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Regular summability techniques in approximation using modified Mellin convolution operators

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Abstract

The paper deals with the approximation properties of the modified Mellin-Gauss-Weierstrass operator in certain weighted spaces under the supremum norm. The primary goal is to apply regular summability methods for more generalized and improved results compared to those found in the existing literature. By employing the weighted logarithmic modulus of continuity, we also investigate the order of convergence in the approximation.

Keywords: Mellin-Gauss-Weierstrass operators, regular summability methods, weighted space, weighted logarithmic modulus of continuity

2020 MSC: 41A25, 41A35

1. Introduction

It is well known that the Mellin transform plays a crucial role in approximation theory, as discussed in (*cf.* [2, 3, 23]). It has important applications in various fields, including mathematics, physics, statistics, economics, engineering and computer science (*cf.* [4, 13, 14]). Mellin analysis has wide application in signal processing especially with regard to exponential sampling (*cf.* [17]). A key operator in the Mellin analysis is the convolution operator. In their study, Kolbe and Nessel first introduced the singular integrals of the Mellin convolution-type operator (*cf.* [21]). Butzer and Jansche [16] extensively studied the Mellin transform, defining Mellin convolution and obtaining notable result. Furthermore, Bardaro and Mantellini [11] have provided a Mellin convolution operators of type

$$T_k(f)(s) = \int_0^\infty K_k(t) f(ts) \frac{dt}{t}, \ s \in \mathbb{R}^+,$$

where the function f lies in the domain of the operator T_k. Here, $K_k : \mathbb{R}^+ \to \mathbb{R}$ is a family of kernel functions that fulfills the condition $\int_0^\infty K_k(t) \frac{dt}{t} = 1$.

In approximation theory, certain summability methods, such as subsequence matrix transformations, are used to accelerate the convergence of sequences (*cf.* [6, 7, 20]). This improvement in convergence speed is one of the key motivations for employing summability techniques. This study extends the application of these techniques to

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approximation by linear operators, building upon previous findings (*cf.* [24]). For approximation by the operators, we consider the regular summability methods introduced by Bell (*cf.* [12]). Let $\mathcal{A} = \{A^v\} = \left\{\left[a_{nk}^v\right]\right\}$ $(n,\ k,\ v \in \mathbb{N})$ be a family of infinite matrices of real or complex numbers. Let $x = \{x_k\}$ be a sequence, the $\mathcal{A}$-transform of x is a sequence $(\mathcal{A}x)_n^v$ which is defined by the $(\mathcal{A}x)_n^v = \sum_{k=1}^{\infty} a_{nk}^v x_k$ $(n,\ v \in \mathbb{N})$ provided that the series converges for every $n,\ v$. Then x is called $\mathcal{A}$-summable to a number L whenever $\lim_{n\to\infty} (\mathcal{A}x)_n^v = L$ uniformly in $v \in \mathbb{N}$. Then we write $\mathcal{A} - \lim x = L$.

A summability method $\mathcal{A}$ is called regular if $\mathcal{A} - \lim x = \lim x_k$ uniformly in v for all $\mathcal{A}$-summable sequences x. A summability method $\mathcal{A} = \{A^v\} = \left\{\left[a_{nk}^v\right]\right\}$ is said to be regular in [1] if and only if the following conditions are satisfied:

(a) For each $n, v \in \mathbb{N}$, $\sum\limits_{k=1}^{\infty} |a_{nk}^v|$ is finite, and there exist positive integers N and M such that $\sup\limits_{n\geq N, v\in\mathbb{N}} \sum\limits_{k=1}^{\infty} |a_{nk}^v| \leq M$.

(b) $\lim\limits_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}^v = \delta$ uniformly in v,

(c) for every $k \in \mathbb{N}$, $\lim\limits_{n\to\infty} a_{nk}^v = \delta_k$ uniformly in v with $\delta_k \equiv 0$ and $\delta \equiv 1$.

In this approach, the nonnegative regularity of a matrix family is essential because it ensures that the standard approximation result is a particular case of the regular summability method. In this paper, we investigate the Mellin convolution type integral operators defined by

$$\mathcal{T}_{n,v}(f)(s) = \sum_{k=1}^{\infty} a_{nk}^v \int_0^{\infty} K_k(t) f(ts) \frac{dt}{t}.$$

An important type of Mellin convolution operator is the Mellin-Gauss-Weierstrass operator whose kernel $K_k(t) = \frac{k}{\sqrt{4\pi}} e^{-\left(\frac{k}{2}\log t\right)^2}$ (*cf.* [16, p. 9, Definition 8]), i.e.,

$$\mathcal{T}_{n,v}(f)(s) = \sum_{k=1}^{\infty} a_{nk}^v \frac{k}{\sqrt{4\pi}} \int_0^{\infty} f(ts) e^{-\left(\frac{k}{2}\log t\right)^2} \frac{dt}{t}.$$

The operator is known as the Mellin-Gauss-Weierstrass convolution operator.

