W^2, \ldots, \eta_W^n)$ are two MqROF sets on $\mathcal{L}$, then the product $H = V \times W$ defined on $\mathcal{L} \times \mathcal{L}$ will be known as generalized cartasian product if

$$\begin{aligned}(V \times W)(\gamma,\mu) &= \left[\left(\tau_V^1, \tau_V^2, \ldots, \tau_V^l; \eta_V^1, \eta_V^2, \ldots, \eta_V^n\right) \times \left(\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \eta_W^1, \eta_W^2, \ldots, \eta_W^n\right)\right](\gamma,\mu) \\ &= (\tau_H^1(\gamma,\mu), \tau_H^2(\gamma,\mu), \ldots, \tau_H^l(\gamma,\mu); \eta_H^1(\gamma,\mu), \eta_H^2(\gamma,\mu), \ldots, \eta_H^n(\gamma,\mu)), \forall(\gamma,\mu) \in \mathcal{L} \times \mathcal{L},\end{aligned}$$

where

$$\tau_H^i(\gamma,\mu) = (\tau_V^i \times \tau_W^i)(\gamma,\mu) = \min\{\tau_V^i(\gamma), \tau_W^i(\mu)\}, \forall i \in N_l$$

and

$$\eta_H^k(\gamma,\mu) = (\eta_V^k \times \eta_W^k)(\gamma,\mu) = \max\{\eta_V^k(\gamma), \eta_W^k(\mu)\}, \forall k \in N_n.$$

Evidently the generalized cartasian product $(V \times W)$ is MqROF set on $\mathcal{L} \times \mathcal{L}$ if

$$0 \leq \{\tau_H^i(\gamma,\mu)\}^q + \{\eta_H^k(\gamma,\mu)\}^q \leq 1,$$

i.e.,

$$0 \leq \{\min\{\tau_V^i(\gamma), \tau_W^i(\mu)\}\}^q + \{\max\{\eta_V^k(\gamma), \eta_W^k(\mu)\}\}^q \leq 1,$$

where $i \in N_l, k \in N_n$.

Theorem 3.15. *Let $\mathcal{L}$ be the Lie Algebra of vectors over the field $\mathcal{F}$. If $V = (\tau_V^1, \tau_V^2, \ldots, \tau_V^l; \eta_V^1, \eta_V^2, \ldots, \eta_V^n)$ and $W = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \eta_W^1, \eta_W^2, \ldots, \eta_W^n)$ are two MqROF Lie subalgebra of $\mathcal{L}$, then the generalized cartasian product $(V \times W)$ is MqROF Lie subalgebra of $\mathcal{L} \times \mathcal{L}$.*

Proof. Let $N_l = \{1, 2, \ldots, l\}, N_n = \{1, 2, \ldots, n\}$ and $\mathcal{L}$ be a Lie algebra of vectors over the field $\mathcal{F}$. Then, $\forall i \in N_l, \forall k \in N_n \forall \gamma = (\gamma_1, \gamma_2), \mu = (\mu_1, \mu_2) \in \mathcal{L} \times \mathcal{L}$ and $c \in \mathcal{F}$, we need to show by Definition 3.3 that the conditions (i), (ii), (iii), (iv), (v) and (vi) are satisfied.

(i) $(\tau_V^i \times \tau_W^i)(\gamma + \mu) \geq \min\{(\tau_V^i \times \tau_W^i)(\gamma), (\tau_V^i \times \tau_W^i)(\mu)\}$.

(ii) $(\eta_V^k \times \eta_W^k)(\gamma + \mu) \leq \max\{(\eta_V^k \times \eta_W^k)(\gamma), (\eta_V^k \times \eta_W^k)(\mu)\}$.

(iii) $(\tau_V^i \times \tau_W^i)(c\gamma) \geq (\tau_V^i \times \tau_W^i)(\gamma)$.

(iv) $(\eta_V^k \times \eta_W^k)\{(c\gamma)\} \leq (\eta_V^k \times \eta_W^k)\{(\gamma)\}$.

(v) $(\tau_V^i \times \tau_W^i)[\gamma,\mu] \geq \min\{(\tau_V^i \times \tau_W^i)(\gamma), (\tau_V^i \times \tau_W^i)(\mu)\}$.

(vi) $(\eta_V^k \times \eta_W^k)[\gamma,\mu] \leq \max\{(\eta_V^k \times \eta_W^k)(\gamma), (\eta_V^k \times \eta_W^k)(\mu)\}$.

Now

$$\begin{aligned}(\tau_V^i \times \tau_W^i)(\gamma+\mu) &= (\tau_V^i \times \tau_W^i)((\gamma_1,\gamma_2)+(\mu_1,\mu_2)) = (\tau_V^i \times \tau_W^i)((\gamma_1+\mu_1),(\gamma_2+\mu_2))\\ &= \min\{\tau_V^i(\gamma_1+\mu_1), \tau_W^i(\gamma_2+\mu_2)\} \geq \min\{\min\{\tau_V^i(\gamma_1), \tau_V^i(\mu_1)\}, \min\{\tau_W^i(\gamma_2), \tau_W^i(\mu_2)\}\}\\ &= \min\{\min\{\tau_V^i(\gamma_1), \tau_W^i(\gamma_2)\}, \min\{\tau_V^i(\mu_1), \tau_W^i(\mu_2)\}\}\\ &= \min\{(\tau_V^i \times \tau_W^i)(\gamma_1,\gamma_2), (\tau_V^i \times \tau_W^i)(\mu_1,\mu_2)\}\\ &= \min\{(\tau_V^i \times \tau_W^i)(\gamma), (\tau_V^i \times \tau_W^i)(\mu)\}\end{aligned}$$

and

$$\begin{aligned}(\eta_V^k \times \eta_W^k)(\gamma+\mu) &= (\eta_V^k \times \eta_W^k)((\gamma_1,\gamma_2)+(\mu_1,\mu_2)) = (\eta_V^k \times \eta_W^k)((\gamma_1+\mu_1),(\gamma_2+\mu_2))\\ &= \max\{\eta_V^k(\gamma_1+\mu_1), \eta_W^k(\gamma_2+\mu_2)\} \leq \max\{\max\{\eta_V^k(\gamma_1), \eta_V^k(\mu_1)\}, \min\{\eta_W^k(\gamma_2), \eta_W^k(\mu_2)\}\}\\ &= \max\{\max\{\eta_V^k(\gamma_1), \eta_W^k(\gamma_2)\}, \max\{\eta_V^k(\mu_1), \eta_W^k(\mu_2)\}\}\\ &= \max\{(\eta_V^k \times \eta_W^k)(\gamma_1,\gamma_2), (\eta_V^k \times \eta_W^k)(\mu_1,\mu_2)\}\}\\ &= \max\{(\eta_V^k \times \eta_W^k)(\gamma), (\eta_V^k \times \eta_W^k)(\mu)\}.\end{aligned}$$

Again,

$$\begin{aligned}(\tau_V^i \times \tau_W^i)(c\gamma) &= (\tau_V^i \times \tau_W^i)\{c(\gamma_1,\gamma_2)\} = (\tau_V^i \times \tau_W^i)(c\gamma_1, c\gamma_2) = \min\{\tau_V^i(c\gamma_1), \tau_W^i(c\gamma_2)\}\\ &\geq \min\{\tau_V^i(\gamma_1), \tau_W^i(\gamma_2)\} = (\tau_V^i \times \tau_W^i)(\gamma_1,\gamma_2) = (\tau_V^i \times \tau_W^i)(\gamma)\end{aligned}$$

and

$$\begin{aligned}(\eta_V^k \times \eta_W^k)\{(c\gamma)\} &= (\eta_V^k \times \eta_W^k)\{c(\gamma_1,\gamma_2)\} = (\eta_V^k \times \eta_W^k)(c\gamma_1, c\gamma_2)\\ &= \max\{\eta_V^k(c\gamma_1), \eta_W^k(c\gamma_2)\} \leq \max\{\eta_V^k(\gamma_1), \eta_W^k(\gamma_2)\}\\ &= (\eta_V^k \times \eta_W^k)(\gamma_1,\gamma_2) = (\eta_V^k \times \eta_W^k)(\gamma).\end{aligned}$$

Finally,

$$\begin{aligned}(\tau_V^i \times \tau_W^i)[\gamma,\mu] &= (\tau_V^i \times \tau_W^i)[(\gamma_1,\gamma_2),(\mu_1,\mu_2)]\\ &\geq \min\{\min\{\tau_V^i(\gamma_1), \tau_W^i(\gamma_2)\}, \{\min\{\tau_V^i(\mu_1), \tau_W^i(\mu_2)\}\}\\ &= \min\{(\tau_V^i \times \tau_W^i)(\gamma_1,\gamma_2)\}, \{(\tau_V^i \times \tau_W^i)(\mu_1,\mu_2)\} = \min\{(\tau_V^i \times \tau_W^i)(\gamma), (\tau_V^i \times \tau_W^i)(\mu)\}\end{aligned}$$

and

$$\begin{aligned}(\eta_V^k \times \eta_W^k)[\gamma,\mu] &= (\eta_V^k \times \eta_W^k)[(\gamma_1,\gamma_2),(\mu_1,\mu_2)]\\ &\leq \max\{\max\{\eta_V^k(\gamma_1), \eta_W^k(\gamma_2)\}, \{\max\{\eta_V^k(\mu_1), \eta_W^k(\mu_2)\}\}\\ &= \max\{(\eta_V^k \times \eta_W^k)(\gamma_1,\gamma_2)\}, \{(\eta_V^k \times \eta_W^k)(\mu_1,\mu_2)\} = \max\{(\eta_V^k \times \eta_W^k)(\gamma), (\eta_V^k \times \eta_W^k)(\mu)\}.\end{aligned}$$

Hence, $(V \times W)$ is MqROF Lie subalgebra of $\mathcal{L} \times \mathcal{L}$. □

4. Multi q-Rung orthopair fuzzy Lie algebra homomorphisms and applications

In this section, the properties of multi q-Rung orthopair fuzzy (resp. MqROF) Lie subalgebras and multi q-Rung orthopair fuzzy Lie ideals under homomorphisms of Lie algebras are investigated. Also some of their applications to solvable and nilpotent multi q-Rung orthopair fuzzy Lie ideals are presented.

Definition 4.1. Let L_1 and L_2 be two Lie algebras over the field $\mathcal{F}$. A linear transformation $\varphi : L_1 \to L_2$ is said to be Lie homomorphism if the relationship $\varphi([\gamma,\mu]) = [\varphi(\gamma), \varphi(\mu)]$ is true, $\forall \gamma, \mu \in L_1$.

Definition 4.2. Let L_1 and L_2 be two Lie algebras over the field $\mathcal{F}$. Then a Lie homomorphism $\varphi : L_1 \to L_2$ is said have natural extension $\varphi : I^{L_1} \to I^{L_2}$ if $\forall W = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \eta_W^1, \eta_W^2, \ldots, \eta_W^n) \in I^{L_1}$ and $\mu \in L_2$, the followings hold for all $i = 1, 2, 3, \ldots, l$;for all $k = 1, 2, 3, \ldots, n$:

$$\varphi(\tau_W^i)(\mu) = \sup\{\tau_W^i(\gamma) : \gamma \in \varphi^{-1}(\mu), \gamma \in L_1\}$$

and

$$\varphi(\eta_W^k)(\mu) = \inf\{\eta_W^k(\gamma) : \gamma \in \varphi^{-1}(\mu), \gamma \in L_1\},$$