Now, we consider the following operators:

$$\mathcal{G}_{n,v}(f)(s) = \sum_{k=1}^{\infty} a_{nk}^v \frac{k}{\sqrt{4\pi}} \int_0^{\infty} f(ta_k(s)) e^{-\left(\frac{k}{2}\log t\right)^2} \frac{dt}{t},$$

and determine a suitable choice of function $a_k(s)$, for which the operator reproduces the function $f(t) = \log^2 t$. Thus, we get $a_k(s) = e^{\sqrt{\log^2(s) - \frac{2}{k^2}}},\ s \geq e^{\frac{\sqrt{2}}{k}}$.

Then, the operators can be written as

$$\mathcal{G}_{n,v}(f)(s) = \sum_{k=1}^{\infty} a_{nk}^v \frac{k}{\sqrt{4\pi}} \int_0^{\infty} f\left(te^{\sqrt{\log^2(s) - \frac{2}{k^2}}}\right) e^{-\left(\frac{k}{2}\log t\right)^2} \frac{dt}{t},$$

$k \geq 1$ and $s > e^{\frac{\sqrt{2}}{k}}$.

Moreover, the modified operator is defined as follows

$$G_{n,v}(f)(s) = \begin{cases} \mathcal{G}_{n,v}(f)(s), & s \in (e^{\sqrt{2}/k},\ \infty), \\ f(s), & s \in (0,\ e^{\sqrt{2}/k}]. \end{cases}$$

This study is motivated by recent results of Özsaraç et al. [24], in which a modified version of the Mellin-Gauss-Weierstrass convolution operator is investigated. By using a regular summability method, we aim to obtain better approximation results than the classical ones. The Bell type summability method includes several well-known convergence methods, such as classical convergence, Cesàro convergence and almost convergence (*cf.* [15, 18, 22]). These convergence methods are especially useful in cases where the classical limit fails and have been extensively applied not only in summability theory but also in approximation theory (*cf.* [6]-[8], [20]).

2. Approximation in supremum norm

Recently, the approximation properties of positive linear integral operators have been studied in certain weighted spaces (*cf.* [9, 25]). For $x \in \mathbb{R}^+$, taking the weight function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\rho(x) = 1 + \log^2(x)$. Then the weighted space of continuous functions $B_2(\mathbb{R}^+)$ introduced in [5] by

$$B_2(\mathbb{R}^+) = \{f : \mathbb{R}^+ \to \mathbb{R} \,\big|\, |f(x)| \leq M\rho(x) \text{ for some } M > 0\}$$

and

$$C_2(\mathbb{R}^+) = C(\mathbb{R}^+) \cap B_2(\mathbb{R}^+),$$

which is natural subspace of $B_2(\mathbb{R}^+)$. It is known that space $C_2^k(\mathbb{R}^+)$ is a subspace of $B_2(\mathbb{R}^+)$ (*cf.* [24]) i.e.,

$$C_2^k(\mathbb{R}^+) = \left\{ f \in C_2(\mathbb{R}^+) \,\middle|\, \lim_{x\to\infty} \frac{|f(x)|}{\rho(x)} = k \right\}.$$

Then $B_2(\mathbb{R}^+)$ forms a normed linear space equipped with the norm

$$\|f\|_L = \sup_{x>0} \frac{|f(x)|}{1 + (\log x)^2}.$$

We also need the following conditions:

(i) There exists a constant $M > 0$ such that $\sup_{n,\, v\in\mathbb{N}} \sum_{k=1}^{\infty} a_{nk}^v = M < \infty$.

(ii) $\mathcal{A} - \lim\left(\int_0^\infty K_k(t)\frac{dt}{t}\right) = 1$.

(iii) $\mathcal{A} - \lim\left(\int_0^\infty K_k(t)(\log t)\frac{dt}{t}\right) = 0$.

Lemma 2.1 (The operator preserves the constant function **1**)**.** $\lim_{n\to\infty} \mathcal{G}_{n,v}(\mathbf{1})(s) = 1,\ n, v \in \mathbb{N}$.

Proof. Using condition (ii), we have

$$\begin{aligned}
\mathcal{G}_{n,v}(\mathbf{1})(s) &= \sum_{k=1}^{\infty} a_{nk}^v \frac{k}{\sqrt{4\pi}} \int_0^\infty \mathbf{1}(ta_k) e^{-\left(\frac{k}{2}\log t\right)^2} \frac{dt}{t} \\
&= \sum_{k=1}^{\infty} a_{nk}^v \frac{k}{\sqrt{4\pi}} \int_0^\infty e^{-\left(\frac{k}{2}\log t\right)^2} \frac{dt}{t} \\
&= 1 \text{ (as } n \to \infty).
\end{aligned}$$

□

Lemma 2.2. $\lim_{n\to\infty} \mathcal{G}_{n,v}(\log)(s) = |\log(s)|$, *for every* $s > 1$.