Definition 4.3. Let $\varphi : L_1 \to L_2$ be a Lie homomorphism between two Lie algebras L_1 and L_2. If

$$W_1 = (\tau_{W_1}^1, \tau_{W_1}^2, \ldots, \tau_{W_1}^l; \eta_{W_1}^1, \eta_{W_1}^2, \ldots, \eta_{W_1}^n)$$

and $W_2 = (\tau_{W_2}^1, \tau_{W_2}^2, \ldots, \tau_{W_2}^l; \eta_{W_2}^1, \eta_{W_2}^2, \ldots, \eta_{W_2}^n)$ are two MqROF sets of L_1 and L_2, then

$$\mu \in L_2, \forall W_2 = (\tau_{W_2}^1, \tau_{W_2}^2, \ldots, \tau_{W_2}^l; \eta_{W_2}^1, \eta_{W_2}^2, \ldots, \eta_{W_2}^n)$$

of Ł$_2$, the inverse image of W under φ we denote by $\varphi^{-1}(W_2)$ is defined by

$$\varphi^{-1}(W_2)(\mu) = (\varphi^{-1}(\tau_{W_2}^1(\mu)), \varphi^{-1}(\tau_{W_2}^2(\mu)), \ldots, \varphi^{-1}(\tau_{W_2}^l(\mu)), \varphi^{-1}(\eta_{W_2}^1(\mu)), \varphi^{-1}(\eta_{W_2}^2(\mu)), \ldots, \varphi^{-1}(\eta_{W_2}^l(\mu)))$$

and

$$\varphi^{-1}(W_2)(\mu) = (\tau_{W_2}^1(\varphi(\mu)), \tau_{W_2}^2(\varphi(\mu)), \ldots, \tau_{W_2}^l(\varphi(\mu)); \eta_{W_2}^1(\varphi(\mu)), \eta_{W_2}^1(\varphi(\mu)), \ldots, \eta_{W_2}^1(\varphi(\mu))$$

Proposition 4.4. *Let $W = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \eta_W^1, \eta_W^2, \ldots, \eta_W^n) \in I^{L_1}$ be MqROF Lie subalgebras and $\varphi : L_1 \to L_2$ be Lie homomorphism between L_1 and L_2. Then $\varphi(W)$ is MqROF Lie subalgebras of L_2.*

Proof. Suppose that $\mu_1, \mu_2 \in L_2$. Then $\{\gamma : \gamma \in \varphi^{-1}(\mu_1 + \mu_2)\} \supseteq \{\gamma_1 + \gamma_2 : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$. Now, for all $i = 1, 2, 3, \ldots, l$ and $k = 1, 2, 3, \ldots, n$, it is sufficient to show that,

(i) $\varphi(\tau_W^i)(\mu_1 + \mu_2) \geq \min\{\varphi(\tau_W^i)(\mu_1), \varphi(\tau_W^i)(\mu_2)\}$.

(ii) $\varphi(\eta_W^k)(\mu_1 + \mu_2) \leq \max\{\varphi(\eta_W^k)(\mu_1), \varphi(\eta_W^k)(\mu_2)\}$.

(iii) $\varphi(\tau_W^i)(c\mu) \geq \varphi(\tau_W^i)(\mu)$.

(iv) $\varphi(\eta_W^k)(c\mu) \leq \varphi(\eta_W^k)(\mu)$.

(v) $\varphi(\tau_W^i)([\mu_1, \mu_2]) \geq \min\{\varphi(\tau_W^i)(\mu_1), \varphi(\tau_W^i)(\mu_2)\}$.

(vi) $\varphi(\eta_W^k)([\mu_1, \mu_2]) \leq \min\{\varphi(\eta_W^k)(\mu_1), \varphi(\eta_W^k)(\mu_2)\}$.

Now

$$\begin{aligned}
\varphi(\tau_W^i)(\mu_1 + \mu_2) &= \sup\{\tau_W^i(\gamma) : \gamma \in \varphi^{-1}(\mu_1 + \mu_2), \gamma \in L_1\} \\
&\geq \{\tau_W^i(\gamma_1 + \gamma_2) : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\} \\
&\geq \sup\{\min\{\tau_W^i(\gamma_1), \tau_W^i(\gamma_2)\} : \gamma_1 \in \varphi^{-1}(\mu_1), \\
\gamma_2 \in \varphi^{-1}(\mu_2)\} &= \min\{\sup\{\tau_W^i(\gamma_1) : \gamma_1 \in \varphi^{-1}(\mu_1)\}, \sup\{\tau_W^i(\gamma_2) : \gamma_2 \in \varphi^{-1}(\mu_2)\}\} \\
&= \min\{\varphi(\tau_W^i)(\mu_1), \varphi(\tau_W^i)(\mu_2)\}.
\end{aligned}$$

Also we obtain,

$$\begin{aligned}
\varphi(\eta_W^k)(\mu_1 + \mu_2) &= \inf\{\eta_W^k(\gamma) : \gamma \in \varphi^{-1}(\mu_1 + \mu_2), \gamma \in L_1\} \leq \{\eta_W^k(\gamma_1 + \gamma_2) : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\} \\
&\leq \inf\{\max\{\eta_W^k(\gamma_1), \eta_W^k(\gamma_2)\} : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\} \\
&= \max\{\inf\{\eta_W^k(\gamma_1) : \gamma_1 \in \varphi^{-1}(\mu_1)\}, \inf\{\eta_W^k(\gamma_2) : \gamma_2 \in \varphi^{-1}(\mu_2)\}\} \\
&= \max\{\varphi(\eta_W^k)(\mu_1), \varphi(\eta_W^k)(\mu_2)\}
\end{aligned}$$

which proves (i) and (ii).