Proof. Let $t^k = x$. Then from condition (iii), we get

$$\begin{aligned}
\mathcal{G}_{n,v}(\log)(s) &= \sum_{k=1}^{\infty} a_{nk}^v \frac{k}{\sqrt{4\pi}} \int_0^\infty (\log ta_k(s)) e^{-\left(\frac{k}{2}\log t\right)^2} \frac{dt}{t} \\
&= \sum_{k=1}^{\infty} a_{nk}^v \frac{k}{\sqrt{4\pi}} \int_0^\infty (\log t + \log a_k(s)) e^{-\left(\frac{k}{2}\log t\right)^2} \frac{dt}{t} \\
&= \sum_{k=1}^{\infty} a_{nk}^v \frac{1}{\sqrt{4\pi}} \int_0^\infty \left(\frac{1}{k}\log x + \log a_k(s)\right) e^{-\left(\frac{1}{2}\log x\right)^2} \frac{dx}{x} \\
&= \sum_{k=1}^{\infty} a_{nk}^v \frac{1}{\sqrt{4\pi}} \int_0^\infty \left(\frac{1}{k}\log x + \sqrt{\log^2 s - \frac{2}{k^2}}\right) e^{-\left(\frac{1}{2}\log x\right)^2} \frac{dx}{x}
\end{aligned}$$

$$= \sum_{k=1}^{\infty} a_{nk}^{v} \frac{1}{\sqrt{4\pi}} \int_0^{\infty} \left(\frac{1}{k}\log x\right) e^{-\left(\frac{1}{2}\log x\right)^2} \frac{dx}{x} + \sum_{k=1}^{\infty} a_{nk}^{v} \frac{1}{\sqrt{4\pi}} \int_0^{\infty} \left(\sqrt{\log^2 s - \frac{2}{k^2}}\right) e^{-\left(\frac{1}{2}\log x\right)^2} \frac{dx}{x}$$

$$= \frac{1}{\sqrt{4\pi}} \sum_{k=1}^{\infty} a_{nk}^{v} \frac{1}{k} \int_0^{\infty} (\log x)\, e^{-\left(\frac{1}{2}\log x\right)^2} \frac{dx}{x} + \frac{1}{\sqrt{4\pi}} \sum_{k=1}^{\infty} a_{nk}^{v} \left(\sqrt{\log^2 s - \frac{2}{k^2}}\right) \int_0^{\infty} e^{-\left(\frac{1}{2}\log x\right)^2} \frac{dx}{x}.$$

Using the following equalities

$$\frac{1}{\sqrt{4\pi}} \int_0^{\infty} e^{-\left(\frac{1}{2}\log x\right)^2} \frac{dx}{x} = 1$$

and

$$\int_0^{\infty} (\log x)\, e^{-\left(\frac{1}{2}\log x\right)^2} \frac{dx}{x} = 0.$$

Using regularity condition (b), we get,

$$\mathcal{G}_{n,v}(\log)(s) = \sum_{k=1}^{\infty} a_{nk}^{v} \sqrt{\log^2 s - \frac{2}{k^2}}$$
$$= |\log(s)| \text{ as } n \to \infty.$$

□

Lemma 2.3 (The operator preserves the function $\log^2$)**.** $\lim_{n\to\infty} \mathcal{G}_{n,v}(\log^2)(s) = \log^2(s),\ n, v \in \mathbb{N}.$

Proof. Let $t^k = x$. Then we obtain

$$\mathcal{G}_{n,v}(\log^2)(s) = \sum_{k=1}^{\infty} a_{nk}^{v} \frac{k}{\sqrt{4\pi}} \int_0^{\infty} (\log t a_k(s))^2\, e^{-\left(\frac{k}{2}\log t\right)^2} \frac{dt}{t}$$

$$= \sum_{k=1}^{\infty} a_{nk}^{v} \frac{k}{\sqrt{4\pi}} \int_0^{\infty} (\log t + \log a_k(s))^2 e^{-\left(\frac{k}{2}\log t\right)^2} \frac{dt}{t}$$

$$= \frac{1}{\sqrt{4\pi}} \sum_{k=1}^{\infty} a_{nk}^{v} \int_0^{\infty} \left(\frac{1}{k}\log x + \sqrt{\log^2 s - \frac{2}{k^2}}\right)^2 e^{-\left(\frac{1}{2}\log x\right)^2} \frac{dx}{x}$$

$$= \frac{1}{\sqrt{4\pi}} \sum_{k=1}^{\infty} a_{nk}^{v} \int_0^{\infty} \left(\frac{1}{k}\log x\right)^2 e^{-\left(\frac{1}{2}\log x\right)^2} \frac{dx}{x}$$

$$+ \frac{2}{\sqrt{4\pi}} \sum_{k=1}^{\infty} a_{nk}^{v} \frac{1}{k} \sqrt{\log^2 s - \frac{2}{k^2}} \int_0^{\infty} (\log x)\, e^{-\left(\frac{1}{2}\log x\right)^2} \frac{dx}{x}$$

$$+ \frac{1}{\sqrt{4\pi}} \sum_{k=1}^{\infty} a_{nk}^{v} \int_0^{\infty} \left(\sqrt{\log^2 s - \frac{2}{k^2}}\right)^2 e^{-\left(\frac{1}{2}\log x\right)^2} \frac{dx}{x}.$$

Now, using the following equalities

$$\frac{1}{\sqrt{4\pi}} \int_0^{\infty} e^{-\left(\frac{1}{2}\log x\right)^2} \frac{dx}{x} = 1$$

and

$$\int_0^{\infty} (\log x)\, e^{-\left(\frac{1}{2}\log x\right)^2} \frac{dx}{x} = 0.$$

We have,

$$\begin{aligned}\mathcal{G}_{n,v}(\log^2)(s) &= \frac{1}{\sqrt{4\pi}}\sum_{k=1}^{\infty} a_{nk}^{v}\frac{1}{k^2}\int_0^{\infty}(\log x)^2\, e^{-\left(\frac{1}{2}\log x\right)^2}\frac{dx}{x} + \sum_{k=1}^{\infty} a_{nk}^{v}\left(\sqrt{\log^2 s - \frac{2}{k^2}}\right)^2\\ &= \frac{1}{2\sqrt{\pi}}\sum_{k=1}^{\infty} a_{nk}^{v}\frac{1}{k^2}4\sqrt{\pi} + \sum_{k=1}^{\infty} a_{nk}^{v}\left(\log^2 s - \frac{2}{k^2}\right)\\ &= \sum_{k=1}^{\infty} a_{nk}^{v}\frac{2}{k^2} + \sum_{k=1}^{\infty} a_{nk}^{v}\left(\log^2 s - \frac{2}{k^2}\right)\\ &= \log^2(s) \text{ as } n\to\infty.\end{aligned}$$

□

Lemma 2.4. *Let $\mathcal{A} = \left\{\left[a_{nk}^{v}\right]\right\}$ be a nonnegative regular summability method. If $f \in C_2(\mathbb{R}^+)$ and condition (i) hold then we have the following inequality*

$$\|G_{n,v}(f)\|_L \le \|f\|_L.$$

Proof. Since $f \in C_2(\mathbb{R}^+)$, by definition of the operator $\mathcal{G}_{n,v}$ we get,

$$\begin{aligned}|\mathcal{G}_{n,v}(f)(s)| &\le \sum_{k=1}^{\infty} a_{nk}^{v}\frac{k}{\sqrt{4\pi}}\int_0^{\infty}|f(ta_k(s))|\, e^{-\left(\frac{k}{2}\log t\right)^2}\frac{dt}{t}\\ &= \sum_{k=1}^{\infty} a_{nk}^{v}\frac{k}{\sqrt{4\pi}}\int_0^{\infty}|f(t)|\, e^{-\left(\frac{k}{2}\log\frac{t}{a_k(s)}\right)^2}\frac{dt}{t}\\ &= \sum_{k=1}^{\infty} a_{nk}^{v}\|f\|_L\frac{k}{\sqrt{4\pi}}\int_0^{\infty} e^{-\left(\frac{k}{2}\log\frac{t}{a_k(s)}\right)^2}(1+\log^2 t)\frac{dt}{t}\\ &= \sum_{k=1}^{\infty} a_{nk}^{v}\|f\|_L\frac{k}{\sqrt{4\pi}}\int_0^{\infty} e^{-\left(\frac{k}{2}\log\frac{t}{a_k(s)}\right)^2}\frac{dt}{t} + \sum_{k=1}^{\infty} a_{nk}^{v}\|f\|_L\frac{k}{\sqrt{4\pi}}\int_0^{\infty}\log^2 t e^{-\left(\frac{k}{2}\log\frac{t}{a_k(s)}\right)^2}\frac{dt}{t}.\end{aligned}$$

From Lemma 2.1 and Lemma 2.3 by simplifying, we get

$$\sum_{k=1}^{\infty} a_{nk}^{v}\frac{k}{\sqrt{4\pi}}\int_0^{\infty} e^{-\left(\frac{k}{2}\log\frac{t}{a_k(s)}\right)^2}(\log t)^2\frac{dt}{t} = (\log s)^2$$

and

$$\sum_{k=1}^{\infty} a_{nk}^{v}\frac{k}{\sqrt{4\pi}}\int_0^{\infty} e^{-\left(\frac{k}{2}\log\frac{t}{a_k(s)}\right)^2}\frac{dt}{t} = 1.$$

If $s \in \left(0,\ e^{\frac{\sqrt{2}}{k}}\right]$; we get

$$|G_{n,v}(f)(s)| = |f(s)| \le \|f\|_L(1+(\log s)^2).$$

If $s \in \left(e^{\frac{\sqrt{2}}{k}},\ \infty\right)$; we have

$$\begin{aligned}|G_{n,v}(f)(s)| &= |\mathcal{G}_{n,v}(f)(s)|\\ &\le \|f\|_L(1+(\log s)^2).\end{aligned}$$

Therefore, we obtain

$$\frac{|G_{n,v}(f)(s)|}{(1+(\log s)^2)} \le \|f\|_L,$$

which gives

$$\|G_{n,v}(f)\|_L \leq \|f\|_L.$$

□

Theorem 2.5. *Let $\mathcal{A} = \left\{\left[a_{nk}^v\right]\right\}$ be a nonnegative regular summability method and the conditions (i)-(iii) holds. Then for every $f \in C_2^k(\mathbb{R}^+)$, we have*

$$\lim_{n\to\infty} \|G_{n,v}(f)(s) - f\|_L = 0 \tag{2.1}$$

uniformly in $v \in \mathbb{N}$.

Proof. It is not hard to show that $\{1,\ \log t,\ \log^2 t\}$ is a Chebyshev system. Now, calculate Wronskian $W(t)$ for all $t > 0$, we have

$$W(t) = \begin{vmatrix} 1 & \log t & \log^2 t \\ 0 & \frac{1}{t} & \frac{2\log t}{t} \\ 0 & -\frac{1}{t^2} & \frac{2(1-\log t)}{t^2} \end{vmatrix} = \frac{2}{t^3}.$$

Since the Wronskian is non zero for all $t > 0$, the system $\{1,\ \log t,\ \log^2 t\}$ is a Chebyshev system on $\mathbb{R}^+$.