For $\mu \in L_2$ and $c \in \mathcal{F}$, we have

$$\{\gamma : \gamma \in \varphi^{-1}(c\mu_1)\} \supseteq \{c\gamma : \gamma \in \varphi^{-1}(\mu)\}.$$

Now, we see that,

$$\begin{aligned}\varphi(\tau_W^i)(c\mu) &= \sup\{\tau_W^i(c\gamma) : \gamma \in \varphi^{-1}(\mu), \gamma \in L_1\} \\ &\geq \sup\{\tau_W^i(c\gamma) : \gamma \in \varphi^{-1}(c\mu), \gamma \in L_1\} \geq \sup\{\tau_W^i(\gamma) : \gamma \in \varphi^{-1}(\mu), \gamma \in L_1\} = \varphi(\tau_W^i)(\mu).\end{aligned}$$

Similarly, we can we find that

$$\begin{aligned}\varphi(\eta_W^k)(c\mu) &= \inf\{\eta_W^k(c\gamma) : \gamma \in \varphi^{-1}(\mu), \gamma \in L_1\} \\ &\leq \inf\{\eta_W^k(c\gamma) : \gamma \in \varphi^{-1}(c\mu), \gamma \in L_1\} \leq \inf\{\eta_W^k(\gamma) : \gamma \in \varphi^{-1}(\mu), \gamma \in L_1\} = \varphi(\eta_W^k)(\mu)\end{aligned}$$

which proves (iii) and (iv).

For, $\mu_1, \mu_2 \in L_2$, we have

$$\{\gamma : \gamma \in \varphi^{-1}(\mu_1 + \mu_2)\} \supseteq \{\gamma_1 + \gamma_2 : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}.$$

Now, we have

$$\begin{aligned}\varphi(\tau_W^i)([\mu_1, \mu_2]) &= \sup\{\tau_W^i(\gamma) : \gamma \in \varphi^{-1}([\mu_1, \mu_2]), \gamma \in L_1\} \\ &\geq \sup\{\tau_W^i([\gamma_1, \gamma_2]) : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\} \\ &\geq \sup\{\min\{\tau_W^i(\gamma_1), \tau_W^i(\gamma_2)\} : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\} \\ &= \min\{\sup\{\tau_W^i(\gamma_1) : \gamma_1 \in \varphi^{-1}(\mu_1)\}, \sup\{\tau_W^i(\gamma_2) : \gamma_2 \in \varphi^{-1}(\mu_2)\}\} \\ &= \min\{\varphi(\tau_W^i)(\mu_1), \varphi(\tau_W^i)(\mu_2)\}.\end{aligned}$$

Similarly, we we find that

$$\begin{aligned}\varphi(\eta_W^k)([\mu_1, \mu_2]) &= \inf\{\eta_W^k(\gamma) : \gamma \in \varphi^{-1}([\mu_1, \mu_2]), \gamma \in L_1\} \leq \inf\{\eta_W^k([\gamma_1, \gamma_2]) : \gamma_1 \in \varphi^{-1}(\mu_1), \\ \gamma_2 \in \varphi^{-1}(\mu_2)\} &\leq \inf\{\max\{\eta_W^k(\gamma_1), \tau_W^k(\gamma_2)\} : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\} \\ &= \max\{\inf\{\eta_W^k(\gamma_1) : \gamma_1 \in \varphi^{-1}(\mu_1)\}, \sup\{\eta_W^i(\gamma_2) : \gamma_2 \in \varphi^{-1}(\mu_2)\}\} = \max\{\varphi(\eta_W^k)(\mu_1), \varphi(\eta_W^k)(\mu_2)\}\end{aligned}$$

which proves (v) and (vi).

Hence, $\varphi(W)$ is MqROF Lie subalgebras of L_2. □

Corollary 4.5. *Let $W = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \eta_W^1, \eta_W^2, \ldots, \eta_W^n) \in I^{L_1}$ be multi q-Rung orthopair fuzzy Lie ideal and $\varphi : L_1 \to L_2$ be Lie homomorphism between L_1 and L_2. Then $\varphi(W)$ is multi q-Rung orthopair fuzzy Lie ideal of L_2.*

Proof. The proof is similar to the proof of Proposition 4.4. □

Proposition 4.6. *Let $W = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \eta_W^1, \eta_W^2, \ldots, \eta_W^n) \in I^{L_2}$ be MqROF Lie subalgebras and $\varphi : L_1 \to L_2$ be Lie homomorphism between L_1 and L_2. Then $\varphi^{-1}(W)$ is MqROF Lie subalgebra of L_1.*

Proof. Suppose that $\mu_1, \mu_2 \in L_1$. Since, W is MqROF Lie subalgebras, so, for all $i = 1, 2, 3, \ldots, l; k = 1, 2, 3, \ldots, n$; we need to show that conditions (i), (ii), (iii), (iv), (v) and (vi) are satisfied.

(i) $\varphi^{-1}(\tau_W^i)(\mu_1 + \mu_2) \geq \min\{\varphi^{-1}(\tau_W^i)(\mu_1), \varphi^{-1}(\tau_W^i)(\mu_2)\}$.

(ii) $\varphi^{-1}(\eta_W^k)(\mu_1 + \mu_2) \leq \max\{\varphi^{-1}(\eta_W^k)(\mu_1), \varphi^{-1}(\eta_W^k)(\mu_2)\}$.

(iii) $\varphi^{-1}(\tau_W^i)(c\mu) \geq \varphi^{-1}(\tau_W^i)(\mu)$.

(iv) $\varphi^{-1}(\eta_W^i)(c\mu) \leq \varphi^{-1}(\eta_W^i)(\mu)$.

(v) $\varphi^{-1}(\tau_W^i)[\mu_1,\mu_2] \geq \min\{\varphi^{-1}(\tau_W^i)(\mu_1), \varphi^{-1}(\tau_W^i)(\mu_2)\}$.

(vi) $\varphi^{-1}(\eta_W^i)[\mu_1,\mu_2] \leq \max\{\varphi^{-1}(\eta_W^i)(\mu_1), \varphi^{-1}(\eta_W^i)(\mu_2)\}$.