The operator $G_{n,v}$ is linear and positive, Lemma 2.4 ensures that it is uniformly bounded in the weighted norm. Therefore, by applying the general result (*cf.* [19]) and Theorem 2.11 in [10], it is sufficient to verify that

$$\lim_{n\to\infty} \|G_{n,v}(\log^m)(\cdot) - \log^m(\cdot)\|_L = 0 \tag{2.2}$$

holds for $m = 0,\ 1,\ 2$.
For $m = 0$, using Lemma 2.1, we get

$$|\mathcal{G}_{n,v}(\log^0)(s) - \log^0(s)| = |\mathcal{G}_{n,v}(\mathbf{1}) - 1| \to 0 \quad \text{as} \quad n \to \infty \text{ (uniformly in } v).$$

For $m = 1$ and $s > 1$, using Lemma 2.2, we have

$$|\mathcal{G}_{n,v}(\log)(s) - \log(s)| \to 0 \quad \text{as} \quad n \to \infty \text{ (uniformly in } v).$$

For $m = 2$, using Lemma 2.3, we can get

$$|\mathcal{G}_{n,v}(\log^2)(s) - \log^2(s)| \to 0 \quad \text{as} \quad n \to \infty \text{ (uniformly in } v).$$

Hence for $s \in \left(e^{\frac{\sqrt{2}}{k}},\ \infty\right)$, relation (2.1) holds. □

Corollary 2.6. *If $s > e^{\frac{\sqrt{2}}{k}}$, then we have*

$$\lim_{n\to\infty} \sup_{s>e^{\frac{\sqrt{2}}{k}}} |\mathcal{G}_{n,v}(f)(s) - f(s)| = 0,$$

holds for any function $f \in C_2^k(\mathbb{R}^+)$.

3. Order of convergence in approximation

In this section, we compute the order of convergence in Theorem 2.5. In [5], the authors defined the logarithmic modulus of continuity for $f \in C_2(\mathbb{R}^+)$ as follows

$$\omega_L(f,\delta) = \sup_{|\log t - \log s| \leq \delta} |f(t) - f(s)|,\ \ \delta > 0$$

and they introduced the following modulus of continuity, which is called as "weighted logarithmic modulus of continuity" and denoted by

$$\Omega(f;\delta) = \sup_{\substack{|\log t|\le\delta \\ s>0}} \frac{|f(ts)-f(s)|}{(1+(\log s)^2)(1+(\log t)^2)}. \tag{3.1}$$

The following properties are associated with the "weighted logarithmic modulus of continuity".

Lemma 3.1. *Let $\delta > 0$, $n \in \mathbb{N}$ then for $f \in C_2(\mathbb{R}^+)$, $\Omega(f;\delta) < \infty$.*

Proof. Since $f \in C_2(\mathbb{R}^+)$, we have

$$\begin{aligned}
|f(ts)-f(s)| &\le \|f\|_L\left(1+\log^2(ts)\right) + \|f\|_L\left(1+\log^2 s\right) \\
&\le 2\|f\|_L(1+\log^2 t+\log^2 s+\log^2 t\log^2 s) + \|f\|_L(1+\log^2 s) \\
&= 2\|f\|_L(1+\log^2 t)(1+\log^2 s) + \|f\|_L(1+\log^2 s).
\end{aligned}$$

Dividing both sides by $(1+\log^2 t)(1+\log^2 s)$ and then taking the supremum over all $s > 0$, with $|\log t| \le \delta$, we obtain

$$\begin{aligned}
\Omega(f;\delta) &< 3\|f\|_L \\
\Omega(f;\delta) &< \infty.
\end{aligned}$$

□

Lemma 3.2. *If $\delta > 0$ and $f \in C_2^k(\mathbb{R}^+)$, then $\lim_{\delta\to 0}\Omega(f;\delta) = 0$.*

Proof. Consider the inequality

$$\begin{aligned}
|f(ts)-f(s)| &= \left|\frac{f(ts)}{(1+\log^2(ts))}(1+\log^2(ts)) - \frac{f(s)}{1+\log^2 s}(1+\log^2 s)\right| \\
&\le |\log^2(ts)-\log^2 s|\frac{|f(ts)|}{(1+\log^2(ts))} + (1+\log^2 s)\left|\frac{f(ts)}{(1+\log^2(ts))} - \frac{f(s)}{1+\log^2 s}\right| \\
&\le |\log t|(|\log t|+2|\log s|)\frac{|f(ts)|}{(1+\log^2(ts))} + (1+\log^2 s)\left|\frac{f(t)}{(1+\log^2(t))} - \frac{f(s)}{1+\log^2 s}\right|.
\end{aligned}$$

Since $f \in C_2^k(\mathbb{R}^+)$, for every $\epsilon > 0$ there exists a positive constant s_0 such that

$$\left|\frac{f(s)}{1+\log^2 s} - k\right| < \frac{\epsilon}{2},\ s > s_0.$$

Let $s_1 > s_0 + \delta$. We obtain

$$\begin{aligned}
\Omega(f;\delta) &\le \sup_{\substack{0<s\le s_1 \\ |\log t|\le\delta}} |f(ts)-f(s)| + \delta(\delta+2)\|f\|_L + \sup_{\substack{s>s_1 \\ |\log t|\le\delta}}\left|\frac{f(t)}{(1+\log^2 t)} - k\right| + \sup_{\substack{s>s_1 \\ |\log t|\le\delta}}\left|\frac{f(s)}{(1+\log^2 s)} - k\right| \\
&\le \omega_L(f;\delta) + \epsilon.
\end{aligned}$$

Since the modulus of continuity $\omega_L(f;\delta)$ tends to zero as $\delta \to 0$ on the compact interval $[0,\ s_1]$. Therefore, we get

$$\lim_{\delta\to 0}\Omega(f;\delta) = 0.$$

□

Lemma 3.3. *For all $f \in C_2(\mathbb{R}^+)$, $\delta > 0$, $n \in \mathbb{N}$, we have*

$$\Omega(f; n\delta) \leq 2n^3(1+\delta^2)\Omega(f;\delta) \quad (n \in \mathbb{N})$$

and for any $\lambda \in \mathbb{R}^+$, we have

$$\Omega(f; \lambda\delta) \leq 2(1+\lambda)^3(1+\delta^2)\Omega(f;\delta).$$

Proof. The inequality is trivial for $n = 1$. Assume $n > 1$, for all $s,\ t > 0$, we have

$$\Omega(f; n\delta) \leq \sup_{\substack{|\log t| \leq \delta \\ s>0}} \frac{|f(t^n s) - f(s)|}{(1+(\log s)^2)(1+(\log t)^2)},$$

on the other hand, an elementary computation show that

$$\begin{aligned}
|f(t^n s) - f(s)| &\leq \sum_{k=1}^{n} |f(t^k s) - f(t^{k-1} s)| \\
&\leq \Omega(f;\delta) \sum_{k=1}^{n} [1+\log^2 t][1+\log^2(t^{k-1} s)] \\
&\leq \Omega(f;\delta) \sum_{k=1}^{n} [1+\log^2 t][1+n^2\log^2(ts)] \\
&\leq 2n^3\Omega(f;\delta)[1+\log^2 t]^2[1+\log^2 s].
\end{aligned}$$

Considering the definition of weighted modulus of continuity and the inequality $|\log t| \leq \delta$, $s > 0$, we get

$$\Omega(f; n\delta) \leq 2n^3(1+\delta^2)\Omega(f;\delta).$$

For the second part, let $\lceil\cdot\rceil$ indicate the integer part of the function or ceiling function. Consider $\lceil\lambda\rceil = n$ where $n \in \mathbb{N}$. By the definition of integer part we have $\lceil\lambda\rceil \leq \lambda \leq \lceil\lambda\rceil + 1$, for every $\lambda > 0$ and $\lceil\lambda\rceil + 1$ is a natural number, it follows that

$$\Omega(f; \lambda\delta) \leq 2(1+\lambda)^3(1+\delta^2)\Omega(f;\delta).$$

□

Lemma 3.4. *For all $f \in C_2(\mathbb{R}^+)$, $\delta > 0$, $n \in \mathbb{N}$, then we have*

$$|f(h) - f(s)| \leq 16(1+(\log s)^2)(1+\delta^2)^2\left(1 + \frac{|\log h - \log s|^5}{\delta^5}\right)\Omega(f;\delta), \quad (h,\ s > 0).$$

Proof. For $h > 0$, taking $ts = h$ in (3.1) and using Lemma 3.3, we get

$$\begin{aligned}
|f(h) - f(s)| &\leq \Omega(f; |\log h - \log s|)(1+(\log s)^2)(1+(\log h - \log s)^2) \\
&\leq 2\left(1 + \frac{|\log h - \log s|}{\delta}\right)^3 (1+\delta^2)(1+(\log s)^2)(1+(\log h - \log s)^2)\Omega(f;\delta),
\end{aligned}$$

which can be expressed in the form

$$|f(h) - f(s)| \leq 16(1+(\log s)^2)(1+\delta^2)^2\Omega(f;\delta) \text{ if } |\log h - \log s| < \delta \tag{3.2}$$

and

$$|f(h) - f(s)| \leq 16(1+(\log s)^2)(1+\delta^2)^2\frac{|\log h - \log s|^5}{\delta^5}\Omega(f;\delta) \text{ if } |\log h - \log s| \geq \delta. \tag{3.3}$$

Taking together (3.2) and (3.3), we have for all $h,\ s \in \mathbb{R}^+$ that

$$|f(h) - f(s)| \leq 16(1 + (\log s)^2)(1 + \delta^2)^2 \left(1 + \frac{|\log h - \log s|^5}{\delta^5}\right) \Omega(f; \delta), \quad (h,\ s > 0).$$

□

Theorem 3.5. *Let $\mathcal{A} = \left\{\left[a_{nk}^v\right]\right\}$ be a non negative regular summability method and $\{\delta_n\}$ be a null sequence of positive real numbers then for every $f \in C_2^k(\mathbb{R}^+)$, we have*

$$\|G_{n,v}(f)(s) - f\|_L = O\left(\Omega(f; \delta_n)\right)$$

as $n \to \infty$ (uniformly in v).