Now, we obtain

$$\begin{aligned}\varphi^{-1}(\tau_W^i)(\mu_1+\mu_2) &= \tau_W^i[\varphi(\mu_1+\mu_2)]\\ &= \tau_W^i[\varphi(\mu_1)+\varphi(\mu_2)] \geq \min\{\tau_W^i(\varphi(\mu_1)), \tau_W^i(\varphi(\mu_2))\\ &= \min\{\varphi^{-1}(\tau_W^i)(\mu_1), \varphi^{-1}(\tau_W^i)(\mu_2)\}\end{aligned}$$

and

$$\begin{aligned}\varphi^{-1}(\eta_W^k)(\mu_1+\mu_2) &= \eta_W^k(\varphi(\mu_1+\mu_2)) = \eta_W^k(\varphi(\mu_1)+\varphi(\mu_2)) \leq \max\{\eta_W^k(\varphi(\mu_1)), \eta_W^k(\varphi(\mu_2))\}\\ &= \max\{\varphi^{-1}(\eta_W^k)(\mu_1), \varphi^{-1}(\eta_W^k)(\mu_2)\}\end{aligned}$$

which shows (i) and (ii). Also for all $\mu \in L_1$ and $c \in \mathcal{F}$, we have,

$$\varphi^{-1}(\tau_W^i)(c\mu) = \tau_W^i[\varphi(c\mu)] = \tau_W^i[c\varphi(\mu)] \geq \tau_W^i(\varphi(\mu)) = \varphi^{-1}(\tau_W^i)(\mu)$$

and

$$\varphi^{-1}(\eta_W^i)(c\mu) = \eta_W^i[\varphi(c\mu)] = \eta_W^i[c\varphi(\mu)] \leq \eta_W^i(\varphi(\mu)) = \varphi^{-1}(\eta_W^i)(\mu)$$

which shows (iii) and (iv).

Now let $\mu_1, \mu_2 \in L_1$, then we find that

$$\begin{aligned}\varphi^{-1}(\tau_W^i)[\mu_1,\mu_2] &= \tau_W^i(\varphi[\mu_1,\mu_2]) = \tau_W^i[\varphi(\mu_1),\varphi(\mu_2)] \geq \min\{\tau_W^i(\varphi(\mu_1)), \tau_W^i(\varphi(\mu_2))\}\\ &= \min\{\varphi^{-1}(\tau_W^i)(\mu_1), \varphi^{-1}(\tau_W^i)(\mu_2)\}\end{aligned}$$

and

$$\begin{aligned}\varphi^{-1}(\eta_W^i)[\mu_1,\mu_2] &= \eta_W^i(\varphi[\mu_1,\mu_2]) = \eta_W^i[\varphi(\mu_1),\varphi(\mu_2)]\\ &\leq \max\{\eta_W^i(\varphi(\mu_1)), \eta_W^i(\varphi(\mu_2))\} = \max\{\varphi^{-1}(\eta_W^i)(\mu_1), \varphi^{-1}(\eta_W^i)(\mu_2)\}.\end{aligned}$$

Hence, $\varphi^{-1}(W)$ is MqROF Lie subalgebra of L_1. □

Corollary 4.7. *Let $W = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \eta_W^1, \eta_W^2, \ldots, \eta_W^n) \in I^{L_2}$ be multi q-Rung orthopair fuzzy Lie ideal and $\varphi : L_1 \to L_2$ be Lie homomorphism between L_1 and L_2. Then $\varphi^{-1}(W)$ is multi q-Rung orthopair fuzzy Lie ideal of L_1.*

Proof. The proof is similar to the proof of Proposition 4.6 □

Theorem 4.8. *Let $V = (\tau_V^1, \tau_V^2, \ldots, \tau_V^l; \eta_V^1, \eta_V^2, \ldots, \eta_V^n)$ and $W = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \eta_W^1, \eta_W^2, \ldots, \eta_W^n)$ be two MqROF Lie subalgebra of $\mathcal{L}_1$ and φ: $\mathcal{L}_1 \to \mathcal{L}_2$ be Lie homomorphism of $\mathcal{L}_1$ onto $\mathcal{L}_2$. Then*

$$\varphi(\ll VW \gg) = \ll \varphi(V)\varphi(W) \gg .$$

Proof. If possible let us suppose that $\varphi(\ll VW \gg) < (\ll \varphi(V)\varphi(W) \gg)$. Then there exists $s \in [0,1]$ such that

$$\varphi(\ll VW \gg)(\gamma) < s < (\ll \varphi(V)\varphi(W) \gg)(\gamma).$$

Then there will exists $\xi_r, \nu_r \in \mathcal{L}_2$ such that $\gamma = \sum_{r=1}^n [\xi_r \nu_r]$ satisfying $\varphi(V)(\xi_r) > s$ and $\varphi(W)(\nu_r) < s$. Because of φ being surjective map there exist $\xi \in \mathcal{L}_1$ such that $\varphi(\xi) = \gamma$ and $\xi = \sum_{r=1}^n [p_r q_r]$ for some $p_r \in \varphi^{-1}(\xi_r), q_r \in \varphi^{-1}(\nu_r)$ where $\varphi(p_r) = \xi_r, \varphi(q_r) = \nu_r, V(p_r) > s, W(q_r) > s$. Since

$$\varphi\left(\sum_{r=1}^n [p_r q_r]\right) = \left(\sum_{r=1}^n \varphi[p_r q_r]\right) = \sum_{r=1}^n [\varphi(p_r)\varphi(q_r)] = \sum_{r=1}^n [\xi_r \nu_r]$$

and hence $\varphi(\ll VW \gg)(\gamma) > s$ which contradicts our assumption.
Similarly if we assume $\varphi(\ll VW \gg) > (\ll \varphi(V)\varphi(W) \gg)$, we shall again arrive at the same contradiction. Hence the only possibility is that

$$\varphi(\ll VW \gg) = \ll \varphi(V)\varphi(W) \gg .$$

This completes the proof. □

Definition 4.9. Let $V = (\tau_V^1, \tau_V^2, \ldots, \tau_V^l; \eta_V^1, \eta_V^2, \ldots, \eta_V^n)$ and $W = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \eta_W^1, \eta_W^2, \ldots, \eta_W^n)$ be two MqROF Lie subalgebra of a Lie algebra $\mathcal{L}$. Then V is said to be same type of W if there exists automorphism f of $\mathcal{L}$ such that $V = W \circ f$, that is, for all $\gamma \in \mathcal{L}$,

$$\tau_V^i(\gamma) = \tau_W^i(f(\gamma)), \eta_V^k(\gamma) = \eta_W^k(f(\gamma)).$$