Proof. Let $f \in C_2^k(\mathbb{R}^+)$ and by definition of the operator $G_{n,v}$ we can get for $s > e^{\frac{\sqrt{2}}{k}}$

$$\begin{aligned}
|G_{n,v}(f)(s) - f(s)| &= |\mathcal{G}_{n,v}(f)(s) - f(s)| \\
&\leq \sum_{k=1}^{\infty} a_{nk}^v \frac{k}{\sqrt{4\pi}} \int_0^{\infty} |f(ta_k(s)) - f(s)|\, e^{-\left(\frac{k}{2}\log t\right)^2} \frac{dt}{t} \\
&\leq \sum_{k=1}^{\infty} a_{nk}^v \frac{k}{\sqrt{4\pi}} \int_0^{\infty} |f(ta_k(s)) - f(a_k(s))|\, e^{-\left(\frac{k}{2}\log t\right)^2} \frac{dt}{t} \\
&+ \sum_{k=1}^{\infty} a_{nk}^v \frac{k}{\sqrt{4\pi}} \int_0^{\infty} |f(a_k(s)) - f(s)|\, e^{-\left(\frac{k}{2}\log t\right)^2} \frac{dt}{t} \\
&\leq I_1 + I_2.
\end{aligned}$$

Now,

$$I_1 = \sum_{k=1}^{\infty} a_{nk}^v \frac{k}{\sqrt{4\pi}} \int_0^{\infty} |f(ta_k(s)) - f(a_k(s))|\, e^{-\left(\frac{k}{2}\log t\right)^2} \frac{dt}{t}.$$

Using Lemma 3.4,

$$I_1 \leq \sum_{k=1}^{\infty} a_{nk}^v \frac{k}{\sqrt{4\pi}} 16\left(1 + (\log a_k)^2\right)(1 + \delta^2)^2 \Omega(f; \delta) \int_0^{\infty} \left\{1 + \frac{|\log(ta_k) - \log a_k|^5}{\delta^5}\right\} e^{-\left(\frac{k}{2}\log t\right)^2} \frac{dt}{t}.$$

Since $\log a_k \leq \log s$, we have

$$I_1 \leq 16\left(1 + (\log s)^2\right)(1 + \delta^2)^2 \Omega(f; \delta) \sum_{k=1}^{\infty} a_{nk}^v \frac{k}{\sqrt{4\pi}} \int_0^{\infty} \frac{|\log t|^5}{\delta^5} e^{-\left(\frac{k}{2}\log t\right)^2} \frac{dt}{t}.$$

Now,

$$\begin{aligned}
\sum_{k=1}^{\infty} a_{nk}^v \frac{k}{\sqrt{4\pi}} \int_0^{\infty} \frac{|\log t|^5}{\delta^5} e^{-\left(\frac{k}{2}\log t\right)^2} \frac{dt}{t} &\leq \frac{1}{\delta^5} \sum_{k=1}^{\infty} a_{nk}^v \frac{k}{\sqrt{4\pi}} \int_0^{\infty} |\log t|^5 e^{-\left(\frac{k}{2}\log t\right)^2} \frac{dt}{t} \\
&= \frac{1}{\delta^5} \sum_{k=1}^{\infty} a_{nk}^v \frac{k}{\sqrt{4\pi}} \int_{-\infty}^{\infty} x^5 e^{-\frac{k^2x^2}{4}} dx \\
&= \frac{1}{\delta^5} \sum_{k=1}^{\infty} a_{nk}^v \frac{k}{\sqrt{4\pi}} \frac{128}{k^6} \\
&= \frac{1}{\delta^5} \sum_{k=1}^{\infty} a_{nk}^v \frac{1}{\sqrt{4\pi}} \frac{64}{k^5} \\
&= \frac{64}{\delta^5 \sqrt{\pi}} M_1,
\end{aligned}$$

choosing $\delta = \delta_n$ we get,

$$I_1 \leq 16\left(1+(\log s)^2\right)(1+\delta_n^2)^2\left(1+\frac{64}{\delta_n^5\sqrt{\pi}}M_1\right)\Omega(f;\delta_n)$$

$$\frac{I_1}{(1+(\log s)^2)} \leq C_1(1+\delta_n^2)^2\Omega(f;\delta_n)$$

$$\frac{I_1}{(1+(\log s)^2)} \leq O\left(\Omega(f;\delta_n)\right) \text{ as } n\to\infty \text{ (uniformly in } v). \tag{3.4}$$