Theorem 4.10. *If $V = (\tau_V^1, \tau_V^2, \ldots, \tau_V^l; \eta_V^1, \eta_V^2, \ldots, \eta_V^n)$ and $W = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \eta_W^1, \eta_W^2, \ldots, \eta_W^n)$ are two MqROF Lie subalgebra of a Lie algebra $\mathcal{L}$ over the field $\mathcal{F}$. Then V will be same type of W iff V is isomorphic to W.*

Proof. Since the sufficient part is very trivial, so we only prove the necessary part. Suppose that V is same type of W. Then there exists an automorphism φ of $\mathcal{L}$ such that for all $\gamma \in \mathcal{L}$, for $i = 1, 2, 3, \ldots, l$;
$k = 1, 2, 3, \ldots, n$;

$$\tau_V^i(\gamma) = \tau_W^i(\varphi(\gamma)), \eta_V^k(\gamma) = \eta_W^k(\varphi(\gamma)).$$

Let $f : V(\mathcal{L}) \to W(\mathcal{L})$ be a mapping such that $f(\varphi(\gamma)) = W(\varphi(\gamma))$. Then

$$f(\tau_V^i(\gamma)) = \tau_W^i(\varphi(\gamma)), f(\eta_V^k(\gamma)) = \eta_W^k(\varphi(\gamma)).$$

This implies that f is a onto mapping. Now let for all $\gamma, \mu \in \mathcal{L}$ and for all $i = 1, 2, 3, \ldots, l$,

$$f(\tau_V^i(\gamma)) = f(\tau_V^i(\mu)) \Rightarrow \tau_W^i(\varphi(\gamma)) = \tau_W^i(\varphi(\mu)) \Rightarrow \tau_V^i(\gamma) = \tau_V^i(\mu).$$

In the similar fashion we can show that for all $k = 1, 2, 3, \ldots, n$;

$$f(\eta_V^k(\gamma)) = f(\eta_V^k(\mu)) \Rightarrow \eta_V^k(\gamma) = \eta_V^k(\mu).$$

This shows that f is one-one function.
We now show that f is homomorphism. Let $\gamma, \mu \in \mathcal{L}, c \in \mathcal{F}$. Then

$$f(\tau_V^i(\gamma + \mu)) = \tau_W^i(\varphi(\gamma + \mu)) = \tau_W^i(\varphi(\gamma)) + \tau_W^i(\varphi(\mu)))$$

and

$$f(\eta_V^k(\gamma + \mu)) = \eta_W^k(\varphi(\gamma + \mu)) = \eta_W^k(\varphi(\gamma)) + \eta_W^k(\varphi(\mu)).$$

Also, we have

$$f(\tau_V^i(c\gamma)) = \tau_W^i(\varphi(c\gamma)) = c\tau_W^i(\varphi(\gamma))$$

and

$$f(\eta_V^k(c\gamma)) = \eta_W^k(\varphi(c\gamma)) = c\eta_W^k(\varphi(\gamma)).$$

Now we find that,

$$f(\tau_V^i[\gamma, \mu]) = \tau_W^i(\varphi[\gamma, \mu]) = \tau_W^i[\varphi(\gamma), \varphi(\mu)]$$

and

$$f(\eta_V^k(\gamma + \mu)) = \eta_W^k(\varphi(\gamma + \mu)) = \eta_W^k[\varphi(\gamma), \varphi(\mu)].$$

Thus V is isomorphic to W. □

Definition 4.11. Let $W = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \eta_W^1, \eta_W^2, \ldots, \eta_W^n)$ be a MqROF Lie ideal of a Lie algebra $\mathcal{L}$. Define a sequence of MqROF Lie ideal of a Lie algebra $\mathcal{L}$ by

$$W_0 = W, W_1 = [W_0, W_0], W_2 = [W_1, W_1], W_3 = [W_2, W_2], \ldots, W_r = [W_{r-1}, W_{r-1}].$$

Then W_r is called r-th derived MqROF Lie ideal of a Lie algebra $\mathcal{L}$ and the chain or series

$$W_0 \supseteq W_1 \supseteq W_2 \supseteq, W_3 \supseteq, \ldots, W_r \supseteq, \ldots$$

is called derived series of MqROF Lie ideal W of Lie algebra $\mathcal{L}$.

Definition 4.12. A MqROF Lie ideal $W = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \lambda_W^1, \lambda_W^2, \ldots, \lambda_W^m; \eta_W^1, \eta_W^2, \ldots, \eta_W^n)$ is said to be solvable MqROF Lie ideal of Lie algebra $\mathcal{L}$ if there is a possitive integer n such that the chain of MqROF Lie ideals $\{W_0, W_1, W_2, W_3, \ldots, W_r\}$ is such that

$$W_0 \supseteq W_1 \supseteq W_2 \supseteq, W_3 \supseteq, \ldots, W_r \supseteq = \{0,0,0,\ldots,0;0,0,0,\ldots,0;0,0,0,\ldots,0\},$$

where $W_0 = W$ and any natural number $r, W_r = [W_{r-1}, W_{r-1}]$.

Theorem 4.13. *Let $W = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \lambda_W^1, \lambda_W^2, \ldots, \lambda_W^m; \eta_W^1, \eta_W^2, \ldots, \eta_W^n)$ be solvable MqROF Lie ideal of Lie algebra L_1 and $\varphi : L_1 \to L_2$ be homomorphism between Lie algebras L_1 and L_2. Then $\varphi(W)$ is solvable MqROF Lie ideal of L_2.*

Proof. Suppose that $\varphi : L_1 \to L_2$ be homomorphism between L_1 and L_2 and

$$W = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \eta_W^1, \eta_W^2, \ldots, \eta_W^n) \in I^{L_1}$$

be solvable MqROF Lie ideal of L_1. By induction on any possitive integer r we shall prove that $\varphi(W^r) \supseteq [\varphi(W)]^r$. Suppose that $\gamma \in L_2$ and

$$\varphi([W,W]) \supseteq [\varphi(W), \varphi(W)].$$

Then for all $i = 1, 2, 3, \ldots, l$, we have

$$\begin{aligned}
\varphi(\ll \tau_W^i, \tau_W^i \gg)(\gamma) &= \sup\{(\ll \tau_W^i, \tau_W^i \gg)(\gamma) : \varphi(\gamma) = \mu\} \\
&= \sup\{\sup\{(\min(\tau_W^i(\alpha), \tau_W^i(\beta)) : [\alpha,\beta] = \mu, \varphi(\gamma) = \mu, \alpha, \beta \in\in L_1\}\} \\
&= \sup\{(\min(\tau_W^i(\alpha), \tau_W^i(\beta)) : [\alpha,\beta] = \mu, \varphi(\gamma) = \mu, \alpha, \beta \in\in L_1\} \\
&= \sup\{(\min(\tau_W^i(\alpha), \tau_W^i(\beta)) : [\varphi(\alpha), \varphi(\beta)] = \gamma, \alpha, \beta \in\in L_1\} \\
&= \sup\{(\min(\tau_W^i(\alpha), \tau_W^i(\beta)) : [\varphi(\alpha), \varphi(\beta)] = \gamma, \alpha, \beta \in\in L_1\} \\
&= \sup\{(\min(\tau_W^i(\alpha), \tau_W^i(\beta)) : [\nu,\delta] = \gamma, \varphi(\alpha) = \nu, \varphi(\beta) = \delta, \alpha, \beta \in\in L_1\} \\
&\geq \sup\{((\min(\sup_{\alpha \in \varphi^{-1}(\nu)} \tau_W^i(\alpha)), \min(\sup_{\beta \in \varphi^{-1}(\delta)} \tau_W^i(\beta))) : [\nu,\delta] = \gamma, \alpha, \beta \in\in L_1\} \\
&= \sup\{\min\{(\varphi(\tau_W^i(\nu)), \varphi(\tau_W^i(\delta))) : [\nu,\delta] = \gamma, \alpha, \beta \in\in L_1\}\} \\
&= \ll \varphi(\tau_W^i), \varphi(\tau_W^i) \gg (\gamma).
\end{aligned}$$

Also, for all $k = 1, 2, 3, \ldots, n$, we have

$$\begin{aligned}
\varphi(\ll \eta_W^k, \eta_W^k \gg)(\gamma) &= \inf\{(\ll \eta_W^k, \eta_W^k \gg)(\gamma) : \varphi(\gamma) = \mu\} \\
&= \inf\{\inf\{(\max(\eta_W^k(\alpha), \eta_W^k(\beta)) : [\alpha,\beta] = \mu, \varphi(\gamma) = \mu, \alpha, \beta \in\in L_1\}\} \\
&= \inf\{(\max(\eta_W^k(\alpha), \eta_W^k(\beta)) : [\alpha,\beta] = \mu, \varphi(\gamma) = \mu, \alpha, \beta \in\in L_1\} \\
&= \inf\{(\max(\eta_W^k(\alpha), \eta_W^k(\beta)) : [\varphi(\alpha), \varphi(\beta)] = \mu, \alpha, \beta \in\in L_1\} \\
&= \inf\{(\max(\eta_W^k(\alpha), \eta_W^k(\beta)) : [\varphi(\alpha), \varphi(\beta)] = \mu, \alpha, \beta \in\in L_1\} \\
&= \inf\{(\max(\eta_W^k(\alpha), \eta_W^k(\beta)) : [\nu,\delta] = \mu, \varphi(\alpha) = \nu, \varphi(\beta) = \delta, \alpha, \beta \in\in L_1\} \\
&\leq \inf\{((\max(\inf_{\alpha \in \varphi^{-1}(\nu)} \eta_W^k(\alpha)), \max(\inf_{\beta \in \varphi^{-1}(\delta)} \eta_W^k(\beta))) : [\nu,\delta] = \gamma, \alpha, \beta \in\in L_1\} \\
&= \inf\{\max\{(\varphi(\eta_W^k(\nu)), \varphi(\lambda_W^k(\delta))) : [\nu,\delta] = \gamma, \alpha, \beta \in\in L_1\}\} \\
&= \ll \varphi(\eta_W^k), \varphi(\eta_W^k) \gg (\gamma).
\end{aligned}$$

Thus,

$$\varphi([W,W]) \supseteq \varphi(\ll \varphi(W), \varphi(W) \gg) = [\varphi(W), \varphi(W)].$$

Now for $r > 1$,

$$\varphi(W^r) = \varphi\left(\left[W^{r-1}, W^{r-1}\right]\right) \supseteq \left[\varphi(W)^{r-1}, \varphi(W)^{r-1}\right].$$

Hence $\varphi(W)$ is solvable MqROF Lie ideal of L_2. □

Definition 4.14. Let $W = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \eta_W^1, \eta_W^2, \ldots, \eta_W^n)$ be a MqROF Lie ideal of a Lie algebra $\mathcal{L}$. Define a sequence of MqROF Lie ideal of $\mathcal{L}$ by

$$W^0 = W, W^1 = \left[W, W^0\right], W^2 = \left[W, W^1\right], W^3 = \left[W, W^2\right], \ldots, W^r = \left[W, W^{r-1}\right].$$

Then the chain or series

$$W^0 \supseteq W^1 \supseteq W^2 \supseteq, W^3 \supseteq, \ldots, W^r \supseteq, \ldots$$

is called descending central series of MqROF Lie ideal W of Lie algebra $\mathcal{L}$.

Definition 4.15. A MqROF Lie ideal $W = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \lambda_W^1, \lambda_W^2, \ldots, \lambda_W^m; \eta_W^1, \eta_W^2, \ldots, \eta_W^n)$ is said to be nilpotent MqROF Lie ideal of Lie algebra $\mathcal{L}$ if there is a possitive integer n such that the chain of MqROF Lie ideals $\{W^0, W^1, W^2, W^3, \ldots, W^r\}$ is such that

$$W^0 \supseteq W^1 \supseteq W^2 \supseteq, W^3 \supseteq, \ldots, W^r \supseteq, \ldots = \{0, 0, 0, \ldots, 0; 0, 0, 0, \ldots, 0; 0, 0, 0, \ldots, 0\},$$

where, $W^0 = W$ and for any natural number r, $W^r = \left[W, W^{r-1}\right]$.