Now, using definition of the weighted modulus of continuity and Lemma 3.3, we obtain

$$\begin{aligned}
I_2 &= \sum_{k=1}^{\infty} a_{nk}^v \frac{k}{\sqrt{4\pi}}\int_0^{\infty} |f(a_k(s))-f(s)|\, e^{-\left(\frac{k}{2}\log t\right)^2}\frac{dt}{t}\\
&= \sum_{k=1}^{\infty} a_{nk}^v |f(a_k(s))-f(s)|\\
&\leq \sum_{k=1}^{\infty} a_{nk}^v (1+\log^2 s)\left(1+\log^2\frac{a_k(s)}{s}\right)\Omega\left(f;\left|\log\frac{a_k(s)}{s}\right|\right)\\
&\leq (1+\log^2 s)\sum_{k=1}^{\infty} a_{nk}^v\left(1+\frac{4}{k^4\left(\sqrt{\log^2 s-\frac{2}{k^2}}+\log s\right)^2}\right)\Omega\left(f;\frac{2}{k^2\left(\sqrt{\log^2 s-\frac{2}{k^2}}+\log s\right)}\right)\\
&\leq 2(1+\log^2 s)\sum_{k=1}^{\infty} a_{nk}^v\left(1+\frac{4}{k^4}\right)\left(1+\frac{4}{k^4\left(\sqrt{\log^2 s-\frac{2}{k^2}}+\log s\right)^2}\right)\left(1+\frac{1}{\sqrt{\log^2 s-\frac{2}{k^2}}+\log s}\right)^3\Omega\left(f;\frac{2}{k^2}\right)\\
&\leq 2(1+\log^2 s)\sum_{k=1}^{\infty} a_{nk}^v\left(1+\frac{4}{k^4}\right)\left(1+\frac{4}{k^4}\lambda(s)^2\right)(1+\lambda(s))^3\Omega\left(f;\frac{2}{k^2}\right),
\end{aligned}$$

where $\lambda(s)=\frac{1}{\sqrt{\log^2 s-\frac{2}{k^2}}+\log s}$.

$$\frac{I_2}{1+\log^2 s} \leq C_2\sum_{k=1}^{\infty} a_{nk}^v\Omega\left(f;\frac{2}{k^2}\right)$$

$$\frac{I_2}{1+\log^2 s} \leq O\left(\Omega\left(f;\delta_n\right)\right) \text{ as } n\to\infty \text{ uniformly in } v. \tag{3.5}$$

From (3.4) and (3.5) we have,

$$\|G_{n,v}(f)(s)-f\|_L = O\left(\Omega(f;\delta_n)\right)$$

as $n\to\infty$ uniformly in v. □

4. Special cases

In this section, we investigate some special cases of Theorem 2.5 and Theorem 3.5. Using different matrix summability methods we have different approximation results as follows.

Let $\alpha\in(0,1)$, we consider the following Lipschitz class of functions

$$Lip\,(\alpha) = \{f\in C_2(\mathbb{R}^+) : \omega_L(f,\delta) = O(\delta^\alpha) \text{ as } \delta\to 0\}.$$

Corollary 4.1. *Consider the Cesàro matrix summability $\mathcal{A} = \{C_1\} = \{[c_{nk}]\}$ (cf. [18]) given by*

$$c_{nk} = \begin{cases} \frac{1}{n}, & \text{if } k = 1, 2, \ldots, n \\ 0, & \text{otherwise.} \end{cases}$$

Which gives the arithmetic mean convergence. Then we get the following approximation results for $f \in C_2^k(\mathbb{R}^+)$

$$\lim_{n\to\infty} \left\| \frac{G_1(f) + G_2(f) + \cdots + G_n(f)}{n} - f \right\|_L = 0,$$

i.e., the sequence $\{G_n(f)\}$ defined in [24] is uniformly Cesàro summable to f.

And for $f \in Lip(\alpha)$, choosing $\delta_n = \frac{1}{n}$, we have

$$\left\| \frac{G_1(f) + G_2(f) + \cdots + G_n(f)}{n} - f \right\|_L = O(n^{-\alpha}) \text{ as } n \to \infty.$$

Corollary 4.2. *$\mathcal{A} = \{F^v\} = \{[c_{nk}^v]\}$, the almost convergence matrix (cf. [22]) defined by*

$$c_{nk}^v = \begin{cases} \frac{1}{n}, & \text{if } v \le k \le v + n - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then for $f \in C_2^k(\mathbb{R}^+)$ we have the following approximation

$$\lim_{n\to\infty} \left\| \frac{G_v(f) + G_{v+1}(f) + \cdots + G_{v+n-1}(f)}{n} - f \right\|_L = 0, \text{ uniformly in } v$$

and taking $\delta_n = \frac{1}{n}$, for $f \in Lip(\alpha)$, we have

$$\left\| \frac{G_v(f) + G_{v+1}(f) + \cdots + G_{v+n-1}(f)}{n} - f \right\|_L = O(n^{-\alpha}) \text{ as } n \to \infty \text{ uniformly in } v.$$

Corollary 4.3. *If we choose $\mathcal{A} = I$, the identity matrix, in Theorem 2.5, then we obtain the approximation result in (cf. [24, Theorem 4.1]).*

5. Conclusion

In this paper, we applied regular summability methods to modified Mellin convolution operators and analyzed the approximation properties in weighted spaces. The findings of this paper show that the modified Mellin-Gauss-Weierstrass operator provides better approximation behavior in weighted framework than previous findings in the area. The order of convergence given in Theorem 4.2 of [24] is also improved, and the strengthened result is established in Theorem 3.5. The use of regular summability methods expands the operator's applicability and strengthens the convergence analysis. Moreover, certain results obtained in approximation are preserved under regular summability methods. The weighted logarithmic modulus of continuity helps in finer details of this convergence. The present results should help further in understanding how these operators behave under the supremum norm. It also encourages for further refinements in weighted approximation theory.

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