Corollary 4.16. *Let $W = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \eta_W^1, \eta_W^2, \ldots, \eta_W^n)$ be nilpotent MqROF Lie ideal of L_1 and $\varphi : L_1 \to L_2$ be homomorphism between Lie algebras L_1 and L_2. Then $\varphi(W)$ is nilpotent MqROF Lie ideal of L_2.*

Proof. The proof is just similar to the proof of Theorem 4.13. □

Theorem 4.17. *Let $V = (\tau_V^1, \tau_V^2, \ldots, \tau_V^l; \eta_V^1, \eta_V^2, \ldots, \eta_V^n)$ be MqROF Lie ideal of MqROF Lie algebra $\mathcal{L}$. If*

$$W = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \eta_W^1, \eta_W^2, \ldots, \eta_W^n)$$

is MqROF Lie ideal of $\mathcal{L}$, then the MqROF set

$$W^{\#} = (\tau_{W^{\#}}^1, \tau_{W^{\#}}^2, \ldots, \tau_{W^{\#}}^l; \eta_{W^{\#}}^1, \eta_{W^{\#}}^2, \ldots, \eta_{W^{\#}}^n)$$

of quotient Lie algebra $\mathcal{L}_{|V}$ defined by

$$\tau_{W^{\#}}^i(\mu + V) = \sup_{\alpha \in V} \tau_W^i(\mu + \alpha), \forall i = 1, 2, 3, \ldots, l$$

and

$$\eta_{W^{\#}}^i(\mu + V) = \sup_{\alpha \in V} \eta_W^i(\mu + \alpha), \forall k = 1, 2, 3, \ldots, n$$

is MqROF Lie ideal of the quotient Lie algebra $\mathcal{L}_{|V}$ of $\mathcal{L}$ relative to MqROF Lie ideal V.

Proof. From definition it is clear that $W^{\#}$ is well defined. Suppose that $\alpha + V, \beta + V \in \mathcal{L}_{|V}$ and $i = 1, 2, 3, \ldots, l$. Then for each i, we have,

$$\begin{aligned}
\tau_{W^{\#}}^i((\alpha + V) + (\beta + V)) &= \tau_{W^{\#}}^i((\alpha + \beta) + V)) = \sup_{\gamma \in V} \tau_W^i((\alpha + \beta) + \gamma) \\
&= \sup_{\gamma = \theta + \xi \in V} \tau_W^i((\alpha + \beta) + (\theta + \xi)) \\
&\geq \sup_{\theta, \xi \in V} \min\{\tau_W^i(\alpha + \theta), \tau_W^i(\beta + \xi)\} = \min\{\sup_{\theta, \xi \in V} \tau_W^i(\alpha + \theta), \sup_{\theta, \xi \in V} \tau_W^i(\beta + \xi)\} \\
&= \min\{\tau_{W^{\#}}^i(\alpha + V), \tau_{W^{\#}}^i(\beta + V)\}.
\end{aligned}$$

Again let $c \in \mathcal{F}$, then we must have,

$$\begin{aligned}
\tau_{W^{\#}}^i(c(\alpha + V)) = \tau_{W^{\#}}^i(c\alpha + V) &= \sup_{\gamma \in V} \tau_W^i(c\alpha + \gamma) \\
&\geq \sup_{\gamma \in V} \tau_W^i(\alpha + \gamma) = \tau_{W^{\#}}^i(\alpha + V)
\end{aligned}$$

and

$$\begin{aligned}\tau^i_{W^\#}\left(\left[\alpha+V,\tau^i_{W^\#}(\beta+V)\right]\right) &= \tau^i_{W^\#}\left(\left[\alpha+V,\sup_{\gamma\in V}\tau^i_W(\beta+\gamma)\right]\right)=\tau^i_{W^\#}\left([\alpha,\beta]+V\right)\\ &= \sup_{\gamma\in V}\tau^i_W\left([\alpha,\beta]+\gamma\right)\geq\sup_{\gamma\in V}\tau^i_W(\alpha+\gamma)=\tau^i_{W^\#}(\alpha+V).\end{aligned}$$

From which it follows that $\tau^i_{W^\#}$ is MqROF Lie ideal of $\mathcal{L}_{|V}$. Similarly it can be proved that $\eta^k_{W^\#}$ is also MqROF Lie ideal of $\mathcal{L}_{|V}\forall k=1,2,3\ldots,n$. Hence $W^\#$ is MqROF Lie ideal of the quotient Lie algebra $\mathcal{L}_{|V}$ of $\mathcal{L}$ relative to MqROF Lie ideal V. □

5. Conclusion

In this paper, we introduce the concept of multi q-Rung orthopair fuzzy Lie subalgebras and multi q-rung orthopair fuzzy Lie ideals of Lie algebra. Some of their fundamental properties and operations like intersection and generalized Cartesian product of multi q-Rung orthopair fuzzy Lie subalgebras are investigated. Moreover, the relationship between multi Fermatean fuzzy Lie subalgebras and multi q-Rung orthopair fuzzy Lie ideals are established. Lastly, the image and the inverse image of multi q-Rung orthopair fuzzy Lie subalgebras (multi q-rung orthopair fuzzy Lie ideals) under Lie homomorphisms are also studied. The findings offered here add to the larger framework of Lie algebraic theory, offering fresh perspectives for further study of the structural characteristics of BCK and BCI algebra using multi q-Rung orthopair fuzzy sets. The proposed work can be applied to situations involving multiple decision makers, pattern recognition, classification, and multicriteria decision making. Thus, the new theory will be a helpful tool for decision and ranking problems like choosing robots, choosing green suppliers, choosing solid waste landfill sections, diagnosing medical conditions, and identifying patterns, among others.

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